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Constraints on the gravitational constant at large distances*

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D. R. Long and others have speculated that the gravitational force between point masses in the Newtonian regime might not be exactly proportional to $1/r^2$. Distance-dependent deviations from the $1/r^2$ law can be represented by a distance-dependent gravitational "constant," $G(r)$. Long has summarized the experimental evidence which constrains $G(r)$ to be very nearly constant for 5 cm $< r <$ 1 m. This paper presents observational evidence for constancy in the range 10^3 km $< r < 10^8$ km, and points out that the value of $G(r) \equiv G_c$ in this range has not been experimentally determined. Constraints on $G(r)$ in the intermediate distance range 10 m $< r <$ 1 km are so poor that one cannot rule out the possibility that G_c differs greatly from the laboratory value G_0. Models of the earth and sun are used to argue that G_c differs from G_0 by not more than $\sim$40%. Methods of improving the determination of G_c are suggested.

I. INTRODUCTION

It is widely recognized that the universal constant of gravitation, G, is the least well-determined fundamental physical constant; indeed some have questioned whether G is actually constant.[1-6] The usual concern has been the time dependence [1,2] of G, but we consider here whether G varies with distance. We investigate the hypothesis that the gravitational force between point masses in the Newtonian regime ($GM/c^2r \ll 1$) departs from an exact inverse-square law; i.e., we inquire whether the quantity

$$G(r) \equiv F_{\text{grav}}\, r^2/Mm \tag{1}$$

is a constant function of distance. Several authors have provided theoretical bases[3-5] for variations in $G(r)$, and recently Long[6] has presented experimental evidence for a departure from the inverse-square law of gravitation.

We shall consider a gravitation constant, $G(r)$, which is a function only of the distance between two gravitating bodies, but not one which is a function of position in spacetime (e.g., the gravitational constant determined by Cavendish experiments in the Brans-Dicke theory of gravity[7]). Nor are we concerned with the well-established relativistic post-Newtonian corrections (of order GM/c^2r) to Newtonian gravity which are better understood within the framework of relativistic metric theories of gravity[8] rather than as manifestations of a variable $G(r)$.

There are two general sources of data useful for determining the behavior of $G(r)$: laboratory gravitation experiments and celestial mechanics. Laboratory experiments (see Sec. II A) are of relatively high absolute accuracy; that is to say, all quantities on the right-hand side of Eq. (1) may be accurately measured. We shall refer to the approximately constant value of $G(r)$ at laboratory distances as G_0. Observations of the motions of artificial and natural members of the solar system (see Sec. II B) demonstrate that $G(r)$ is constant to high precision for 10^3 km $< r < 10^8$ km; we shall refer to this approximately constant value of $G(r)$ as G_c.

While the product G_cM has been observationally determined with great precision for many bodies in the solar system, the value of G_c can be extracted only if the mass M of one of these bodies is known. The masses found in standard astrophysical reference works cannot be used for this purpose since they are obtained from G_cM on the assumption that $G_c = G_0$.

After discussing the experimental and observational evidence for the constancy of $G(r)$ in Sec. II, we derive upper and lower limits on G_c. In Sec. III, G_c is constrained by geophysical arguments, and in Sec. IV by considerations of solar structure. In the final section we suggest methods which might be used to determine G_c more precisely.

II. EXPERIMENTAL AND OBSERVATIONAL CONSTRAINTS ON $G(r)$

A. Laboratory experiments

During the last three centuries many attempts have been made to determine the value of Newton's universal constant of gravitation, G. The earlier attempts used parts of the Earth (e.g., mountains) as gravitational sources, and their results typically agree with the modern value to within a factor of 3.[9,10] Experiments of this type are plagued by unseen and unmeasurable local density inhomogeneities in the Earth's crust. Later laboratory experiments which used carefully constructed gravitational sources proved capable of relatively great precision[11] (1–0.1%). Long[12] has compiled the results of a number of the more precise laboratory determinations; while the data are consistent (at the 3σ level) with a constant value of $G(r)$, they do not rule out changes of up to 1% when r varies from 5 cm to 1 m. More recently, Long[6] has claimed positive detection of a variation in $G(r)$; this result is, however, unconfirmed.

For purposes of exposition and analysis we shall adopt a specific form for the gravitational potential $V(r)$ at a distance r from a point source:

$$V(r) = -(G_c Mm/r)(1+\alpha e^{-\mu r}) \quad (\alpha > -1) . \tag{2}$$

We shall refer to α as the "amplitude" of the variation and to μ^{-1} as the "range." The form of Eq. (2) has been suggested theoretically[3-5] and the specific value $\alpha = \frac{1}{3}$ has been proposed.[4,5] The gravitational force which results from Eq. (2) is

$$F_{\text{grav}} = (G_c Mm/r^2)\left[1+\alpha(1+\mu r)e^{-\mu r}\right]$$

and thus

$$G(r) = G_c\left[1+\alpha(1+\mu r)e^{-\mu r}\right] . \tag{3}$$

Note that when $\mu r \ll 1$, $G(r) \simeq G_c(1+\alpha)$ and when $\mu r \gg 1$, $G(r) \simeq G_c$; this is consistent with our definition of G_c in Sec. I since we shall later require $\mu^{-1} \ll 10^3$ km. If we also require $\mu^{-1} > 10$ m we have $G_0 \simeq G_c(1+\alpha)$ and we are confronted with the interesting possibility that $G_0 \neq G_c$. The form of $G(r)$ in Eq. (3) is consistent with the laboratory data discussed above if $\mu^{-1} > |50\alpha|^{1/2}$ m; note that even $|\alpha| \simeq 1$ is allowed if $\mu^{-1} > 10$ m.

Unfortunately, conventional laboratory experiments are inherently insensitive to variations with range $\mu^{-1} > 10$ m, so we turn to the field of celestial mechanics to determine $G(r)$ for larger distances.

B. Celestial mechanics

Our discussion of large-scale gravitation will be confined to the solar system where, in recent years, precision tracking of planets and spacecraft has made possible many important tests of gravitational theories. Much larger gravitational systems exist, of course, but the masses and mass distributions of these systems are so poorly known[13] that we can say only that gravitational forces appear to exist at distances up to a few Mpc with $G(r)$ within an order of magnitude of G_0.

Orbital precession provides the most sensitive test of the constancy of $G(r)$. It is well known that an inverse-square force law leads to closed elliptical orbits and that any deviation from such a law will, in general, cause orbital precession. This is a secular effect, so even a small variation in $G(r)$ becomes detectable after a long period of time. In the case of a variable $G(r)$ we may use standard techniques of perturbation theory to calculate the precession of a two-body orbit due to a perturbing force

$$\delta\vec{F} = -\delta G(r) Mm\vec{r}/r^3 ,$$

where $\delta G(r) = G(r) - G(a)$ and a is the semimajor axis of the unperturbed orbit. The angular precession $\delta\omega$ per orbit is given, to first order in $\delta G(r)$, by[14]

$$\delta\omega = [eG(a)]^{-1}\int_0^{2\pi}\delta G(r)\cos v\, dv , \tag{4}$$

where r is given by the unperturbed relation for an orbit of eccentricity e: $r = a(1-e^2)/(1+e\cos v)$. Expanding $G(r)$ in a Taylor series yields an approximate result useful for small e:

$$\frac{\delta\omega}{2\pi} \simeq -\frac{aG'(a)}{2G(a)} + O(e^2) , \tag{5}$$

where the prime denotes differentiation. It is evident from the form of Eq. (4) that the precession test is less sensitive to oscillatory variations[3] in $G(r)$ than to monotonic variations over the orbit.

Planetary orbits are observed to precess, but almost all of the physical precession is accounted for by the purely Newtonian mutual perturbations of the planets. The residual rates of precession (observed rate minus Newtonian rate) for Mercury, Mars, and Icarus are shown in Table I (the observed rates are obtained from Refs. 15–18).

TABLE I. Rates of post-Newtonian apsidal precession.

System	Periastron (km)	Apastron (km)	$\delta\omega/2\pi$ Predicted[a]	$\delta\omega/2\pi$ Observed	Ref.
Sun					
Mercury	0.5×10^8	0.7×10^8	7.98×10^{-8}	8.00[b] $\pm 0.04\times10^{-8}$	15
Mars	2.3×10^8	2.5×10^8	2.0×10^{-8}	$2.0 \pm 0.9\times10^{-8}$	17
Icarus	0.3×10^8	3.0×10^8	8.7×10^{-8}	$8.2 \pm 0.7\times10^{-8}$	18
Binary Pulsar	$\sim7\times10^5$	$\sim3\times10^6$	$\sim1\times10^{-5}$	1.0×10^{-5}	20

[a] Calculated from the general-relativistic formula $\delta\omega/2\pi = 3GM_\odot/[c^2a(1-e^2)]$.

[b] Assumes a solar quadrupole moment appropriate for uniformly rotating Sun; this is observationally supported by Ref. 16.

TABLE II. Planetary mass determinations.

Planet	$G_c M_\odot / G_c M$ [a]	$G_c M_\odot / G(r) M$	r (km)	$[G(r) - G_c]/G_c$
Earth + Moon	328 900 ± 20	328 900.5 ± 0.1 [b]	4×10^5	$(0 \pm 6) \times 10^{-5}$
Venus	408 520 ± 100	408 524 ± 1 [c]	1×10^4	$(0 \pm 3) \times 10^{-4}$
Mercury	6 025 000 ± 15 000	6 023 700 ± 300 [d]	3×10^3	$(0 \pm 3) \times 10^{-3}$

[a] See Ref. 21. [b] See Ref. 22. [c] See Ref. 23. [d] See Ref. 24.

These "post-Newtonian" precessions are in good agreement with the predictions of general relativity[19] (see Table I), so we shall adopt the point of view that variations in $G(r)$ can be related only to differences between the observed and predicted rates of precession. According to Eq. (5) the agreement between the observed and theoretically predicted motions rules out variations in $G(r)$ larger than a few parts in 10^8 for 0.3×10^8 km $< r < 3\times10^8$ km. Similarly, the observed rate of precession of the orbit of the binary pulsar (see Table I, Ref. 20) provides evidence that $G(r)$ is constant to within a few parts in 10^5 for 7×10^5 km $< r < 3\times10^6$ km.

The strictest limit on variations in $G(r)$ which we have obtained is the above limit of $rG'(r)/G(r) \lesssim 10^{-8}$ for distances $r \sim 10^8$ km. We shall denote this nearly constant value of $G(r)$ as G_c, and in the remainder of the paper we relate G_c to $G(r)$ at smaller distances.

Planetary mass determinations provide evidence for the constancy of $G(r)$ down to distances $r\sim3\times10^3$ km. These determinations are actually measures of $G(r)M$, where r is the distance between a planet of mass M and a body whose carefully observed motion is perturbed by the planet. For historical reasons the masses are reported as fractions of the solar mass; the commonly quoted inverse mass is thus $m^{-1} = G_c M_\odot / G(r) M$ [$G_c M_\odot$ appears here because the value of $G(r) M_\odot$ is derived from analyses of planetary orbits with radii of order 10^8 km]. In Table II we present two mass determinations each for the three inner planets. The first value (obtained from Ref. 21) of each pair was derived from perturbations of bodies at distances of $\sim10^8$ km, so $G(r) = G_c$. The second mass for the Earth-Moon system (Ref. 22) was derived from lunar laser-ranging data, so r is the Earth-Moon separation, which is roughly constant. Mariner 10 tracking data provided the final mass determinations (Refs. 23 and 24). In these cases, the minimum distances between the planets and Mariner 10 are taken as r since most of the change in the spacecraft velocity occurs in the regions near the planets.[25] The fractional differences between the masses of each pair are equal to $[G(r) - G_c]/G_c$ and are listed in Table II.

It would appear from the results of Tables I and II that $G(r) = G_c$ to within 0.03% for 10^4 km $< r < 3\times10^8$ km. Long's[6] proposed form

$$G(r) = G_0[1 + 0.002 \ln(r/1 \text{ cm})],$$

which was fitted to his determinations of $G(r)$ at laboratory distances, clearly cannot continue to hold for $r > 10^4$ km since it disagrees with the observed high degree of constancy of $G(r)$ at these larger distances.

The absence of gravitational data near the surfaces of celestial bodies with radii less than 10^3 km precludes direct determination of $G(r)$ at smaller distances. It is possible, however, to infer the behavior of $G(r)$ at these distances, since the mass within 10^3 km of a point just outside the surface of a large body contributes significantly to the gravitational acceleration there. It is helpful, in discussing the behavior of $G(r)$ at these distances, to use the proposed potential given by Eq. (2). The gravitational acceleration g at an altitude h above the surface of a sphere of radius R and uniform density ρ is then

$$g(h) = [G_c M/(R+h)^2] \times [1 + \alpha f(x)(1 + x + \mu h) e^{-\mu h}] \tag{6}$$

with $f(x) = 3e^{-x}(x \cosh x - \sinh x)/x^3$ and $x = \mu R$. When $\mu^{-1} \gg R$ and $\mu^{-1} \gg h$

$$g(h) \simeq G_c M(1+\alpha)/(R+h)^2,$$

and when $\mu^{-1} \ll R$ and $\mu^{-1} \ll h$,

$$g(h) \simeq [G_c M/(R+h)^2] \times [1 + 3\alpha e^{-\mu h}(x + \mu h)/2x^2].$$

From the above, it is clear that the values of α which are consistent with a given observation depend strongly on the range μ^{-1} when it is smaller than either R or h, and only weakly when the range is much greater than both R and h.

In Fig. 1 we present upper limits on $|\alpha|$, as a function of μ, which are consistent with the mass determinations for Venus and Mercury presented in Table II. These limits are obtained by equating the uncertainties in $[G(r) - G_c]/G_c$ to the term $|\alpha| f(x)(1 + x + \mu h) e^{-\mu h}$ from Eq. (6), where h is taken to be the minimum altitude achieved during the encounters.[26] It is evident from Fig. 1 that if $\mu^{-1} < 10^2$ km the form of $G(r)$ implicit in Eq. (6) is consistent with the observed constancy of $G(r)$ even

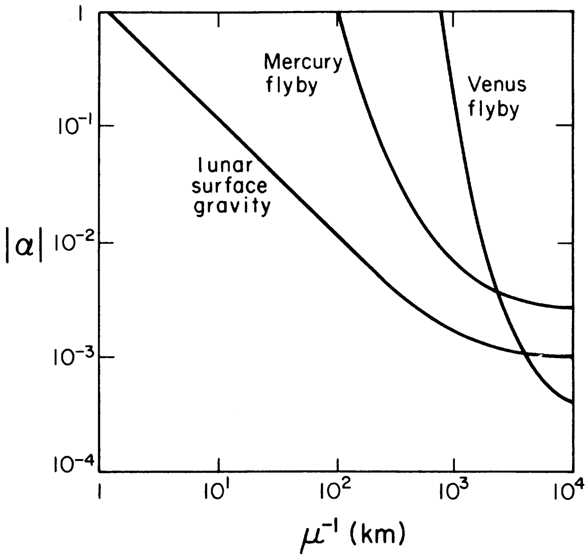

FIG. 1. Upper limits on the amplitude $|\alpha|$ of the variation in $G(r)$ [see Eq. (2)] as a function of the range μ^{-1}

when $|\alpha| \simeq 1$. The limits on $|\alpha|$ can be significantly improved for shorter ranges only if h is reduced.

C. Lunar surface gravity

Fuji[4] has pointed out that surface gravity measurements provide strong constraints on $|\alpha|$ when the range μ^{-1} is small compared to the radius.[27] We feel that measurements of the lunar surface gravity provide the best limits when $\mu^{-1} < 10^3$ km. The density is probably nearly constant throughout the moon,[28] so Eq. (4) should provide a good approximation to the surface gravity:

$$g = (G_c M/R^2)[1 + \alpha(1+x)f(x)] .$$

This is expected to be only approximately correct for several reasons:

(i) Local density inhomogeneities generate local gravity anomalies[29] as large as 0.1%.

(ii) The lunar surface is not exactly spherical.

(iii) The center of mass is offset from the center of figure[30] by a distance $d \simeq 10^{-3}R$.

Problems (i) and (ii) can be handled by averaging a number of measurements at dispersed sites and applying the free-air and Bouger corrections.[31] However, all the available surface gravity measurements[32] are on the near side of the moon and will be systematically affected by problem (iii). The fact that the center-of-mass offset could be caused by many different nonspherical density distributions, each of which may cause some systematic gravity anomaly of order $\Delta g/g \simeq d/R \simeq 10^{-3}$ on the near side of the moon, means that the *a priori* uncertainty in the gravity measurements should be ~0.1%. The observed values[32] do agree with $g = G_c M/r^2$ to within 0.1% (r is the distance to the center of mass), so we infer that

$$|\alpha| f(x)(1+x) \leq 10^{-3}, \quad x = \mu R .$$

This limit on $|\alpha|$ is also shown in Fig. 1 as a function of μ.

Note that the limits derived here together with the limits in Sec. II A cannot restrict $|\alpha| < 1$ if 10 m $< \mu^{-1} < 1$ km. Thus, it is entirely possible, in the face of all the evidence favoring constancy of $G(r)$ presented thus far, that G_c differs significantly from G_0. However, if we could estimate the mass of some cosmic body with a measured value of $G_c M$ we could estimate G_c directly and compare it with G_0; we address this problem in the next section.

III. THE DETERMINATION OF G_c FROM GEOPHYSICAL CONSIDERATIONS

The mass of the Earth is conventionally determined by measuring $G_c M_\oplus$, setting $G_c = G_0$, and solving for $M_\oplus$. In this section we shall estimate $M_\oplus$ directly and solve for G_c. We estimate $M_\oplus$ by the use of simple density distributions in the Earth's interior. Very detailed models of the Earth have been developed by geophysicists using a wealth of seismic data.[33] We cannot use these models to estimate the mass of the Earth, however, because they are always constrained to have a mass of $G_c M_\oplus / G_0$. This "known" mass of the Earth is a very important constraint on models, because there is no accurate way currently known of estimating the density distribution from seismic data alone.

We shall attempt only crude estimates of $M_\oplus$. Our models of the Earth are required to obey only the following constraints:

(i) The density must decrease outward.

(ii) The dimensionless moment of inertia, $I^* \equiv I/MR^2$, must equal 0.33, the observed value for the Earth.[34]

(iii) The central density, ρ_c, must be no greater than 15 g cm^{-3}.

(iv) The surface density, ρ_s, must be no less than 3.3 g cm^{-3}.

While shear stresses in solid regions may cause (i) to be violated in the real Earth, the expected strength[35] of the material is too low to permit significant violations at depths greater than ~100 km. The density constraints are obtained from the observed seismic velocities in the Earth's central and outer regions with the use of semiempirical relations[36,37] between seismic velocity and density.

For a given composition there is a one-to-one relation between seismic velocity and density; for a given velocity, the higher the mean atomic number of the material the higher the density. All elements with Z greater than iron are rare,[38] and if we assume that the Earth's core is mostly iron we obtain the upper bound on the central density of $\sim 15\ \mathrm{g\,cm^{-3}}$ from the observed seismic velocity in the core. It seems likely that the mean atomic number of the material in the outer mantle of the Earth is close to that of the rocks in the crust; with this assumption, it follows that $\rho_s \cong 3.3\ \mathrm{g\,cm^{-3}}$ (we ignore the crust, which is the very thin outer layer with a density of $\sim 2.6\ \mathrm{g\,cm^{-3}}$).

We now derive the minimum and maximum masses which are consistent with constraints (i)–(iv) above. It follows directly from (i) that $\rho \geq \rho_s$ and thus $M \geq \frac{4}{3}\pi\rho_s R^3$; but if ρ_c were as small as ρ_s, I^* would be 0.40 instead of 0.33. In order to reduce I^* to the observed value we must add mass to the central region of the model. The amount of additional mass is minimized by concentrating it in the central region so that its contribution to I is minimized. Thus, the density distribution which minimizes M consistent with the stated constraints has the form $\rho = \rho_c$ for r less than some radius R_c, and $\rho = \rho_s$ for $r > R_c$. The density distribution which maximizes the mass is obtained by an analogous argument: First let $\rho = \rho_c$ throughout, then reduce ρ to ρ_s in the outer region to reduce I^* to the required value while minimizing the reduction in M. Density distributions of this type have

$$M = \tfrac{4}{3}\pi\rho_s R^3(1 + as^3)\,,$$

$$I = \tfrac{8}{15}\pi\rho_s R^5(1 + as^5)\,,$$

where $s = R_c/R$ and $a = (\rho_c - \rho_s)/\rho_s$. In order that such a model have a particular value of I^*, we require that

$$a = (1 - 2.5I^*)/[s^3(2.5I^* - s^2)]\,. \tag{7}$$

For each value of $a > (2/3I^*)^{3/2}(1 - 2.5I^*)/I^*$ there are two values of s which satisfy Eq. (7); they correspond to the minimum and maximum mass models introduced above. With our chosen values of ρ_s, ρ_c, and I^* we find that $0.76 < M_\oplus/M_0 < 2.0$, where $M_0 = G_c M_\oplus / G_0$, and thus

$$0.50 < G_c/G_0 < 1.32\,.$$

In the following section similar constraints are found through the use of solar models.

IV. DETERMINATIONS OF G_c FROM CONSIDERATIONS OF SOLAR STRUCTURE

While the solar-neutrino problem continues to remain an embarrassment for solar modelers,[39] Newman and Fowler[40] have shown that elementary requirements of hydrostatic equilibrium, energy conservation, etc., can make considerations of solar structure a powerful tool for constraining hypothetical deviations from the laws of physics as observed in the laboratory or inferred from terrestrial measurements. Ulrich[41] has considered the possibility that the gravitational constant on the solar scale may differ from that of the laboratory, and has shown that solar models of somewhat reduced neutrino-counting rate can result. We extend Ulrich's work to show, in the manner of Newman and Fowler for the proton-proton rate, that values of G_c very different from G_0 are incompatible with solar observations.

A solar model is a model of a solar mass star which attains the solar luminosity and the solar radius after evolving for $t_\odot \approx 4.7\times 10^9$ years from the zero-age main sequence. In an effort to quench the troublesome solar neutrinos, many solar models have been constructed in recent years with some input parameter altered. In general the result is a star which does not reach solar conditions at solar age. To recover solar conditions the initial luminosity must be adjusted by altering the primordial composition, while the radius is adjusted by changing the mixing length parameter of the convection theory until $L_\odot, R_\odot$ is again achieved at $t_\odot$.

We have constructed a sequence of solar models with the product $G_c M$ fixed at its observed value $G_0 M_\odot$ ($M_\odot$ denotes the conventional value $G_c M/G_0$ of the solar mass), but with $G_c/G_0 = (M/M_\odot)^{-1}$ allowed to vary, from model to model.[42] In our models decreasing G_c/G_0 (increasing $M/M_\odot$) depressed the central temperature and density, and the luminosity. The initial mean molecular weight had to be increased to compensate for this effect, as Ulrich[41] found. Increasing G_c/G_0 (decreasing $M/M_\odot$) elevated T_c, ρ_c, and the luminosity, and thus the initial mean molecular weight had to be decreased to recover solar conditions at $t_\odot$. Varying G_c had little effect on the radius, so significant adjustments in the mixing length parameter were not found to be necessary. For each value of G_c/G_0 the model was reevolved from the zero-age main sequence to $t_\odot$ after adjusting the initial composition.

The initial helium mass fraction, Y_0, required to compensate for various values of G_c/G_0 is shown in Fig. 2. By beginning with a star of essentially pure hydrogen we could accommodate G_c/G_0 as large as 1.85; larger values of G_c produce $L(t_\odot) > L_\odot$ and are therefore excluded. While one might think that it would be very difficult for the sun to be powered by hydrogen-burning nuclear reactions with a very small hydrogen mass fraction, the very large solar mass required for $G_c/G_0 \ll 1$ allows the total amount of fuel available to be adequate.[43]

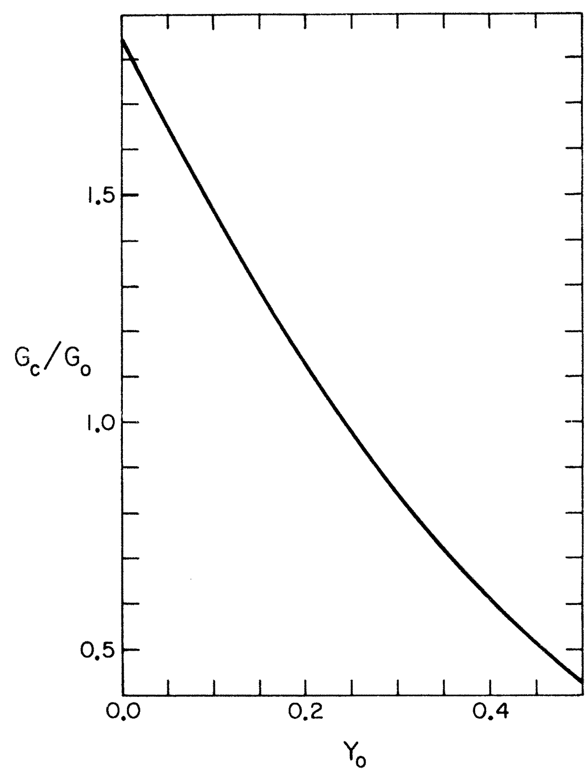

FIG. 2. Initial helium mass fraction Y_0 needed to compensate for trial values of G_c/G_0.

The predicted neutrino-counting rate for the ^{37}Cl experiment decreases with reduced G_c/G_0—as Ulrich[41] found—for values of G_c/G_0 near 1. For low G_c/G_0 the helium mass fraction is so large that PP II and PP III completions of the proton-proton chain begin to dominate, and $\sum \sigma\varphi$ again rises. The smallest neutrino-counting rate achieved was about 2.3 solar neutrino units near $G_c/G_0 \approx 0.5$.

The value of G_c inferred from our solar models is a function of the adopted helium abundance. If this abundance is unrestricted, $0 \leq Y_0 \leq 1$, we find that G_c/G_0 lies in the range ~0.05 to 1.85, a rather loose constraint. Better limits on G_c can be obtained if we specify the present surface solar helium abundance Y, a quantity which is quite difficult to determine.[44] Ross and Aller[45] have surveyed the available estimates and conclude that $0.13 \leq Y \leq 0.28$; this implies that $0.90 \leq G_c/G_0 \leq 1.36$. However, all direct abundance determinations sample only the surface material, and several authors[46] have suggested that the solar surface has been contaminated by accretion. Even if this is the case, it is possible to make a plausible indirect estimate of Y_0 because most estimates of the helium abundance in many different kinds of objects lie in a relatively restricted range.[47] Searle and Sargent[48] have argued that the helium mass fraction is universally[49] in the range 0.22 to 0.34; this implies that $0.75 \leq G_c/G_0 \leq 1.06$. We consider it quite likely that $0.1 \leq Y_0 \leq 0.4$ in the protosun. If so, then our solar models constrain G_c such that

$$0.6 \leq G_c/G_0 \leq 1.5 .$$

V. DISCUSSION

While a considerable body of evidence demonstrates that $G(r)$ is very nearly constant for distances $10^3\text{ km} < r < 3\times10^8\text{ km}$, the actual value of that constant is only poorly determined. We feel that our lowest reliable upper bound on G_c is that which follows from the lower limit on $M_\oplus$ derived in Sec. III, while the solar models discussed in Sec. IV coupled with the restriction $Y_0 > 0.10$ provide the highest reliable lower bound on G_c. The resulting allowed range $0.6 < G_c/G_0 < 1.3$ is rather broad, and does not even rule out Fuji's suggested value $G_c/G_0 = 0.75$. Fortunately, there appear to be several ways of determining G_c more precisely.

A precise estimate of the mass of a body for which G_cM is known or can be measured would be of obvious value. Both the upper and lower limits on $M_\oplus$, and hence G_c, could be tightened if the full range of available seismological information concerning the Earth's interior were used to determine the density distribution. The major problem here is finding a way to unambiguously determine the composition of the various regions of the Earth. If the question of the helium abundance of the Sun could be settled definitively the work of Sec. IV would provide a much more precise determination of G_c.

Gravitation experiments in the intermediate distance range between 1 m and 10 km would remove the large uncertainty in $G(r)$ in this range. Such experiments may also provide a good determination of G_c if they are conducted with a source-detector distance so large that $G(r)$ can be related to G_c. Second-generation gravitational radiation detectors, especially those sensitive to frequencies below 1 kHz, may be capable of detecting time varying Newtonian gravitational sources at distances up to 10 km. Available indirect evidence (see the discussion of Fig. 1) indicates that $G(r)$ at such distances differs from G_c by no more than ~10%. Spacecraft tracking during close encounters of celestial bodies with radii $\lesssim 5$ km (e.g., the moons of Mars, small asteroids) might enable one to relate G_c directly to $G(r)$ at distances of a few km.

While Newtonian gravitation has been tested as yet only for laboratory distance scales ($5\text{ cm} \lesssim r \lesssim 1\text{ m}$) and solar system distances ($10^3\text{ km} \lesssim r \lesssim 10^8$ km), the behavior of $G(r)$ for $1\text{ m} \lesssim r \lesssim 10^3$ km is susceptible to experimental determination. We hope that our present efforts will arouse interest in the problem, and that the large uncertainty in $G(r)$ at these distances will be reduced in the near future.

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[43]Solar models were constructed with Y_0 as large as 0.73 (corresponding to $G_c/G_0 = 0.15$), but we consider values of Y_0 greater than 0.5 to be astrophysically implausible and have omitted them from Fig. 2. These results may be recovered from the accurate empirical formula $G_c/G_0 = 0.115\ \mu^{-4}$, where the initial mean molecular weight μ is given by $\mu = (2X_0 + 3Y_0/4 + Z_0/2)^{-1}$ with Z_0/X_0 fixed at 0.02.
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