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# Probability distribution for radiation from a black hole in the presence of incoming radiation*

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Using only results from quantum field theory in curved space-time, we derive the formula for the probability that $k$ particles in a given mode will emerge from a Kerr black hole if $j$ particles in that mode are sent in at late times. Our formula agrees with the formula previously derived by Page and by Bekenstein and Meisels using other arguments. In particular, this proves that a Schwarzschild black hole responds to incoming radiation exactly as a blackbody does.

It is now well established that, as first shown by Hawking,[1] spontaneous particle creation in the strong gravitational field outside a Schwarzschild black hole results in a steady flux of particles at large distances from the black hole with an *exactly* thermal spectrum.[2,3] Thus, a Schwarzschild black hole "emits" radiation just like an ordinary blackbody at temperature $T=\hbar\kappa/2k_B\pi$, where $\kappa$ denotes the surface gravity of the black hole. This analogy between black holes and blackbodies is particularly striking in view of the close analogy, noted prior to the quantum calculations, between the laws of thermodynamics and those of black-hole mechanics.[4,5] These analogies have suggested a new law of physics—the generalized second law of thermodynamics[5]—as well as a number of other intriguing ideas.[6]

However, the fact that a Schwarzschild black hole spontaneously emits particles in exactly the same manner as a blackbody does not, of course, imply that it must continue to behave just like a blackbody in the presence of incoming radiation. Using thermodynamic and information-theoretic arguments, Bekenstein and Meisels[7] recently obtained a formula for the probability that $k$ particles in a given mode will emerge from a Kerr black hole if $j$ are incident. Their formula agrees with a formula derived by Page[8] for stimulated emission from a hot amplifier, i.e., from a collection of oscillators at temperature $T$. Thus, in particular, if the Bekenstein-Meisels formula is valid, it implies that a Schwarzschild black hole does indeed behave exactly like a blackbody even in the presence of incoming radiation.

The Bekenstein-Meisels formula was derived using maximum-entropy arguments as well as other *ad hoc* assumptions concerning the black-hole emission process. On the other hand, in quantum field theory, the complete information concerning the state of the system is contained in the quantum state vector. A formula for the "out" state vector corresponding to a situation where $j$ particles are incoming was derived by one of us[9] using quantum field theory alone. In Ref. 9 a formula was derived for the expected number of outgoing particles when $j$ particles are sent in. However, the detailed probability distribution $P(k|j)$ was calculated only for the simple case of a superradiant mode in the limit $\kappa\to 0$.

In this paper, we shall calculate $P(k|j)$ for all modes of a Hermitian scalar field from the quantum state vector and show that it agrees with the previously derived formula of Bekenstein and Meisels[7] and Page.[8] Thus, the additional assumptions which entered the Bekenstein-Meisels derivation can be justified from quantum field theory. The analogy between black holes and blackbodies is confirmed.

We consider a Hermitian scalar field in the Kerr spacetime geometry. We shall follow the notation and conventions of Refs. 2 and 9 except that we will use the notation $\otimes$ rather than the index notation to denote symmetrized tensor-product states. We wish to consider the effect of sending in $j$ particles at late times in the state $\gamma$ corresponding to a wave packet with angular quantum numbers $l,m$ and frequencies peaked about $\omega$. (The effect of sending in particles at "early times," i.e., during the collapse phase, was treated in Ref. 9.) The formula given by Bekenstein and Meisels[7] for the probability that $k$ particles in this mode will emerge is

$$P(k|j)=\frac{(1-x)x^k(1-|R|^2)^{j+k}}{(1-|R|^2x)^{j+k+1}}\times\sum_{m=0}^{\min(j,k)}\frac{(j+k-m)!}{(j-m)!\,(k-m)!\,m!}\times\left[\frac{(|R|^2-x)(1-|R|^2x)}{(1-|R|^2)^2x}\right]^m, \qquad (1)$$

where $R$ is the classical reflection amplitude for the mode, and

$$x=\exp[-\pi(\omega-m\Omega)/\kappa], \qquad (2)$$

where $\Omega$ is the angular velocity of the black hole. We shall now calculate $P(k|j)$ from the quantum "out" state vector and show that the result agrees with Eq. (1).

Let $\rho$ denote the outgoing-particle state associated with the mode $\gamma$; let $\sigma$ denote the corresponding "late-time-horizon state" and let $\tau$ denote the corresponding "early-time-horizon state" constructed in the manner described in Ref. 2. Consider, first, the case of a nonsuperradiant mode. Then we have[2]

$$C\gamma = T\sigma + R\rho \, , \tag{3a}$$

$$D\gamma = 0 \, , \tag{3b}$$

where $T$ and $R$ denote the classical transmission and reflection amplitudes for the mode $\gamma$, and $C$ and $D$ are the operators[2] describing classical scattering, from which the quantum $S$ matrix is constructed.

The formula for the "out" state vector describing spontaneous emission in the mode of interest is[2]

$$\Psi_0 = N\left( |0\rangle + \sum_{n=1}^{\infty} \frac{(2n)!^{1/2}}{n! 2^n} \overset{n}{\otimes} \epsilon \right) , \tag{4}$$

where the two-particle state $\epsilon$ is given by

$$\epsilon = 2x(\lambda \otimes \tau) \, , \tag{5}$$

where $\lambda = t\rho + r\sigma$, where $t$ and $r$ are the classical transmission and reflection amplitudes for a wave packet with the same frequency and angular dependence as $\gamma$ but incoming from the past horizon of the vacuum Kerr solution. The time-reflection symmetry of the Kerr solution implies the following relations between $t$, $r$ and $T$, $R$:

$$t^* = \frac{t}{tT - rR} \, , \tag{6a}$$

$$r^* = \frac{-R}{tT - rR} \, . \tag{6b}$$

For nonsuperradiant modes we also have $|t|^2 + |r|^2 = |T|^2 + |R|^2 = 1$. This implies that $t = T$ and $r = -TR^*/T^*$. (For superradiant modes $|r|^2 - |t|^2 = |R|^2 - |T|^2 = 1$, and we obtain $t = -T$ and $r = -TR^*/T^*$.) It is easy to check that the normalization constant $N$ in Eq. (4) with $\epsilon$ given by Eq. (5) is $|N|^2 = 1 - x$.

From Eq. (2.14) of Ref. 9, the state vector describing emission when $j$ particles in the state $\gamma$ are incident is

$$\Psi_j = \frac{1}{\sqrt{j!}} \left( \prod^{j} a^\dagger(\gamma) \right) \Psi_0 = \frac{1}{\sqrt{j}} \left( \prod^{j} [b^\dagger(C\gamma) - b(D\gamma)] \right) \Psi_0 \, , \tag{7}$$

where $a^\dagger$ denotes the "in" creation operator and $b$ and $b^\dagger$ denote the "out" annihilation and creation operators. Using Eqs. (3), (4), and (5) we can expand the right-hand side of Eq. (7) as a sum of states of the form $\overset{n}{\otimes} \tau \overset{k}{\otimes} \rho \overset{n-k+j}{\otimes} \sigma$.

The probability $P(k|j)$ that $k$ particles in the state $\rho$ will be seen to emerge from the black hole is given in terms of $\Psi_j$ by

$$P(k|j) = \sum_H |\langle k, H | \Psi_j \rangle|^2 \, , \tag{8}$$

where $|k, H\rangle$ denotes the tensor product of a state describing $k$ particles in the state $\rho$ with a "horizon state" $H$, and the sum runs over a complete orthonormal basis of horizon states.

From the above formula for $\Psi_j$ we find

$$P(k|j) = (1-x) \sum_{n=0}^{\infty} \left| \sum_{m=0}^{\min(j,k)} \binom{j}{m} R^m T^{j-m} \binom{n}{k-m} t^{k-m} r^{n-k+m} \right|^2 \frac{x^n (2n+j)!}{j!(n!)^2} \| \overset{n}{\otimes} \tau \overset{k}{\otimes} \rho \overset{n-k+j}{\otimes} \sigma \|^2 \, , \tag{9}$$

where any binomial coefficient $\binom{p}{q} = p!/q!(p-q)!$ is understood to be zero if $q < 0$ or $q > p$. Expanding the square in Eq. (9), rearranging the sum, and using the relations between $t$, $r$, $T$, and $R$, we obtain

$$P(k|j) = (1-x) x^k (1 - |R|^2)^{j+k} \sum_{m,l=0}^{\min(j,k)} \frac{[-|R|^2/(1-|R|^2)]^{m+l} j! k!}{m! l! (j-l)! (k-l)! (j-m)! (k-m)!} \sum_{n=0}^{\infty} \frac{n!(n-k+j)!}{(n-k+m)!(n-k+l)!} (x|R|^2)^{n-k} \, . \tag{10}$$

Writing $u = x|R|^2$, we recognize the sum over $n$ as the series expansion of

$$(k-m)! u^{-l} (d^{j-l}/du^{j-l}) [u^{j-m}/(1-u)^{k-m+1}] \, .$$

Writing $y = 1 - u$, binomially expanding $u^{j-m} = (1-y)^{j-m}$, and performing the differentiation term by term we obtain

$$P(k|j) = K \sum_{m,l,p} k! j! \left[ \frac{|R|^2}{(1-|R|^2)} \right]^{m+l} (-1)^{m+l+p} \frac{(1-|R|^2 x)^{m+l+p}}{(|R|^2 x)^l} \frac{(j+k-m-l-p)!}{m! l! (k-l)! (j-l)! (k-m-p)! p! (j-m-p)!} \, , \tag{11}$$

where here and in the following the sum ranges over all non-negative values of the summation variables such that all the integers in the factorials are non-negative, and where

$$K=\frac{(1-x)x^k(1-|R|^2)^{j+k}}{(1-|R|^2x)^{j+k+1}}. \tag{12}$$

Our task now is to show that this rather complicated finite sum is, in fact, identical to Eq. (1). To facilitate comparison we change independent variables from $|R|^2$ and $x$ to $u$ and $v$, defined by

$$u=|R|^2x, \tag{13a}$$

$$v=\frac{(|R|^2-x)(1-|R|^2x)}{(1-|R|^2)^2x}. \tag{13b}$$

In terms of $u$ and $v$, we have

$$|R|^2/(1-|R|^2)=[u+(u+uv)^{1/2}]/(1-u). \tag{14}$$

Substituting this variable change in Eq. (11) and binomially expanding $[u+(u+uv)^{1/2}]^{m+l}$, we find

$$P(k|j)=K\sum_i\sum_m\sum_p\sum_l\frac{k!\,j!(k+j-m-l-p)!\,(m+l)!\,(-1)^{m+l+p}u^{m-i}(1-u)^p(u+uv)^{i/2}}{p!\,m!\,l!\,(k-l)!\,(j-l)!\,(k-m-p)!\,(j-m-p)!\,(m+l-i)!\,i!}. \tag{15}$$

The sum over $l$ can now be rewritten as

$$\sum_l\frac{(m+l)!\,(j+k-m-l-p)!\,k!\,(-1)^l}{(m+l-i)!(j-l)!\,(k-l)!\,l!}=\frac{d^{k-m-p}}{d\beta^{k-m-p}}\beta^{j-m-p}\frac{d^i}{d\alpha^i}\alpha^m(\beta-\alpha)^k\bigg|_{\substack{\alpha=1\\ \beta=1}}$$

$$=\sum_q\frac{i!\,m!\,(-1)^q(k-m-p)!\,(j-m-p)!\,k!}{q!\,(i-q)!\,(q-m-p)!\,(j-q)!\,(k-q)!\,(m-i+q)!}. \tag{16}$$

(The first equality may be verified by binomially expanding and then differentiating, while the second equality follows by differentiating using the Leibnitz rule.) When this replacement is made, a number of cancellations in the factorials occur, and the sum over $p$ can be recognized as the binomial expansion of $[1-(1-u)]^{q-m}=u^{q-m}$. Thus, we obtain

$$P(k|j)=K\sum_{i,q}\frac{k!\,j!\,u^{q-i}(u+uv)^{i/2}(-1)^q}{q!\,(k-q)!\,(2q-i)!\,(i-q)!\,(j-q)!}\times\sum_m\frac{(2q-i)!(-1)^m}{(m-i-q)!\,(q-m)!}. \tag{17}$$

The sum over $m$ can now be recognized as the binomial expansion of $(1-1)^{2q-i}$, which vanishes unless $i=2q$. Thus, we obtain

$$P(k|j)=K\sum_{q=0}^{\min(j,k)}\frac{k!\,j!(1+v)^q}{q!^2(k-q)!(j-q)!}. \tag{18}$$

Binomially expanding $(1+v)^q$, we get

$$P(k|j)=K\sum_{s=0}^{\min(j,k)}\sum_{q=s}^{\min(j,k)}\frac{k!\,j!\,v^s}{(q-s)!s!q!(k-q)!(j-q)!}. \tag{19}$$

Finally, by binomially expanding both sides of the equation

$$(1+\alpha)^k(1+\alpha)^{j-s}=(1+\alpha)^{k+j-s} \tag{20}$$

and equating the coefficient of $\alpha^j$, one can prove the identity

$$\sum_{q=s}^{\min(j,k)}\frac{k!(j-s)!}{q!(k-q)!(j-q)!(q-s)!}=\frac{(k+j-s)!}{j!(k-s)!}. \tag{21}$$

This shows that

$$P(k|j)=K\sum_s\frac{(k+j-s)!\,v^s}{s!(k-s)!(j-s)!}. \tag{22}$$

Recalling the definitions of $K$ and $v$, Eqs. (12) and (13b), it now can be easily seen that Eq. (22) is, in fact, identical to the formula of Bekenstein and Meisels[7] and Page[8] [Eq. (1)].

For the case of a superradiant mode, a number of modifications must be made in the above calculation of $P(k|j)$. The state vector describing spontaneous emission is still of the general form given by Eq. (4), but now we have [see Ref. 2, Eq. (5.4)]

$$\epsilon=\frac{2}{\gamma}\sigma\otimes\left(t\rho+\frac{1}{\sqrt{x}}\tau\right). \tag{23}$$

Furthermore, in place of Eq. (3) we have

$$C\gamma=R\rho, \tag{24a}$$

$$D\gamma=T\bar{\sigma}. \tag{24b}$$

From Eqs. (7) and (8) we now obtain

$$P(k|j)=\frac{x-1}{|R|^2x}\sum_{n=0}^{\infty}\|\overset{n}{\otimes}\sigma\overset{k}{\otimes}\rho\overset{n-k+j}{\otimes}\tau\|^2\frac{(2n+j)!}{(n!)^2j!}x^{-(n-k+j)}\left|\sum_{m=0}^{\min(j,k)}\frac{(-1)^mT^{j-m}R^mt^{k-m}}{\gamma^{n+j-m}}\binom{n+j-m}{k-m}\binom{j}{m}\right|^2. \tag{25}$$

Expanding the square and performing the summation over $n$ in a manner similar to the nonsuperradiant case, we again get precisely Eq. (11) which, of course, may again be reduced to Eq. (22). Thus, the validity of the formula of Bekenstein and Meisels and Page is proven for superradiant modes also.

We note that Bekenstein and Meisels obtained their formula for $P(k|j)$ by considering a black hole in equilibrium with incoming thermal radiation and assuming that the probability distribution in that situation is the one that maximizes the entropy subject to a constraint on the expected number of outgoing particles. Hence, by reversing the steps of their argument, it follows that their maximum-entropy assumption is indeed valid. Of course, if the incoming distribution is not thermal, then the outgoing distribution will not be the one of maximum entropy.

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