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Roy's equations and the study of inelastic effects on the pion form factor

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We study the inelastic effects on the pion form factor using the Muskhelishvili-Omnes integral equation. The $\pi\pi$ phase shifts are required to satisfy Roy's equations which follow from crossing symmetry. Effects of the $I = 1$, $J = 1$ resonance ρ' are manifested remarkably in our results.

I. INTRODUCTION

Electron-positron colliding-beam experiments provide useful information regarding the pion electromagnetic form factor in the timelike region of the momentum transfer. It follows from unitarity that the phase of the pion form factor is equal to $I=1$, $J=1$ $\pi\pi$ phase shifts, δ_1^1 (modulo π) for $q^2 < 16m_\pi^2$. However, phenomenologically the form factor is also fitted with δ_1^1 for larger values of q^2. Thus it is of interest to study the manifestation of inelastic effects in the pion form factor.

It is well known that ρ appears as a resonance in the $I=J=1$ channel, and its coupling to the photon is expressed through the vector-dominance hypothesis. This hypothesis has been used[1] to study the pion form factor in the timelike region.

On the other hand, unitarity could act as a constraint in the study of the phase of the form factor. Such a constraint is introduced through the Muskhelishvili-Omnès (MO) equation.[2] Recently, using complete unitarity, Pham and Truong[3] (PT) have obtained an interesting equation in studying the phase of the pion form factor. They use the experimental phase shifts[4] and the amplitude of Gounaris-Sakurai[1] and Frazer-Fulco[5] to determine the phase and the inelastic effects in the form factor.

In the present paper, to study the phase of the pion form factor, we use a set of $\pi\pi$ phase shifts which satisfies the crossing constraints in the form of Roy's equations,[6] derived from first principles. Moreover, the phases satisfy correct analyticity in the complex energy plane. And since unitarity is already an input through the MO equation, we feel that the ingredients in our calculations are more refined, and thus the results are expected to be more reliable.

In Sec. II we define relevant quantities and write down the Muskhelishvili-Omnès equation. Section III contains a brief account of the solutions of Roy's equations. In Sec. IV we discuss our calculations and results and compare them with those of PT.

II. MO EQUATION FOR PION FORM FACTOR

We define

$$\langle 0|J_\mu|\pi^+(\vec{p})\pi^-(\vec{p}\,')_{\rm in}\rangle = (p-p')_\mu F(s) \ , \tag{1}$$

where $s=(p+p')^2$. Equation (1) can also be written as

$$F(s) = \frac{2}{s-4m_\pi^2}\langle 0|\vec{J}\cdot\vec{p}|\pi^+(\vec{p})\pi^-(-\vec{p})_{\rm in}\rangle \ , \tag{2}$$

with the normalization $F(0)=1$. $\vec{p}$ is the momentum of the π^+ in the c.m. of the $\pi^+\pi^-$ system. It is well known that $F(s)$ has a cut from $s=4m_\pi^2$ to ∞ and $F(s)$ is regular, analytic in the complex s plane. Thus a dispersion relation could be written down for $F(s)$. We write a once-subtracted dispersion relation for $F(s)$ as

$$F(s) = 1 + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{{\rm Im}F(s')}{s'(s'-s)}\,ds' \ . \tag{3}$$

One can then use the standard Lehmann-Symanzik-Zimmermann (LSZ) technique and the MO equations, and follow the formalism of PT to obtain for the phase, ϕ, of the form factor,

$$\phi = \delta_\rho - \frac{\pi}{2}(1-\alpha)\frac{M^2\Delta^2}{(s-M^2)^2+M^2\Delta^2} \tag{4}$$

and for $|K(s)|$.

$$|K(s)| = \exp\left[-\frac{\pi}{2}(1-\alpha)M\Delta\left(\frac{M^2 s}{(s-M^2)^2+M^2\Delta^2} - \frac{1}{M^2}\right)\right], \tag{5}$$

where M, Δ, α are parameters used to take into account the effects of higher-mass intermediate states in $\pi\pi$ scattering, and where $|K(s)|$ represents the deviation of the modulus of the form factor, $|F(s)|$, from its Breit-Wigner form. To evaluate M, Δ, α, we follow in this work what we hope will be a more refined approach than that of PT, as discussed in the next section.

III. ROY'S EQUATIONS AND MULTIPLICITY OF THEIR SOLUTIONS

It is well known[7] that unitarity, crossing, and analyticity are powerful constraints on the behavior of the scattering amplitude and may alone define a unique solution for it. Failure to exploit them fully has led all systems of equations (for on-shell S-matrix elements) to nonunique solutions. This nonuniqueness appears in partial-wave analysis either in the form of ambiguities or in the form of various allowed shapes for the partial waves, from which one does not have the latitude to prefer one to the other solutions.

Recently, Roy[6] has expressed the crossing properties of the scattering amplitude in terms of the physical region partial waves in a form which has proved to be very useful for practical applications[8-10] in pion-pion scattering. Roy's equation for the $I=1$ P-wave amplitude, $g(s)$, is

$$\mathrm{Re}g(s)=(s-4m_\pi^2)\left[\int_{4m_\pi^2}^{\infty}dx\,K'(s,x)\frac{\mathrm{Im}g(x)}{x-4m_\pi^2}+\mathrm{DT}\right]. \tag{6}$$

The term DT is the driving term, fixed by the S-wave scattering lengths and a sum of integrals over the absorptive parts of all the other partial waves. The kernel $K'(s,x)$ is known; it contains the principal values $P(1/(x-s))$ as a singular term.

Most careful numerical calculations[8,11] show that given the analyticity of $g(s)$ the range of possible solutions of Roy's equations is rather large, once DT and the scattering lengths are fixed. The experimental phase shifts happen to be one of these multiple solutions. The dimension of the multiplicity of solutions is given by[12]

$$n_1=\left[\frac{2}{\pi}\,\delta(s_1)\right], \tag{7}$$

where $\delta(s_1)$ is the phase shift at the inelastic threshold. A nonzero value of n_1 results from two facts[12]: (a) The P-wave Roy equation (i.e., usable form of physical-region crossing constraints) combined with unitarity and analyticity requirements does not constrain $g(s)$ above the inelastic threshold s_1. (b) The very existence of the ρ meson implies $\delta(s_1)>\pi/2$ and $n_1\neq 0$.

Thus instead of using the experimental phase shifts to fix M, Δ, α, we use the multiple solutions of Roy's equations.[12] These solutions have been obtained by assuming elastic unitarity in the interval $4m_\pi^2\leq s\leq s_0$, where $s_0\simeq 60m_\pi^2$. However, imposition of analyticity properties of the scattering amplitude on these solutions of Roy's equations extends[12] the domain of their validity to the internal $4m_\pi^2\leq s\leq s_1$, independent of the choice of s_0 $(s_0<s_1)$. Thus it is hoped that these $I=J=1$ phases allowed by such formal properties of the scattering amplitude as unitarity, crossing symmetry, and analyticity will reflect the inelastic effects more accurately.

IV. CALCULATIONS AND RESULTS

Pomponiu and Wanders[12] have obtained four bands of phase shifts for the $I=J=1$ wave. We use them in Eq. (4). For δ_ρ we use the formalism of Gounaris and Sakurai. A least-χ^2 fit to all four bands of phase shifts,[12] taken simultaneously, gives $M=1.542\pm 0.009$ GeV, $\Delta=0.169\pm 0.001$ GeV, and $\alpha=0.28252\pm 0.1$. The ratio χ^2/NDF was 0.85.

The results for $\phi(s)$ and $|K(s)|$ up to 2.5 GeV are plotted in Fig. 1 along with those of PT. It is interesting to note that in the lowest-order approximation used by us, the modulus of the form factor deviates from the usual Breit-Wigner form only by 6 to 7% for energies up to 1 GeV. This is perhaps due to the fact that although the inelastic 4π channel opens up at 0.56 GeV, still the inelasticity is negligible up to the $K\bar{K}$ threshold ≈ 1 GeV.

It is only after 1 GeV that the differences between our results and those of PT are perceptible. Although qualitatively they agree in the sense that in both calculations $\phi(s)$ and $|K(s)|$ show a downward trend after 1 GeV and reach a minimum around 1.6 GeV, still quantitatively $\phi_{\min}(s)$ occurs at 1.55 GeV in our case and PT obtain $\phi_{\min}(s)$ at 1.6 GeV. This produces a minimum at 1.45 GeV for $|K(s)|$ in

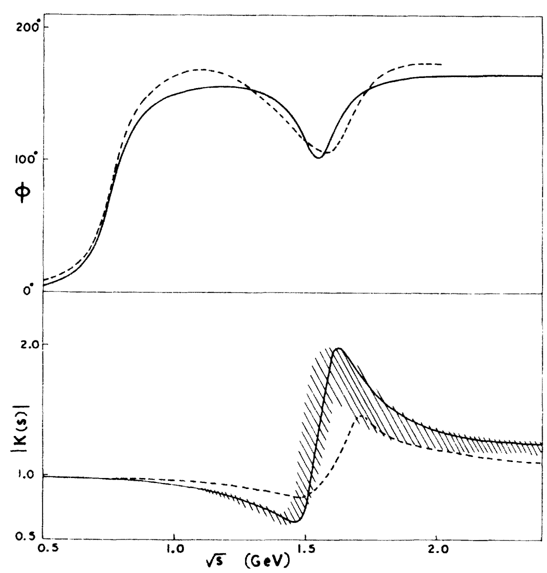

FIG. 1. $\phi(s)$ and $K(s)$ up to 2.5 GeV. The straight solid lines represent this analysis. The dashed lines represent the PT analysis. The slanted lines indicate a band of $|K(s)|$ values allowed by Roy's equations.

our case to be compared with PT's minimum at 1.55 GeV.

Beyond this $\phi(s)$ increases rapidly to a value around 170°. This manifests itself as a peak for $|K(s)|$ at 1.625 GeV. We consider this a striking result since there exists a $J=1$, $I=1$, resonance, ρ', at 1.62 GeV in $\pi\pi$ scattering. It must be noted, however, that PT get a peak in $|K(s)|$ at 1.7 GeV. We attribute this improvement over the result of PT to the ingredients of our calculation, which are derived from first principles.

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