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Wave scattering theory and the absorption problem for a black hole

Norma Sánchez
Departement d'Astrophysique Fondamentale, Observatoire de Paris, 92190 Meudon, France
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The general problem of scattering and absorption of waves from a Schwarzschild black hole is investigated. A scattering absorption amplitude is introduced. The unitarity theorem for this problem is derived from the wave equation and its boundary conditions. The formulation of the problem, within the formal scattering theory approach, is also given. The existence of a singularity in space-time is related explicitly to the presence of a nonzero absorption cross section. Another derivation of the unitarity theorem for our problem is given by operator methods. The reciprocity relation is also proved; that is, for the scattering of waves the black hole is a reciprocal system. Finally, the elastic scattering problem is considered, and the elastic scattering amplitude is calculated for high frequencies and small scattering angles.

INTRODUCTION

This article represents a contribution to a line of research that was initiated by Matzner,[1] continued by Persides[2] and myself,[3] and on which much remains to be done in the future.

In this scattering problem, the exact solutions for the scattering parameters have not been found even in partial waves. The complexity of the exact radial wave solutions[2] makes it very difficult. Approximate analytical results for the phase shifts, absorption coefficients, and elastic and inelastic cross sections have been reported previously by us.[3]

However, in scattering theory it is not only the explicit expressions for the cross sections (and other scattering parameters) that are interesting, but also general properties satisfied by them. In this paper, we consider this last problem for the specific case of scattering of waves by a black hole. The existence of a singularity in space-time is related explicitly to the presence of a nonzero absorption cross section, and general properties of the wave scattering amplitudes from a Schwarzschild geometry are established.

The wave equation in Schwarzschild space-time has singularities at $r=0$ and $r=r_s$, and the radial behavior of the solutions near these points is the same as the behavior of the quantum-wave solutions near the origin for an inverse-square attractive nonrelativistic potential. As is known (see, for example, Ref. 4), the physical solution for such potentials cannot be determined uniquely from a regularity condition, and additional assumptions are required to specify it. In this black-hole problem, the physical solution of the wave equation is such that it has only purely ingoing waves on the horizon $r=r_s$.[1]

In Sec. I, we introduce a regularization in order to have the physical solution well defined even for $r=r_s$. It is based on the analytic continuation of the solution in the variable r_s.

If we take $\epsilon=\mathrm{Im}\, r_s$ positive, it can be seen that the physical solution is regular at the horizon. This suggests that we define the physical solution as the $\epsilon\to 0+$ limit of the regular solution.

In Sec. II, an absorption scattering amplitude, whose modulus squared gives the differential absorption cross section, is introduced by means of the behavior of the physical solution for $r\to r_s+$. We relate the total absorption cross section obtained from this amplitude to the parameters connected to the asymptotic behavior of the solution for $r\to\infty$, and also to the $r\to 0+$ behavior of the solution.

Within this context, we establish general properties of the wave scattering amplitudes from a Schwarzschild geometry. We derive a unitarity relation which takes into account the absorption of waves by the black hole. This relation generalizes for our case the well known unitarity theorem in elastic potential scattering theory. It is well known that one can describe an absorption process in one-channel scattering theory by using a complex (nonsingular) potential, that is, a non-Hermitian Hamiltonian. We find that, in our problem, although the effective Hamiltonian is real, it is not Hermitian, because of its singularity at the origin (this is shown in Sec. III). Moreover, in the $\epsilon\to 0+$ limit, the Hamiltonian is Hermitian and the current density is conserved for all $r\neq 0$ (even at $r=r_s$). We will see (in Sec. III) that the divergence of the current density is proportional to a Dirac δ function.

In Sec. III, we formulate the scattering problem by a black hole in the formal scattering theory approach. The use of this powerful formalism has permitted a better insight into the absorption problem. In this formalism, it is convenient to work with an "effective Hamiltonian" which plays a role similar to that of the true Hamiltonian in the time-independent quantum-scattering theory. We find that this effective Hamiltonian is Hermitian except at the origin. This non-Hermitian character is due entirely to the singularity present at the origin of the Schwarzschild space-time. The expression for the difference between the effective Hamiltonian and its adjoint is derived. It is a distribution concentrated at the origin [Eq. (46)]. An absorption matrix is introduced as a measure of the difference between the unit operator and the product $SS^\dagger$ (S stands for the elastic S matrix). We derive the relation between this absorption matrix and the anti-Hermitian part of the effective Hamiltonian, which was previously found. Thus, the relation between the singularity at the origin of the Schwarzschild space-time and the presence of absorption processes is explicitly shown.

In Sec. IV, we show that for the scattering of waves the black hole behaves reciprocally. We give another independent proof of the reciprocity theorem by formal operator methods. Although the effective Hamiltonian is neither symmetric, nor time-reversal-invariant, the reciprocity relation holds because of the equality of some matrix elements of the effective Hamiltonian and its transpose.

In Sec. V, the elastic scattering problem is considered. We calculate from the scattering integral equation the elastic scattering amplitude for high frequencies k and small deviation angles θ. The differential elastic cross section obtained from this amplitude gives the Rutherford law plus corrections. In the limit $k\to\infty$, the geometrical optical result is recovered.

I. GENERAL CONSIDERATIONS

We begin by considering a complex scalar field in a curved space-time. It satisfies the equation

$$g^{\mu\nu}\psi_{;\mu\nu}=0\,, \tag{1}$$

as does its complex conjugate,

$$g^{\mu\nu}\psi^*_{;\mu\nu}=0\,, \tag{2}$$

where the semicolon denotes covariant differentiation.

Multiplying (1) by ψ^*, (2) by ψ, and subtracting, one obtains

$$\left[\sqrt{-g}\,g^{\mu\nu}\left(\psi^*\frac{\partial\psi}{\partial x^\nu}-\psi\frac{\partial\psi^*}{\partial x^\nu}\right)\right]_{,\mu}=0\,. \tag{3}$$

Here the comma denotes ordinary differentiation. This is a conservation law associated with the invariance of the Lagrangian under the transformation

$$\psi\to\psi e^{i\alpha}\quad(\alpha \text{ is a real constant}).$$

It is clear that Eq. (3) is only valid at points where the metric tensor is nonsingular.

We integrate Eq. (3) over a three-dimensional volume and use Gauss's theorem to obtain

$$\oint(-\sqrt{-g}\,g^{ik}j_k)dS_i=-\int\frac{\partial}{\partial\tau}(\sqrt{-g}\,j_\tau)dV\,, \tag{4}$$

where

$$\begin{aligned} j_k&=\frac{1}{2i}\left(\psi^*\frac{\partial\psi}{\partial x^k}-\psi\frac{\partial\psi^*}{\partial x^k}\right),\\ j_\tau&=\frac{1}{2i}\left(\psi^*\frac{\partial\psi}{\partial\tau}-\psi\frac{\partial\psi^*}{\partial\tau}\right), \end{aligned} \tag{5}$$

and

$$d\tau=\sqrt{g_{00}}\,dx^0\,. \tag{6}$$

The four quantities j^τ, j^k are the components of a conserved current density. In certain cases it is possible to relate the spatial current $\vec{j}$ to the Poynting vector $\vec{S}$.

From now on we will consider coordinate frames where $g_{0k}=0$. The general expression for $\vec{S}$ is then

$$S_k=\partial_0\psi\partial_k\psi^k+\partial_0\psi^*\partial_k\psi\,; \tag{7}$$

it can be obtained from the energy-momentum tensor of the scalar field.

If one considers a single-frequency solution

$$\psi=e^{-i\omega t}\Psi \tag{8}$$

from (5) and (7), one gets

$$\vec{S}=2\omega\vec{j}\,.$$

Thus, in order to express the energy flux both vectors are equivalent.

We shall consider Eq. (1) for static gravitational fields, and make the temporal separation (8). Then

$$\frac{g_{00}}{\sqrt{|g|}}\partial_i(\sqrt{|g|}\,g^{ik}\partial_k\Psi)-k^2\Psi=0\,, \tag{9}$$

where

$$t=x^0\,,$$

$$k=\omega\,.$$

We write (9) as

$$H\Psi=k^2\Psi\,, \tag{10}$$

where H is the energy-dependent operator

$$H\equiv\partial_i(\sqrt{|g|}\,g^{ik}\partial_k)+k^2(1-\sqrt{|g|}\,g^{00})\,. \tag{11}$$

Now we write Eq. (10) in the Schwarzschild metric. It gives

$$\left[-\vec{\nabla}^2+\frac{r_s}{r}\left(\partial_r{}^2+\frac{1}{r}\partial_r-\frac{k^2}{1-r_s/r}\right)\right]\Psi=k^2\Psi. \quad (12)$$

The behavior of the physical solution of Eq. (12) for $r\to r_s+$ reads

$$\Psi=(r-r_s)^{-ikr_s}g(\theta,\varphi)[1+O(r-r_s)]; \quad (13)$$

it has a divergent phase for $r=r_s$.

One can use a regularization in order to have Ψ well defined also at the Schwarzschild radius. We make an analytic continuation taking r_s complex. We write

$$r_s=a+i\epsilon\,,\quad \epsilon>0\,. \quad (14)$$

The plus sign in (14) guarantees that solution (13) is finite in $r=r_s$. Thus, with (14), the operator (12) also contains the boundary condition, and the physical solution can be obtained as the $\epsilon\to 0+$ limit of the regular solution.

We proceed to consider the analytic continuation of the current $\vec{j}$ [Eq. (5)] as a function of r_s. In order to preserve the validity of the conservation law [Eq. (3)] for complex r_s, we define

$$\vec{j}_\epsilon=\frac{1}{2i}\left(1-\frac{r_s}{r}\right)\hat{e}_r[\Psi^*(r_s^*)\partial_r\Psi(r_s)-\Psi(r_s)\partial_r\Psi^*(r_s^*)]$$
$$+\frac{1}{2i}[\Psi^*(r_s^*)\vec{\nabla}_\Omega\Psi(r_s)-\Psi(r_s)\vec{\nabla}_\Omega\Psi^*(r_s^*)]\,, \quad (15)$$

where $\vec{\nabla}_\Omega$ denotes the angular part of the gradient. The component $j_{r\epsilon}$ can be written as

$$j_{r\epsilon}=\frac{1}{2i}\left(1-\frac{r_s}{r}\right)W_r[\Psi^*(r_s^*)\,,\Psi(r_s)]\,,$$

where W_r is the radial Wronskian. It is easily seen that for $r\neq r_s$ the $\epsilon\to 0+$ limit of the right-hand side of Eq. (15) coincides with Eq. (5).

II. THE ABSORPTION PROBLEM

The asymptotic behavior of the solution of the wave equation [Eq. (12)], which describes the scattering of a plane wave by a Schwarzschild black hole, reads (see Appendix A)

$$\Psi_{\vec{k}}=e^{i\vec{k}\cdot\vec{r}-ikr_s\ln[kr(1-\cos\theta)]}$$
$$+\frac{f(\theta)}{r}e^{ikr+ikr_s\ln 2kr}+O\left(\frac{1}{r^2}\right), \quad (16)$$

where $\vec{k}$ stands for the wave vector of the incident wave ($\cos\theta=\hat{k}\cdot\hat{r}$), and we have taken into account the Coulomb tail of the interaction. Here $f(\theta)$ is the elastic scattering amplitude whose modulus squared gives the differential elastic cross section. As we will see in a moment, the function $g(\theta)$ in Eq. (13) results in the absorption scattering amplitude.

In order to find a general expression for the absorption cross section, we consider the current density given by Eq. (16). The differential flux absorbed by the black hole can be written as

$$d\Phi_{\text{abs}}=2ka^2\left(\lim_{r\to a+}j_{r\epsilon}\right)d\Omega\,. \quad (17)$$

This flux gives the energy absorbed by unit proper time τ, and unit solid angle, as one can see from Eq. (4). Then

$$\frac{d\sigma_{\text{abs}}}{d\Omega}=\frac{1}{\Phi_{\text{inc}}}\frac{d\Phi_{\text{abs}}}{d\Omega}$$
$$=a^2\,|g(\theta)|^2\,, \quad (18)$$

where the incident flux Φ_{inc} is $2k^2$.

The absorption scattering amplitude can be expanded in partial waves,

$$g(\theta)=\sum_{l=0}^{\infty}g_l P_l(\cos\theta)\,. \quad (19)$$

The partial-wave absorption coefficients g_l can be related to the imaginary part of the phase shifts $\delta_l(k)$.

We can write

$$\Psi(\vec{r})=\sum_{l=0}^{\infty}\mathcal{R}_l(r)P_l(\cos\theta)\,, \quad (20)$$

where $\mathcal{R}_l(r)$ satisfies the radial wave equation

$$r(r-r_s)^2\frac{d^2\mathcal{R}_l}{dr^2}+(r-r_s)(2r-r_s)\frac{d\mathcal{R}_l}{dr}$$
$$+[k^2r^3-l(l+1)(r-r_s)]\mathcal{R}_l=0\,. \quad (21)$$

From Eqs. (13), (19), and (20), it follows that

$$\mathcal{R}_l(r)\underset{r\to r_s+}{\sim}g_l(r-r_s)^{-ikr_s}[1+0(r-r_s)]\,. \quad (22)$$

The Wronskian of two radial solutions behaves like

$$W[\mathcal{R}_l^*,\mathcal{R}_l]=\frac{K_l}{r(r-r_s)}\,, \quad (23)$$

as can be proved easily from Eq. (21). The constant K_l can be evaluated from the behavior of $\mathcal{R}_l(r)$ near the Schwarzschild radius [Eq. (22)]

$$K_l=-2ikrs^2\,|g_l\,|^2\,. \quad (24)$$

This constant can also be obtained from the asymptotic behavior of $\mathcal{R}_l(r)$ for $r\to\infty$,

$$\mathcal{R}_l(r)=\left[\frac{A_l}{r}\sin\left(kr-l\,\frac{\pi}{2}+kr_s\ln 2kr+\delta_l(k)\right)\right]$$
$$+O\left(\frac{1}{r^2}\right)$$

with

$$A_l=i^l\,\frac{(2l+1)}{k}\,e^{i\delta_l}\,. \quad (24')$$

Equating both results, we obtain

$$|g_l|^2=\frac{|A_l|^2}{2a^2}\sinh 2\beta_l\,, \tag{25}$$

where

$$\beta_l=\mathrm{Im}\,\delta_l\,.$$

This formula shows explicitly the relation of the imaginary part of the phase shifts to the partial-wave absorption amplitudes.

We now proceed to express the total absorption cross section σ_{abs} in a partial-wave expansion. From Eqs. (18) and (19), it follows that

$$\sigma_{\text{abs}}=a^2\int |g(\theta)|^2 d\Omega$$

$$=4\pi a^2\sum_{l=0}^{\infty}\frac{|g_l|^2}{2l+1}\,. \tag{26}$$

Substituting Eq. (25) into Eq. (26), we see that we have obtained, solely from the wave equation (and the boundary conditions), the standard expression for the total inelastic cross section,[5]

$$\sigma_{\text{abs}}=\frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)(1-e^{-4\beta_l})\,. \tag{27}$$

We wish to point out that this expression is derived in standard textbooks[5] from assumptions on the unitarity of the multichannel S matrix, without reference to wave equations. We also recall that the absorption cross section has been defined by means of the behavior of the solution for $r\to r_s+$. In Eq. (27), we give its expression in terms of parameters related to the asymptotic behavior of $r\to\infty$. We can also obtain an expression for the total absorption cross section in terms of parameters related to the behavior of the solution near the origin.

In the neighborhood of $r=0$, the behavior of two linearly independent radial solutions is[2]

$$\mathcal{R}_{1l}=1+O(r)\,, \tag{28}$$

$$\mathcal{R}_{2l}=\ln r[1+O(r)]+1+O(r)\,. \tag{29}$$

We have chosen $\mathcal{R}_{1l}$ and $\mathcal{R}_{2l}$ as real functions. As we know, in the neighborhood of $r=r_s$ two linearly independent radial solutions are given by Eq. (22) and their complex conjugate, which we denote $\mathcal{R}_{l(+)}$ and $\mathcal{R}_{l(-)}$, respectively. These functions are related by

$$\mathcal{R}_{l(+)}=C_l\mathcal{R}_{1l}+D_l\mathcal{R}_{2l}\,, \tag{30}$$

$$\mathcal{R}_{l(-)}=C_l^*\mathcal{R}_{1l}+D_l^*\mathcal{R}_{2l}\,. \tag{31}$$

It is easily seen that the constants C_l, D_l are expressed as

$$C_l=-\frac{r}{r_s}(r-r_s)W[\mathcal{R}_{2l},\mathcal{R}_{l(+)}]\,,$$

$$D_l=\frac{r}{r_s}(r-r_s)W[\mathcal{R}_{1l},\mathcal{R}_{l(+)}]\,.$$

We evaluate the constant K_l [Eq. (23)] with the solutions given by Eqs. (30) and (31) and, using Eq. (24), we obtain

$$|g_l|^2=\frac{1}{kr_s}\,\mathrm{Im}(C_l^*D_l)\,. \tag{32}$$

That is to say,

$$\sigma_{\text{abs}}=\frac{4\pi r_s}{k}\sum_{l=0}^{\infty}\mathrm{Im}\,\frac{(C_l^*D_l)}{(2l+1)}\,, \tag{33}$$

which gives the total absorption cross section in terms of the Wronskians between the radial solutions defined near the origin and near the horizon. These Wronskians and their properties have been studied by Persides.[2]

We will now derive the unitarity theorem which relates the absorption cross section to the elastic scattering amplitude. We consider two solutions $\Psi_{\vec{k}}(\vec{r})$ and $\Psi_{\vec{k}'}(\vec{r})$, whose asymptotic behaviors are given in Eq. (17). We denote the elastic scattering amplitudes for $\Psi_{\vec{k}}$ and $\Psi_{\vec{k}'}$ as $f(\vec{k},\vec{k}_r)$ and $f(\vec{k}',\vec{k}_r)$, respectively, where $|\vec{k}|=|\vec{k}'|$ and $\vec{k}_r=|\vec{k}|\hat{r}$.

It follows from the wave equation (12) that we have the identity

$$\vec{\nabla}\cdot\Big[\Psi_{\vec{k}}\vec{\nabla}\Psi^*_{\vec{k}'}-\Psi^*_{\vec{k}'}\vec{\nabla}\Psi_{\vec{k}} - \hat{e}_r\frac{r_s}{r}(\Psi_{\vec{k}}\partial_r\Psi_{\vec{k}'}-\Psi^*_{\vec{k}'}\partial_r\Psi_{\vec{k}})\Big]=0\,.$$

We integrate this expression over the volume that extends for $r=r_s$ to a large spherical surface $(r=R)$. Using Green's theorem, we find

$$\oint_R W_R[\Psi_{\vec{k}},\Psi^*_{\vec{k}'}]\,dS_R - \oint_{R=r_s}\Big(1-\frac{r_s}{R}\Big)W_R[\Psi_{\vec{k}},\Psi^*_{\vec{k}'}]\,dS_R=0\,.$$

From the asymptotic form (16) of the solution and the behavior (13), one obtains in the $R\to\infty$ limit, by using the stationary-phase method

$$\frac{i}{2i}[f(\vec{k},\vec{k}')-f^*(\vec{k}',\vec{k})] =\frac{k}{4\pi}\int f(\vec{k},\vec{k}_r)f^*(\vec{k}',\vec{k}_r)\,d\Omega_r-\frac{ka^2}{4\pi}\int|g(\theta)|^2 d\Omega\,. \tag{34}$$

This is the unitarity theorem for the scattering of waves by a black hole. The first term on the right-hand side takes into account the elastic scattering and the second term takes into account the absorption process. It must be pointed out that Eq. (34) has been derived from the wave equation and its boundary conditions without any further assumptions.

It is well known in potential scattering theory that absorption processes are related to non-Hermitian Hamiltonians. This suggests that we study the properties of our effective Hamiltonian [Eq. (11)]. In the following section we will find that this Hamiltonian is not Hermitian, although it is real. This non-Hermitian character is the reason for Eq. (34). We are thus motivated to derive a more abstract unitarity relation by using the results and methods of formal scattering theory.

III. FORMAL APPROACH

Let us derive the scattering integral equation from the wave equation and the boundary conditions. It is convenient to split the effective Hamiltonian H, Eq. (12), into two parts,

$$H = H_0 + H' \ ,$$

so that H_0 is an exactly soluble differential operator.

We define the "free" Green's function as the solution of the equation

$$(H_0 - k^2) G_{(+)}(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \ , \tag{35}$$

which at large distances contains only outgoing spherical waves. The wave equation can be written as

$$(H_0 - k^2)\Psi(\vec{r}) = -H'\Psi(\vec{r}) \, . \tag{36}$$

From Eqs. (35) and (36) and integrating over the domain between the spherical surfaces S at $r = r_s$ and S' at $r = R \to \infty$, we obtain

$$\Psi(\vec{r}) = \Psi(0) + \oint_S \left(\Psi \, \frac{\partial G}{\partial r'} - G \, \frac{\partial \Psi}{\partial r'}\right) dS' + \sum_{r_s} G_{(+)}(\vec{r}, \vec{r}\,') H'(r') \Psi(\vec{r}\,') \, d^3 r' \ , \tag{37}$$

where $\Psi_{(0)}$ is a solution of the homogeneous equation

$$(H_0 - k^2)\Psi_{(0)} = 0 \tag{38}$$

that has been added in order to obtain the general solution of Eq. (36). It is easy to see that the contribution of the integral over S' vanishes in the limit $R \to \infty$. (From now on, we will understand that the limit $\epsilon \to 0+$ has been taken.)

The integral equation (37) does not have the form of the Lippmann-Schwinger equations owing to the contribution of the surface integration over S. By operator methods these equations read[6]

$$\Psi_{(+)} = \Psi_{(0)} + G_{(+)} H' \Psi_{(+)} \ , \tag{39}$$

$$\Psi_{(-)} = \Psi_{(0)} + G_{(-)} H' \Psi_{(-)} \ , \tag{40}$$

where $G_{(\pm)}$ are the Green's operators

$$G_{(\pm)} = (k^2 \pm i0 - H_0)^{-1} \, .$$

The function $\Psi_{(+)}$ is that solution of the scattering problem, which has incoming plane waves and outgoing scattered waves. The term $+i0$ in $G_{(+)}$ guarantees that only outgoing scattered waves exist. Similarly $\Psi_{(-)}$ is that solution which has incoming spherical waves and outgoing plane waves.

Also, Eqs. (39) and (40) can be expressed as

$$\Psi_{(+)} = \Psi_{(0)} + \mathcal{G}_{(+)} H' \Psi \ , \tag{41}$$

$$\Psi_{(-)} = \Psi_{(0)} + \mathcal{G}_{(-)} H' \Psi_{(0)} \tag{42}$$

in terms of the Green's operators for H,

$$\mathcal{G}(\pm) = (k^2 \pm i0 - H)^{-1} \, .$$

In order to use operator methods, within formal scattering theory, it is more convenient to have an integral scattering equation of the Lippmann-Schwinger form. In order to recast Eq. (37) into the form of Eq. (39), we will extend the three-dimensional integration over all positive values of the r coordinate. Because of the behavior of the solution at the origin,[2] the surface integral

$$\oint_{r'=\delta} \left(\Psi \, \frac{\partial G}{\partial r'} - G \, \frac{\partial \Psi}{\partial r'}\right) dS'$$

vanishes in the limit $\delta \to 0$.

Also, we will label the physical solution with the subscript $(+)$. The solution $\Psi_{(+)}$ satisfies the physical boundary condition at $r \to \infty$ at $r \to r_s +$.

Then we obtain

$$\Psi_{(+)}(\vec{r}) = \Psi_{(0)} + \int G_{(+)}(\vec{r}, \vec{r}\,') H'(r') \Psi_{(+)}(\vec{r}\,') d^3 r', \tag{43}$$

which has the desired Lippmann-Schwinger form. Also, we define a scalar Hermitian product as

$$(\psi, \phi) = \int \psi^* \phi \, d^3 r \, . \tag{44}$$

Now we will study the non-Hermitian character of the operator H, with respect to the scalar product (44).

By integrating the identity

$$\phi H \xi - \xi H \Phi = \nabla \cdot \vec{\mathcal{P}}(\phi, \xi) \tag{45}$$

for ϕ, ξ, which are two solutions of the wave equation (12), where

$$\vec{\mathcal{P}}(\phi, \xi) = \left(1 - \frac{r_s}{r}\right) (\phi \partial_r \xi - \xi \partial_r \phi) \hat{e}_r + \phi \vec{\nabla}_\Omega \xi - \xi \vec{\nabla}_\Omega \phi \ ,$$

and from the fact that

$$\lim_{\delta \to 0} \oint_{r=\delta} \mathcal{P}_r(\phi, \xi) da = \left[\left(1 - \frac{r_s}{r}\right) 4\pi r^2 (\phi \partial_r \xi - \xi \partial_r \phi) \right]_{r \to 0+} \ ,$$

we obtain the difference $H - H^\dagger$ by operator methods:

$$H - H^\dagger = \left(1 - \frac{r_s}{r}\right)\left[\delta(r)\frac{\partial}{\partial r} - \left(\delta(r)\frac{\partial}{\partial r}\right)^\dagger\right]. \tag{46}$$

It is seen that the effective Hamiltonian is non-Hermitian only at the origin. This non-Hermiticity comes from its singular character at that point, which reflects the singularity present there in the Schwarzschild space-time.

Furthermore, it can be proved from Eqs. (17), (20), and (23) that the flux through a sphere with center at the origin is independent of its radius (R). This means that the current density is conserved everywhere except at the origin. In other words,

$$\vec{\nabla}\cdot\vec{j} = \Phi_{\text{abs}}\,\delta(\vec{r}). \tag{47}$$

Now we will proceed to derive in a formal way the generalized unitarity relation.

It is convenient to take for H_0 a Hermitian operator, in such a way that the non-Hermitian piece of H coincides with the non-Hermitian piece of H'. For example, we take

$$H_0 = -\vec{\nabla}^2 - \frac{1}{2}\left[\delta(r)\frac{\partial}{\partial r} - \left(\delta(r)\frac{\partial}{\partial r}\right)^\dagger\right].$$

It is easy to check from Eq. (44) that this operator is Hermitian.

From the standard relation

$$(\Psi_{0\alpha}, S\Psi_{0\beta}) = (\Psi_{0\alpha}, \Psi_{0\beta}) - 2\pi i\,\delta(k_\alpha^2 - k_\beta^2)T_{\alpha\beta}(k^2), \tag{48}$$

which is between the S and T matrices, and the Lippmann-Schwinger equations (39) and (41), we obtain the usual expression for the matrix T,

$$T = H' + H'\mathcal{G}_{(+)}H', \tag{49}$$

where we have used the property

$$G_{(+)} - G_{(-)} = 2\pi i\,\delta(k^2 - H_0).$$

We wish to point out that Eq. (49) is the same formal expression as that in the Hermitian case.

In Eq. (48), $\Psi_{0\alpha}, \Psi_{0\beta}$ are the free states of H_0 [Eq. (38)] characterized by labels α, β. In terms of $G_{(+)}$, Eq. (49) reads

$$T = H' + H'G_{(+)}T, \tag{50}$$

$$T = H' + TG_{(+)}H'. \tag{51}$$

From these integral equations and using Eq. (49) we derive the following relation:

$$T - T^\dagger = -2\pi i T\delta(k^2 - H_0)T^\dagger + (1 + \mathcal{G}_{(+)}H')^\dagger(H' - H'^\dagger)(1 + \mathcal{G}_{(+)}H'). \tag{52}$$

This is the optical theorem, generalized for a non-Hermitian Hamiltonian as our effective Hamiltonian.

From Eq. (48) and using Eq. (52), we obtain

$$SS^\dagger = 1 - 2\pi i\,\delta(k^2 - H_0)(1 + \mathcal{G}_{(+)}H')^\dagger \times (H' - H'^\dagger)(1 + \mathcal{G}_{(+)}H'). \tag{53}$$

Thus

$$\mathcal{A} = 2\pi i(1 + \mathcal{G}_{(+)}H')^\dagger(H' - H'^\dagger)(1 + \mathcal{G}_{(+)}H') \tag{54}$$

can be considered as an absorption operator, which measures the difference between the unit operator and the product $SS^\dagger$. The matrix elements of $\mathcal{A}$ between the $\Psi_{(0)}$ states give the scattering absorption cross section. This can be seen by integration of the identity (45). Taking $\phi = \Psi_{(+)}^*$, $\xi = \Psi_{(+)}$, and using Eqs. (44) and (47), one obtains

$$(\Psi_{(+)}, H\Psi_{(+)}) - (\Psi_{(+)}, H^\dagger\Psi_{(+)}) = 2ik\sigma_{\text{abs}}. \tag{55}$$

From Eq. (41) and Eq. (55) it follows that

$$(\Psi_{(0)}, \mathcal{A}\Psi_{(0)}) = 4\pi k\sigma_{\text{abs}}, \tag{56}$$

which shows explicitly the connection between the singularity at $r = 0$ and the absorption of waves by the black hole.

IV. RECIPROCITY RELATION

We consider two wave solutions $\Psi_{\vec{k}}(\vec{r})$ and $\Psi_{-\vec{k}'}(\vec{r})$ whose elastic scattering amplitudes are $f(\vec{k}, \vec{k}_r)$ and $f(-\vec{k}', \vec{k}_r)$, respectively, and whose asymptotic behaviors have the form of Eq. (16).

By integrating the identity

$$\Psi_{-\vec{k}'}H\Psi_{\vec{k}} - \Psi_{\vec{k}}H\Psi_{-\vec{k}'} = 0$$

and following lines similar to those in Eq. (34), one obtains

$$f(\vec{k}, \vec{k}') = f(-\vec{k}', -\vec{k}). \tag{57}$$

This is the reciprocity relation. A physical interpretation of Eq. (57) is that the reciprocity symmetry ensures equality of the elastic scattering amplitude when source and detector are interchanged.

It is readily shown that if the Hamiltonian is both Hermitian and time-reversal-invariant, the system is reciprocal.[6] But although Hermiticity and time-reversal invariance of the Hamiltonian are sufficient, they are not necessary for reciprocity. This black-hole problem, where neither Hermiticity nor time-reversal invariance [the boundary condition, Eq. (13) is not time-reversal-invariant] is fulfilled, is a clear example. Another sufficient condition (but not necessary) for reciprocity is the symmetry of the Hamiltonian.[7] We shall consider in a formal way the reciprocity relation and the conditions which guarantee it for our problem.

We take the transpose of both sides of Eq. (51) and, subtracting it from Eq. (50), we obtain

$$T-T^T=(1+\mathcal{G}_{(+)}H')^T(H'-H'^T)(1+\mathcal{G}_{(+)}H') . \quad (58)$$

The matrix elements of Eq. (58) between the free functions are given by

$$(\Psi_{0\alpha},(T-T^T)\Psi_{0\beta})=(\Psi_{(-)\alpha},(H'-H'^T)\Psi_{(+)\beta}) . \quad (59)$$

If the condition

$$(\Psi_{(-)\alpha},(H'-H'^T)\Psi_{(+)\beta})=0 \quad (60)$$

holds then

$$(\Psi_{0\alpha},T\psi_{0\beta})=(\psi_{0\alpha},T^T\Psi_{0\beta}) . \quad (61)$$

In order to obtain the reciprocity relation, it is convenient to transform the right-hand side of Eq. (61) by an antiunitary operation Q:

$$Q\Psi=\Psi_Q ,$$

$$QH=H^*Q .$$

Equation (61) reads

$$(\Psi_{0\alpha},T\Psi_{0\beta})=(\Psi_{0\beta Q},T\Psi_{0\alpha Q}) ,$$

which is the reciprocity relation. Condition (60) is also expressed as

$$(\Psi_{(-)\alpha},H'\Psi_{(+)\beta})=(\Psi_{(+)\beta Q},H'\Psi_{(-)\alpha Q}) . \quad (62)$$

For $\alpha=\vec{k}$, $\beta=\vec{k}'$, and for Q, the operation of complex conjugation, Eq. (62) gives formula (57). In our case, condition (60) holds, because from Eq. (46) and Eq. (20)

$$(\Psi_{(-)}(H-H^T)\Psi_{(+)})=\sum_l \frac{4\pi}{2l+1} \lim_{r\to 0+} r(r-r_s)W[R_l,R_l]$$

$$=0 . \quad (63)$$

Thus, reciprocity is derived as a consequence of the equality of the matrix elements of H and H^T [Eq. (63)], although H is not a symmetric operator.

V. ELASTIC SCATTERING

In order to obtain the elastic scattering amplitude, we separate explicitly the Coulomb tail of the effective Hamiltonian. Thus, we take

$$H_0=-\vec{\nabla}^2-\frac{2k^2r_s}{r} . \quad (64)$$

Then

$$H'=\vec{\nabla}\cdot\left(\frac{r_s}{r}\hat{e}_r\partial_r\right) + k^2\frac{r_s}{r}\frac{1-2r_s/r}{1-r_s/r} . \quad (65)$$

For this case, the Green's function of Eq. (35) is the Coulomb Green's function, whose asymptotic behavior for $r\to\infty$ is[8]

$$G_{(+)}(\vec{r},\vec{r}')=-\frac{e^{ikr+ikr_s\ln 2kr}}{r}\mathcal{F}(\vec{r}'\hat{r}) , \quad (66)$$

where

$$\mathcal{F}(\vec{r}',\hat{r})=e^{\pi k r_s/2}\frac{\Gamma(1-ikr_s)}{4\pi}e^{-ik\vec{u}_r\cdot\vec{r}'} \times F(ikr_s;1;ik(r'+\vec{r}'\cdot\vec{u}_r)) .$$

$F(\alpha;\gamma;Z)$ is the confluent hypergeometric function of the first kind, and

$$\vec{u}_r=\vec{r}/r .$$

The solution for the homogeneous equation (38) is

$$\Psi_{(0)}=e^{-\pi k r_s/2}\Gamma(1-ikr_s)e^{ikz} \times F(ikr_s;1;ikr(1-\cos\theta)) \quad (67)$$

In order to find the elastic scattering amplitude, we are interested in the behavior of the solution of Eq. (43) at $r\to\infty$.

Using Eq. (66) and the asymptotic behavior of Eq. (67), it follows that

$$f_s(\theta)=f_0(\theta)-\int_0^\infty \Psi(\vec{r}')H'(r')\mathcal{F}(\vec{r};\hat{r})d^3r' , \quad (68)$$

where

$$f_0(\theta)=\frac{r_s}{2\sin^2\theta/2}e^{2ikr_s[\ln\sin\theta/2-\arg\Gamma(ikr_s)]} \quad (69)$$

is the scattering amplitude for H_0. We compute now the first-order contribution to $f_s(\theta)$, taking $\Psi(\vec{r})=\Psi_{(0)}(\vec{r})$ in the exact expression for $f_s(\theta)$ given by Eq. (68). This is a high-frequency and small-scattering-angle approximation. For Eqs. (65). (66), and (67) we evaluate the integral (68) for that case. We obtain (see Appendix B)

$$|f(\theta)|^2=\frac{16M^2}{\theta^4}+\tfrac{15}{4}\pi\frac{M^2}{\theta^3}+\frac{(15\pi)^2}{256}\frac{M^2}{\theta^2}+\frac{81}{64}\frac{(2M)^{3/2}}{\theta}\left(\frac{\pi}{k}\right)^{1/2} +\frac{81}{178}\left(\frac{\pi}{k}\right)^{1/2}(2M)^{3/2}\left(1+\frac{15\pi}{64}\right)+\frac{1}{4k^2} . \quad (70)$$

This expression for the differential elastic cross section shows the well-known Rutherford behavior for $\theta\to 0$ plus corrections. It can also be pointed out that in the $k\to\infty$ limit, expression (70) gives the optical geometrical result.[9]

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APPENDIX A

We consider the asymptotic behavior for $r\to\infty$ of the partial-wave expansion Eq. (20). It is

$$\Psi \underset{r\to\infty}{\sim} \frac{i}{2kr} \sum_{l=0}^{\infty} (2l+1)P_l(\cos\theta)[(-1)^l e^{-i(kr+kr_s\ln 2kr)} - e^{2i\delta_l} e^{i(kr+kr_s\ln 2kr)}], \tag{A1}$$

where we have used Eq. (24′).

For $\theta \neq 0$, Eq. (A1) can be written as

$$\Psi \underset{r\to\infty}{\sim} \frac{i}{2kr} \sum_{l=0}^{\infty} (2l+1)P_l(\cos\theta)\left[(-1)^l e^{-i(kr+kr_s\ln 2kr)} + \frac{f(\theta)}{r} e^{i(kr+kr_s\ln 2kr)}\right]. \tag{A2}$$

On the other hand, we consider the function

$$e^{ikr\cos\theta} Z(+ikr_s, 1, ikr(1-\cos\theta)),$$

where Z stands for the confluent hypergeometric function of the second kind.

For $r\to\infty$, the asymptotic behavior is[10]

$$e^{ikr\cos\theta} Z \underset{r\to\infty}{\sim} e^{\pi kr_s/2} e^{i[kr\cos\theta - kr_s\ln kr(1-\cos\theta)]} \times \left[1+O\left(\frac{1}{kr}\right)\right]. \tag{A3}$$

By using the integral expression of Z, we write

$$e^{ikr\cos\theta} Z(ikr_s, 1, ikr(1-\cos\theta)) = \frac{1}{\Gamma(ikr_s)} \int_0^\infty t^{ikr_s-1} (1+t)^{-ikr_s} e^{-ikrt} e^{ikr(1+t)}\, dt. \tag{A4}$$

With

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos\theta),$$

where $j_l(kr)$ is the Bessel function, the right-hand side of Eq. (A4) is equal to

$$\sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\theta)\, \mathcal{C}_l(kr),$$

where

$$\mathcal{C}_l(kr) = \frac{1}{\Gamma(ikr_s)} \int_0^\infty t^{-1+ikr_s} (1+t)^{-ikr_s} e^{-ikrt} j_l(kr(1+t))\, dt. \tag{A5}$$

We evaluate $\mathcal{C}_l(kr)$ for $kr\to\infty$ and obtain

$$e^{ikr\cos\theta} Z(+ikr_s, 1, ikr(1-\cos\theta)) = \frac{i}{2kr} \sum_{l=0}^{\infty} (2l+1)P_l(\cos\theta)(-1)^l e^{\pi kr_s/2} \times \left[1+O\left(\frac{1}{kr}\right)\right]. \tag{A6}$$

Comparing Eq. (A6) with (A3), one sees that

$$e^{i[kr\cos\theta - kr_s\ln kr(1-\cos\theta)]} = \frac{i}{2kr} \sum_{l=0}^{\infty} (-1)^l (2l+1) P_l(\cos\theta) e^{i(kr+kr_s\ln 2kr)}.$$

By replacing this expansion in Eq. (A2), we obtain Eq. (16) (Sec. II).

APPENDIX B

By using known properties of the confluent hypergeometric functions, the integral in Eq. (64) can be written as

$$\int_0^\infty \mathcal{F}(\hat{r}, r') k^2 \frac{r_s}{r'}\left[-\left(1-\frac{i}{kr'}\right)F(1) + 2(1-ikr_s)(1-\cos\theta)\left(1+\frac{1}{2kr'}\right)F(2) - (1-ikr_s)\left(1-\frac{ikr_s}{2}\right)(1-\cos\theta)^2 F(3)\right] e^{ikr'} d^3\vec{r}', \tag{B1}$$

where

$$F(A) = F(A-ikr_s; A; -ikr(1-\cos\theta')), \quad \text{for } A=1,2,3$$

$$\vec{u}_r \cdot \vec{r}' = r'[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\psi-\psi')].$$

Using[10]

$$\int_0^\infty e^{-\lambda z} z^{\gamma-1} F(\alpha;\gamma;pz) F(\alpha';\gamma;p'z)\, dz = \Gamma(\gamma)\lambda^{\alpha+\alpha'-\gamma}(\lambda-p)^{-\alpha}(\lambda-p')^{-\alpha'} F\left(\alpha,\alpha';\gamma;\frac{pp'}{(\lambda-p)(\lambda-p')}\right),$$

the first term of the integral (B1) is equal to

$$ikr_s\left[I(\alpha) + \frac{d}{d\alpha} I(\alpha)\right]_{\alpha=1}, \tag{B2}$$

where

$$I(\alpha)=\int_0^\infty [-ik(1+\alpha)]^{-ikr_s}(s-p')^{ikr_s-1}F\left(ikr_s, 1-ikr_s; 1; \frac{pp'}{(s-p)(s-p')}\right)\sin\theta' d\theta' dp' ,$$

with

$$p=ik(1+\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\psi') ,$$

$$p'=-ik(1-\cos\theta') ,$$

$$s-p=-ik(1+\alpha) ,$$

$$s-p'=ik[1-\alpha-\cos\theta'(1-\cos\theta)+\sin\theta\sin\theta'\cos\psi'] .$$

The second and third terms of the integral (B1) can be reduced to an expression similar to that of (B2).

By means of the relations[10]

$$F(\alpha,\beta;\beta;z)=(1-z)^{-\alpha} ,$$

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-1)^{\alpha}z^{-\alpha}F\left(\alpha,\alpha+1-\gamma;\alpha+1-\beta;\frac{1}{z}\right)$$
$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-1)^{\beta}z^{-\beta}F\left(\beta,\beta+1-\gamma;\beta+1-\alpha;\frac{1}{z}\right)$$

and using the stationary-phase method, we obtain the asymptotic expression for $f(\theta)$ for high frequencies and small angles. The modulus squared of this amplitude is given by Eq. (66).

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