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Dynamical fit to low-energy π-N phase shifts and determination of the threshold parameters*

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For the description of low-energy πN scattering, [1/1] Padé approximants have had limited success starting from Lagrangian-induced power series. We have shown elsewhere that, from a formal power series whose generating kernel can in principle be approximated by a kernel of finite rank N, we can construct a democratic approximant A^N with N perturbative terms which provides as good an approximation to the true solution as a Padé approximant $[N/N]$ with $2N$ perturbative terms. Here we use the two available orders of perturbative terms g^2 and g^4 of the Lagrangian $g\bar{\psi}\gamma_5\psi\phi$ to construct a democratic approximant $A^{N=2}$. We apply it to the low-energy πN phase-shift analysis of Carter, Bugg, and Carter and show empirically that a reasonably good fit can be obtained in the low-energy region with the two available orders of perturbative terms. Extrapolating this fit to threshold we determine scattering lengths and effective ranges for S and P waves which are in reasonably good agreement with more conventional dispersion-relation determinations. The method indicates how the concept of Lagrangian can be made dynamically relevant in a strong-interaction context.

I. INTRODUCTION

Since the early work of Hamilton,[1] determinations of threshold parameters for pion-nucleon scattering have usually shunned extrapolation to threshold from energy regions where reliable data are available. This has been largely due to doubts regarding the validity of parametric expansions in powers of q^2 so far away from threshold; one exception perhaps is the analysis of Rittenberg *et al.*[2] Basically three different approaches have coexisted to describe the low-energy behavior of πN scattering, and all have an experimental input to some extent.

The first and most extensively used is based on dispersion relations, this approach claims to be model-independent and integrates a large amount of experimental data. Considerable sophistication has been achieved in this direction originally with fixed-t dispersion relations (see Pilkuhn *et al.*[3] for a review of pre-1973 work on the subject) and still recently.[4,5] Then were developed forward dispersion relations,[6,7,8] interior dispersion relations,[9] and also weighted-dispersion-relation techniques.[10] These analyses usually predict scattering lengths, or at least constraints upon their values, and also, of late, parameters at the crossing-symmetry point in the unphysical region according to a parametrization due to Hohler, Jakob, and Strauss.[11]

In spite of these considerable refinements in the determination of low-energy parameters many practical problems remain in analysis through dispersion relations, due in part to the use of data from different experiments spanning vast energy ranges, and the need for hypotheses about behavior at high energies.[12] (In order to avoid model input at high energies, Nielsen and Oades[13] have used finite-contour fixed-t dispersion relations.) Although a fair consensus has been achieved about the general area where values should lie for scattering lengths, instabilities are still being felt (see for example the values of $a_1 + 2a_3$ predicted by Samaranayake and Woolcock[7]). Further, dispersion-relation methods being free from dynamical content, it is very difficult to calculate electromagnetic corrections since dispersion relations are notably difficult to apply to problems involving photons (see in this respect the work of Hamilton, Tromborg, and co-workers[14]). Electromagnetic effects become very important at very low energies (below 40 MeV pion kinetic energy) and are needed for comparison of data with theoretical predictions.

The other two approaches for predicting low-energy behavior of πN scattering have a large dynamical content with more or less of an experimental input: the current-algebra approach and the Lagrangian-based predictions. Typical of the current algebra with various particle-exchange terms is the large number of adjustable parameters, seven in the model of Olsson and Osypowski.[15] The motivation there is phenomenological, in the sense that it makes no reference to a Lagrangian, the aim being to cover a large energy range accounting for resonances, etc. Again it is difficult to see how electromagnetic effects could be calculated in that context for the very-low-energy region.

Models based on a fundamental Lagrangian for the nuclear forces would have the definite advantage of being amenable to electromagnetic corrections through the minimal-coupling rule on the charged hadrons. They may also be deemed by many as more satisfying as dynamical models go.

The Lagrangian yields Feynman diagrams and a power series which is purely formal in view of the strong-coupling constants. This important difficulty has been tackled since Padé-approximant (PA) techniques have gained popularity.[16] The Padé approximant recuperates the information contained in the first few perturbative terms and aims at reconstructing the "true" solution. It should be pointed out that its use is in general purely heuristic since convergence cannot be proven. The difficulty is compounded by the fact that usually only the first two perturbative terms are available via very tedious Feynman-diagram calculations.

Using the traditional interaction Lagrangian

$$\mathcal{L}=g\bar{\psi}\gamma_5\psi\phi \tag{1.1}$$

and a [1/1] Padé approximant, Mignaco *et al.*[17] find that they are unable to arrive at a satisfactory model when checked against experimental data. However, with the addition of a $\lambda\phi^4$ term and using λ as an adjustable parameter they achieved reasonable agreement with experiment for the $T=\frac{3}{2}$ partial waves but unfortunately not for the $T=\frac{1}{2}$ waves. Other Padé calculations starting from different Lagrangians have been carried out lately with the nonlinear σ model.[18,19,20] The latest of these calculations[20] by Lin and Willey arrives at encouraging results; however, their answers are different from either of the previous two calculations. There is an adjustable parameter and the differences seem to lie in the nonunique regularization procedure necessary with that Lagrangian.

The efforts so far to construct [1/1] Padé approximants with only two orders of perturbative terms are probably too ambitious when attempting to fit all the data with one adjustable parameter. It seems to us that the PA puts undue emphasis on the coupling constant, since for large coupling constants the power series is purely formal. The working hypothesis in this article will be that the traditional Lagrangian $\mathcal{L}=g\bar{\psi}\gamma_5\psi\phi$ has been abandoned too soon and that an approximant which makes more economical use of the scarce "perturbative terms" may show more promise than the PA. From the two available orders of perturbative terms of the above Lagrangian we construct the "democratic approximant" to the K-matrix elements corresponding to the S and P waves. The democratic approximant[21] stresses the energy dependence of the perturbative terms instead of the dependence on the coupling constant as the PA does. We have shown elsewhere[21] that, to the extent that the generating kernel of the formal power series can be approximated by a kernel of finite rank, the democratic approximant describes as well the energy dependence with N perturbative terms as does the Padé approximant with $2N$ perturbative terms. The reason for this is that the PA uses up precious perturbative terms in its attempt at reconstructing the dependence on the coupling constant. This seems to us bad management of the few available perturbative terms when considering that, in the finite-rank kernel approximation, the solution tends to become independent of the coupling constant as it gets large. Nevertheless it was shown also that the error is formally of order λ^{N+1} for the democratic approximant as for the PA when N perturbative terms are available (see Ref. 21 for a detailed discussion).

Thus, in that context, the democratic approximant using two perturbative terms should do as well in predicting the energy dependence as a [2/2] PA using four perturbative terms. The price we have to pay for this is to supply additional empirical information to make up for our lack of predictions regarding coupling-constant dependence. This takes the form of knowledge of the true solution and its derivatives at a single point or, equivalently, knowledge of the true solution over some finite region. The democratic approximant then effectively uses the perturbative terms as best-fit functions whose weights have to be determined empirically. Since it is a linear combination of perturbative terms, the analytical properties of the power series are preserved; however, unitarity is not. Hence we shall use the power-series expansion of the K matrix or its inverse instead of that of the S matrix.

We cannot strictly demonstrate the applicability of the democratic approximant to the present problem; but this remark is equally applicable to users of the PA. However, we do suggest in Sec. II a heuristic way of testing whether the number of perturbative terms is adequate using the energy dependence of the given perturbative terms and knowledge of the "solution" through some region. We then apply this method, in Sec. III, to the phase-shift analysis of Carter, Bugg, and Carter[22] (88 MeV to 310 MeV); it stems from a coherent set of data of good accuracy and has been corrected for Coulomb and mass-difference effects in a model-dependent way. Electromagnetic effects in those regions are relatively small and apparently not very dependent on the particular model.[23,24] A best-fit criterion is used to determine the weight of each perturbative term after we have established that the two available orders of perturbative terms satisfy reasonably well the heuristic test developed in Sec. II. Extrapolating this fit to threshold, scattering lengths and effective ranges can be predicted through the theoretically calculated energy dependence of the perturbative terms at threshold.

Section IV examines the results and compares them with determinations of scattering lengths obtained through other methods.

It should be remembered that the purpose of this article is not to give a full dynamical account of low-energy πN scattering starting from the two available perturbative terms as was tried in the past with PA. Its aims are more modest: (i) to show heuristically that the first two orders of perturbative terms of the traditional Lagrangian can reasonably be used as best-fit functions to the low-energy πN data and (ii) to use this fit in the democratic-approximant context to extrapolate data to threshold and make semiempirical predictions for the threshold parameters.

II. THE APPROXIMATION

In a previous article[21] we developed an approximant, starting from a formal Born series, which we have called "democratic" because all available perturbative terms are *a priori* on an equal footing, in contrast to having weights and roles dependent on their order of appearance in the formal power series, as is the case with Padé approximants. Briefly the argument goes as follows: The power series generated from an integral equation

$$(I-\lambda K)f=g \tag{2.1}$$

can be written

$$f(x,\lambda)-g(x)=\lambda f_1(x)+\lambda^2 f_2(x)+\cdots , \tag{2.2}$$

where iteration gives $f_j=K^j g$. For a finite-rank kernel of arbitrary rank L, the Fredholm solution can be written as

$$f^L(x,\lambda)-g(x)=\sum_{i=1}^{L} C_i^L(\lambda) f_i^L(x) . \tag{2.3}$$

We noted that as λ becomes large, the coefficients $C_i^L(\lambda)$ tend asymptotically towards constants independent of λ. Thus for series obtained from finite-rank kernels at any rate, and those from operators that may be approximated by finite-rank kernels (e.g., completely continuous operators), it makes sense for large λ to build an approximant which treats all "perturbative" terms on an equal footing. Thus we proposed

$$\tilde{A}^N(x)\equiv A^N(x)-g(x)=\sum_{i=1}^{N} a_i^N f_i(x) , \tag{2.4}$$

where $A^N(x)$ aims at reconstructing the true solution $f(\lambda,x)$. This approximant has no use for the large coupling constant λ; it regards λ as a mere formal device in the process of obtaining a Born series. The democratic approximant uses the x dependence of perturbative terms as the only significant ingredients. The coefficients a_i^N contain the λ dependence in some complicated way which we need not unravel if we have sufficient empirical information on the true solution in some region or at some point x_0. In our article[21] we studied a_i^N determined through knowledge of $f(x)$ and $(N-1)$ derivatives at one point x_0, then

$$a_i^N=\left.\frac{W_N(f_1,\ldots,f_{i-1},f-g,f_{i+1},\ldots,f_N)}{W_N(f_1,\ldots,f_{i-1},f_i,f_{i+1},\ldots,f_N)}\right|_{x_0} , \tag{2.5}$$

where W_N are Wronskians, and a detailed error analysis is given for this choice of a_i^N. In this article we shall use information over a finite range of values of x (i.e., the energy) to determine the coefficients. It is well to emphasize the contrast with Padé approximants. Whereas the PA stresses the λ dependence and considers the $f_i(x)$ as coefficients in the power expansion in λ, the democratic approximant stresses the x dependence of the $f_i(x)$. Thus these two approximants are meant for use on very different objects. The diagonal PA satisfies unitarity and can be used to approximate the amplitudes f_l [where $\mathrm{Im}f_l=(f_l)^2$], but it introduces x dependence in the denominator thus upsetting the analytical properties in x of the formal power series and presumably of the true solution. The democratic approximant preserves analyticity but, being a linear combination, it is not suitable for approximating unitary functions; it is best used on functions with smooth dependence on the x variable such as $q^{2l+1}\cot\delta_l$ or its inverse as we shall see later on.

In practive, whatever may be the operator generating the Born series, it may be possible to define heuristically a democratic approximant by consideration of the following algorithm. Take

$$\tilde{f}(x)\equiv f(\lambda;x)-g(x)=\lambda f_1(x)+\lambda^2 f_2(x)+\cdots . \tag{2.6}$$

First divide by $f_1(x)$ on both sides:

$$\tilde{f}/f_1=\lambda+\lambda^2 f_2/f_1+\lambda^3 f_3/f_1+\cdots . \tag{2.7}$$

If the kernel were of finite rank $=1$, all the ratios on the right-hand side (rhs) would be constant, and

$$\tilde{f}=C_1^{L=1} f_1 . \tag{2.8}$$

Thus if the ratios at hand are nearly constant, the following ansatz may be made for an approximant to $\tilde{f}$:

$$\tilde{A}^{N=1}=a_1^{N=1} f_1 . \tag{2.9}$$

If these ratios of perturbative terms are not constant we first differentiate everywhere and divide by one of the new coefficients of λ^j, say $(f_2/f_1)'$:

$$(\tilde{f}/f_1)'/(f_2/f_1)' = \lambda^2+\lambda^3 W_2(f_1,f_3)/W_2(f_1,f_2)+\cdots . \tag{2.10}$$

Thus if the kernel were of rank = 2, all the terms on the rhs would be constant, and

$$\tilde{f}=C_1^{N=2}f_1+C_2^{N=2}f_2. \tag{2.11}$$

Again, if the ratios of derivatives are nearly constant, an approximant may be defined using this near constancy as the dominant feature:

$$\tilde{A}^{N=2}=a_1^{N=2}f_1+a_2^{N=2}f_2. \tag{2.12}$$

This repeated process of division and differentiation can be carried on in theory until practical constancy for the ratios of Wronskians on the rhs is achieved, thus heuristically justifying an approximant with a finite linear combination of Born terms. In practice this will be limited by the number of perturbative terms available; however, we shall be encouraged in the use of the approximant if successive division and differentiation show a trend towards constancy.

We can also consider the case where the true solution $\tilde{f}$ is known over a given range of the independent x variable. If, for example, we should find that

$$(\tilde{f}/f_1)'/(f_2/f_1)'\simeq \text{constant}, \tag{2.13}$$

we can say that at least over this range (and hopefully beyond) $\tilde{f}$ can be approximated as

$$\tilde{A}^{N=2}=a_1^{N=2}f_1+a_2^{N=2}f_2. \tag{2.14}$$

Note that in this case the coefficients are determined by a best fit, not as in Eq. (2.5).

We have therefore sought for the problem at hand a function $\tilde{f}(W)$:

(1) with smooth dependence in W in the region of interest, threshold to low energies,

(2) for which reasonably accurate data were available in a portion of the range of interest, and

(3) which satisfied, approximately, criterion (2.13).

If this last criterion is reasonably satisfied then the coefficients $a_i^{N=2}$ can be determined empirically through the known region, and a prediction for the values of $\tilde{f}$ and $\tilde{f}'$ at threshold can be made since $f_1(W)$ and $f_2(W)$ can be evaluated theoretically at threshold. Thus the compromise between data extrapolation and a mild theoretical input is realized to estimate values for the scattering length and effective range.

The search for a function which satisfied the three conditions mentioned above led us to the obvious candidates: $q^{2l+1}\cot\delta_l$ and its inverse. Both have been studied in detail by Rasche and Woolcock[25] in a Schrödinger-equation square-well model and found to have *a priori* approximately equal energy regions for a valid expansion in terms of scattering length and effective range (or, alternatively, curvature). We have chosen $q^{2l+1}\cot\delta_l$ in all instances except $P(1,1)$, where we took $q^{-2l-1}\tan\delta_l$ because of the change in the sign of the δ in the vicinity of $W=1205$ MeV. We give the leading power expansions for these two functions in terms of the first two perturbative terms of the partial-wave amplitude

$$f_l\equiv e^{i\delta_l}\sin\delta_l$$
$$=g^2\mathrm{Re}\,f_l^{(2)}+g^4(\mathrm{Re}\,f_l^{(4)}+i\,\mathrm{Im}\,f_l^{(4)})+\cdots. \tag{2.15}$$

Writing $\lambda\equiv g^2$, $f_1\equiv\mathrm{Re}\,f_l^{(2)}(W)$, and $f_2=\mathrm{Re}\,f_l^{(4)}(W)$ we have

$$q^{-2l-1}\tan\delta_l=\lambda(q^{-2l-1}f_1)+\lambda^2(q^{-2l-1}f_2)+O(\lambda^3), \tag{2.16a}$$

$$\lambda^2q^{2l+1}\cot\delta_l=\lambda(q^{2l+1}/f_1)-\lambda^2(q^{2l+1}f_2/f_1^2)+O(\lambda^3). \tag{2.16b}$$

After division by the first Born term in each case this reads, in a democratic-approximant context using two perturbative terms,

$$\tan\delta/f_1\simeq A_0+A_1f_2/f_1, \tag{2.17a}$$

$$f_1\cot\delta\simeq B_0+B_1f_2/f_1. \tag{2.17b}$$

The terms $\mathrm{Re}\,f_l^{(2)}(W)$ and $\mathrm{Re}\,f_l^{(4)}(W)$ have been obtained in a numerical evaluation[26] of all contributing diagrams to the πN scattering of order g^2 and g^4 starting from the traditional Lagrangian, $g\bar{\psi}\gamma_5\psi\phi$. In that calculation we noted an interesting fact: Whereas the terms $f_1\equiv\mathrm{Re}\,f_l^{(2)}$ and $f_2\equiv\mathrm{Re}\,f_l^{(4)}$ have individually a fairly pronounced dependence on W (c.m. energy), it is remarkable that the ratios $X(W)\equiv f_2/f_1$ are much less strongly characterized in the variable W. In fact, for S waves ($T=1,3$), the ratios $X(W)$ are practically straight lines over the whole range of W calculated (1100 to 2000 MeV). This straight-line trend is maintained for the P waves also, albeit over a shorter energy range, with the exception of $P(3,1)$, which tapers off near threshold. Unfortunately we do not have the terms in $\lambda^3\equiv g^6$ at hand to verify if constancy of the rhs is achieved as described in Eq. (2.10), thus justifying a two-term approximant. However, we can test for the criterion (2.13) by actually using the known values of the function $\tilde{f}$ in the range ($W=1152$ to 1320 MeV) provided by the detailed data of Carter, Bugg, and Carter (CBC).[22]

Since the ratios f_2/f_1 are practically straight lines in W for the low-energy regions, it is sufficient for Eqs. (2.17a), (2.17b) to be good approximations that $f_1\cot\delta$ or $[f_1^{-1}\tan\delta$ for $P(1,1)]$ be also approximately straight lines as a function of W. Examination of these functions (Figs. 1 to 6), as computed from CBC[22] data, show that they are

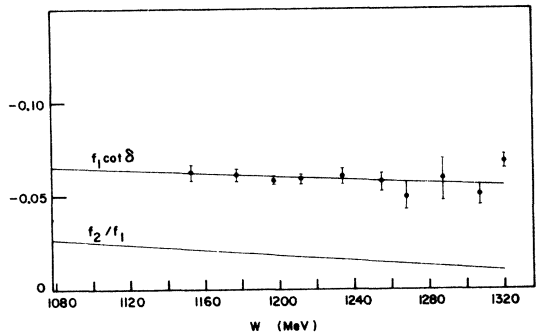

FIG. 1. For partial wave $S(1,1)$ we show the experimental quantities $f_1\cot\delta$ using CBC phase shifts (Ref. 16) and a fit using a linear regression in the ratio f_2/f_1.

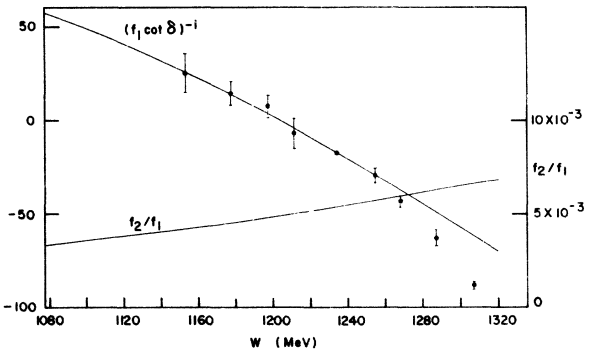

FIG. 3. For partial wave $P(1,1)$ we show the experimental quantities $(f_1\cot\delta)^{-1}$ using CBC phase shifts (Ref. 16) and a fit using a linear regression in the ratio f_2/f_1.

compatible to a satisfactory degree with a straight line at the low-energy end of the range made available.

Putting emphasis on the data at this low-energy end, we will determine the coefficients B_0 and B_1 [or A_1 and A_0 for $P(1,1)$] as accurately as possible, and estimate the errors in the extrapolation to threshold of the function $f_1\cot\delta$ (or $f_1^{-1}\tan\delta$).

III. FITTING WITH A LINEAR REGRESSION

We proceed to find the best possible fit to the available data points of each partial wave for the function $Y \equiv \mathrm{Re}\, f_l^{(2)} \cot\delta_l$ [or its inverse $\tan\delta_l/\mathrm{Re} f_l^{(2)}$ for $P(1,1)$] with an approximant linear in $X(W)$ $\equiv \mathrm{Re}\, f_l^{(4)}/\mathrm{Re}\, f_l^{(2)}$

$$\hat{Y}(W) = B_0 + B_1 X(W) . \tag{3.1}$$

This is a linear regression for $Y(W)$ in the independent variable $X(W)$ whose values have been calculated at the energy points $W = W_i$ of the CBC data.[22]

Now it is unlikely that the "true" $Y(W)$ will be fully described by (3.1) over a range of energies which extends well into the inelastic region where more dynamical input should be necessary. In fact the true $Y(W)$ in our model should include extra terms in $\mathrm{Re}\, f_l^{(6)}/\mathrm{Re}\, f_l^{(2)}$, etc. We must therefore guard against forcing the linearity in $X(W_i)$ beyond its range of validity thereby introducing a bias in the estimated values $\hat{Y}(W)$ due to lack of fit.

In practice this region of validity for (3.1) is not easy to define; we have used a method which keeps as wide a range as possible on the low-energy end of the data while minimizing model bias. Considering that the criterion usually retained for fitting N points with p parameters is to choose the number of degrees of freedom $(N-p)$ so as to minimize the residual mean square

$$s^2 = \sum_{i=1}^{N} (Y_i - \hat{Y}_i)^2/(N-p) , \tag{3.2}$$

we have allowed into the regression the first $n \leq N$ points which made s^2 a minimum since in our model $p = 2$ is fixed. We have found s^2 to be a smoothly varying function of n. Relative minima

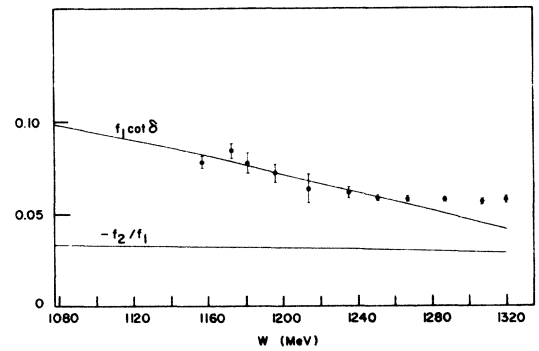

FIG. 2. For partial wave $S(3,1)$ we show the experimental quantities $f_1\cot\delta$ using CBC phase shifts (Ref. 16) and a fit using a linear regression in the ratio f_2/f_1.

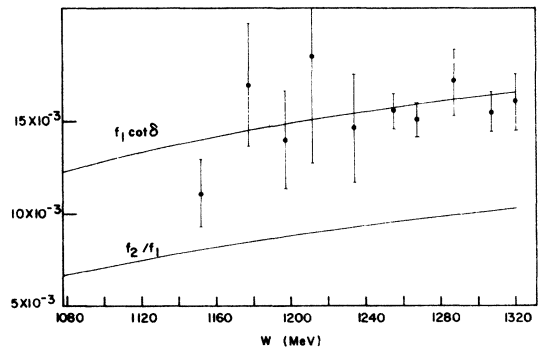

FIG. 4. For partial wave $P(1,3)$ we show the experimental quantities $f_1\cot\delta$ using CBC phase shifts (Ref. 16) and a fit using a linear regression in the ratio f_2/f_1.

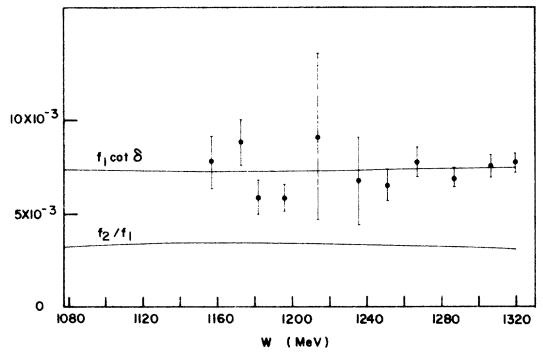

FIG. 5. For partial wave $P(3,1)$ we show the experimental quantities $f_1\cot\delta$ using CBC phase shifts (Ref. 16) and a fit using a linear regression in the ratio f_2/f_1.

were found for $n = 6, 6, 8$ in the case $S(1,1)$, $P(1,1)$, $S(3,1)$, which were associated with strong correlation coefficients between X and Y. Minima were obtained for $n = N$ for $P(1,3)$ and $P(3,1)$ corresponding to situations of weak correlation coefficients. $P(3,3)$ has a striking fit with just the first three energy points having a correlation coefficient practically equal to one due to the effective linearity in W of both X and Y in that low-energy region.

We have systematically used the first n points of lowest energies as being most likely to be compatible with the linear approximation (free of inelastic effects), as well as being closer to the threshold where we wish to extrapolate. We have not otherwise discriminated between data points; the errors quoted do vary from point to point in the CBC[22] data but not substantially in general. The quoted errors were used to define an estimate $\bar{\sigma}^2$ of the variance σ^2 due to "pure error" in the data for each partial wave; $\bar{\sigma}^2$ is the average of the quoted variance over the number of points used.

This rough estimate of the variance due to pure error in the data enables us to check our approximant for bias. The residual mean square s^2 achieved through regression should be such that $s^2 \leq \sigma^2$, otherwise we may regard the regression as model-biased.[27]

A glance at Table I shows that in all cases $s^2 < \bar{\sigma}^2$, an encouraging sign. Had we insisted on the use of all N available points, the bias induced by linearity in $X(W)$ would have swelled s^2 to values much larger than $\bar{\sigma}^2$ in some cases, particularly for $P(3,3)$, where very accurate data are available. In all subsequent estimates of standard errors we have therefore used the residual mean square s^2 as the basis of computing estimates.[27] For the various formulas relevant to computations of estimates of standard errors and their relations with the scattering length and effective range see the Appendix.

IV. RESULTS AND COMMENTS

Table I gives the parameters obtained in our analysis. We see that in all instances we have found a minimum residual mean square $s_{\min}^2$ smaller than the average variance due to pure error $\bar{\sigma}^2$ as calculated from the CBC data,[22] thus removing any *a priori* rejections on the grounds of model bias.

The first n data points (on the low-energy end) used to achieve this determination turned out to be the maximum number of available points in the case of $P(1,3)$ and $P(3,1)$ due to the bad scattering of data points (see Figs. 4 and 5). This is also reflected in the low correlation coefficients R_{XY} for these two partial waves. Weak correlation can also be expected in cases where the data points $f_1\cot\delta(W)$ and the linear variable $X(W) = f_2/f_1$ of the regression are both nearly constant as functions of the energy [see Fig. 1 for $S(1,1)$].

The standard deviations given in Table I are calculated according to the usual linear-regression criterion, using s^2 as a yardstick, and they therefore give a measure of deviations for estimated values on the basis of a two-term approximant. One should remember that the calculated deviations for the constant term B_0 and the slope coefficient B_1 of the regression are in general greater than those obtained for the scattering length and the effective range ($\frac{1}{2}r_t$) because the linear variable $X(W) \equiv f_2/f_1$ at threshold is not substantially different from its values in the data range. Since it is in terms of the variable $X(W)$ that the regression is carried out, estimates at threshold for $f_1\cot\delta$ do not constitute a distant extrapolation in

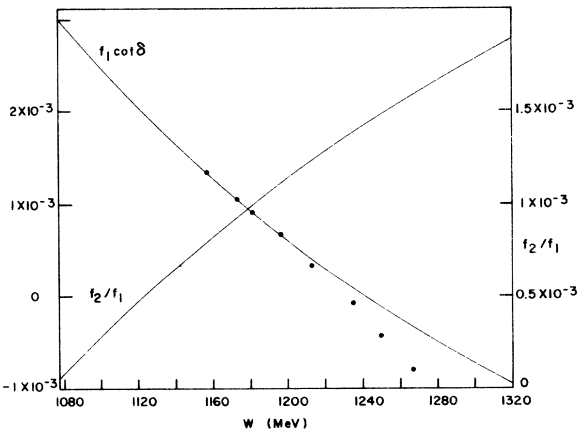

FIG. 6. For partial wave $P(3,3)$ we show the experimental quantities $f_1\cot\delta$ using CBC phase shifts (Ref. 16) and a fit using a linear regression in the ratio f_2/f_1.

TABLE I. Low-energy parameters through the regression $f_1 \cot\delta \simeq B_0 + B_1 f_2/f_1$.

Partial wave	$s_{\min}^2$	$\bar{\sigma}^2$	n	R_{XY}	B_0	B_1	a_l	$\frac{1}{2}r_l$
$S(1,1)$	2.4×10^{-6}	9.3×10^{-6}	6	0.752	-0.0489 ± 0.0048	0.615 ± 0.270	0.135 ± 0.005	0.712 ± 0.128
$S(3,1)$	1.5×10^{-5}	2.4×10^{-5}	8	−0.938	-0.243 ± 0.047	-10.11 ± 1.53	-0.113 ± 0.005	0.842 ± 0.171
$P(1,1)$ [a]	6.1	57.0	6	−0.994	176.5 ± 9.6	$(-3.59\pm0.19)\times10^4$	-0.0575 ± 0.0033	-19.1 ± 1.1
$P(1,3)$	0.37×10^{-5}	1.33×10^{-5}	10	0.418	$(4.66\pm8.31)\times10^{-3}$	1.15 ± 0.89	-0.0243 ± 0.0049	-32.0 ± 1.4
$P(3,1)$	1.35×10^{-6}	6.6×10^{-6}	11	−0.055	$(8.71\pm8.44)\times10^{-3}$	-0.43 ± 2.58	-0.0400 ± 0.0021	-13.9 ± 3.2
$P(3,3)$	1.33×10^{-14}	3.5×10^{-11}	3	−0.999 999	$(3.048\pm0.001)\times10^{-3}$	-2.137 ± 0.001	0.2022 ± 0.0001	-0.0662 ± 0.0005

[a] For $P(1,1)$ $\hat{Y}$ approximates $(f_1\cot\delta)^{-1}$.

the variable $X(W)$ (see the ratio f_2/f_1 in Figs. 1 to 5). The only exception is for $P(3,3)$, where the variable ratio $X(W) = f_2/f_1$ is almost zero at threshold. It is interesting to spectulate in this case on the relation between this fact and the nearness of the resonance.

Table II is for comparison with representative different methods of determining scattering lengths. Our value for $S(1,1)$ differs from the dispersion-relation values,[3,7] but is compatible with solution II of Langbein.[5] At any rate, the near constancy of $f_1\cot\delta$ is rather compelling in view of the lack of features of f_2/f_1 (see Fig. 1). The scattering lengths for $S(3,1)$, $P(1,3)$, and $P(3,1)$ are all very close to the recommended values of Pilkuhn *et al.*[3] and those of Samaranayake and Woolcock,[7,8] as well as being compatible with other less stringent determinations.[5,28]

Partial wave $P(1,1)$ represents a special problem is that the low-lying zero of the phase shift caused us to use a different function for our approximation, i.e., $(f_1\cot\delta)^{-1}$. We have nonetheless obtained a good fit, but the result is quite different from dispersion-relation values[3,7,8] though compatible with solution I of Langbein.[5]

Our determination of the $P(3,3)$ scattering length rests precariously on three data points at 1157.2, 1173.0, and 1181.5 MeV, but the very low residual mean square s^2 and the striking correlation coefficient between $f_1\cot\delta(W)$ and $f_2/f_1 \equiv X(W)$ make it interesting. This value, $a_l = 0.202$, is in very good agreement with that of Ref. 8 as well as being compatible with that of the statistical approach of Lichard,[26] and the algebraic-function fit Langbein[5] (solution I). Note that this value was obtained using the straightforward π^+p data[22]; we have also tried using this same π^+p data to include a fourth point at 1195.8 MeV, and the correlation coefficient was still fairly good but the value obtained, 0.198, appeared model-biased because $s^2 > \bar{\sigma}^2$. Similarly, we have carried out the analysis on the $P(3,3)$ from π^-p data,[22] this time using the first three points available at 1152.3, 1177.1, and 1196.8 MeV; again the value obtained, 0.218, seemed unreliable because the residual s^2 was much greater than the pure-error variance $\bar{\sigma}^2$. It appears that already around 1190 MeV the effect of the resonance is felt, and that a more reliable determination which distinguishes between π^-p and π^+p would require more data points below that energy or more perturbative terms (i.e., $\mathrm{Re}\,a_l^{(6)}$) to describe the dynamics.

Our results indicate that the Lagrangian $\bar{\psi}\gamma_5\psi\phi$ is dynamically relevant to a heuristic description of the low-energy πN data when used with a democratic-approximant approach. This is due to the remarkable straight-line (or near-straight-line)

TABLE II. Comparative determinations of scattering lengths.

Partial wave	This determination	Recommended by Pilkuhn *et al.* (Ref. 3)	Forward dispersion relations (Refs. 7, 8)	Statistical analytic approach (Ref. 28)	Algebraic-function fit to fixed-t dispersion relations, solutions I and II of Ref. 5
$S(1,1)$	$+0.135 \pm 0.005$	$+0.178 \pm 0.078$	$+0.1814 \pm 0.0102$	$+0.208 \pm 0.020$	(I) $+0.178 \pm 0.012$
					(II) $+0.149 \pm 0.019$
$S(3,1)$	-0.113 ± 0.005	-0.112 ± 0.022	-0.0892 ± 0.0064	-0.091 ± 0.017	(I) -0.105 ± 0.006
					(II) -0.092 ± 0.010
$P(1,1)$	-0.0575 ± 0.0033	-0.082 [a]	-0.0845 ± 0.0102	-0.109 ± 0.035	(I) -0.067 ± 0.017
					(II) -0.089 ± 0.020
$P(1,3)$	-0.0243 ± 0.0049	-0.029 [a]	-0.0266 ± 0.0063	-0.045 ± 0.035	(I) -0.017 ± 0.012
					(II) -0.039 ± 0.016
$P(3,1)$	-0.0400 ± 0.0021	-0.043 [a]	-0.0429 ± 0.0071	-0.063 ± 0.022	(I) -0.061 ± 0.010
					(II) -0.043 ± 0.011
$P(3,3)$	$+0.2022 \pm 0.0001$	$+0.215$ [a]	$+0.2041 \pm 0.0045$	$+0.186 \pm 0.022$	(I) $+0.198 \pm 0.008$
					(II) $+0.214 \pm 0.009$

[a] No error quoted.

behavior in W of ratio $\mathrm{Re}f_l^{(4)}/\mathrm{Re}f_l^{(2)}$ coupled to a similar trend for the semiempirical $\mathrm{Re}f_l^{(2)}\cot\delta_l$ at low energies. This fact resulted in a ratio of the slopes of these two quantities that is nearly constant for many of the partial waves studied over a good range of energies, thus warranting an approximation with the two perturbative terms available. That such fits were possible without undue model bias was empirically justified by the fact that the residual mean square was in all cases less than the average variance due to pure error quoted in the experimental analysis. This is not trivial in spite of the restriction to lowest-energy points that we had to impose in order to minimize the residual mean square. These fits, when extrapolated to threshold using the theoretical behavior of the perturbative terms, gave semiempirical determinations for scattering lengths and effective ranges. Those determinations have calculated errors which cannot be taken at face value for the moment in view of the fitting procedure. Nonetheless the values predicted for the scattering lengths are remarkably close to those obtained by dispersion-relation calculations in several cases. Considering the limited data available and that only two orders of perturbative terms were used, this is encouraging, and with more refinements the method might prove an interesting alternative to dispersion-relation methods over the long run.

We stress the concept of dynamical relevance of the Lagrangian $\bar{\psi}\gamma_5\psi\phi$ when used in a democratic-approximant approach instead of the notion of complete dynamical explanation, which is traditionally sought when starting from a Lagrangian. It may not be as satisfying as a Lagrangian theory in QED, say, but in a strong-interaction context with rapidly diverging power series, we contend that it is a way to give meaning to a Lagrangian and put it to use as a predictive device. A question remains with this semiempirical use of a Lagrangian and its formal perturbative series: how to choose one Lagrangian as more "dynamically relevant" than another. One way to decide would be to repeat the analysis described here with different Lagrangians; obviously the Lagrangian which needs less perturbative terms to achieve the same goodness of fit to the given data may be judged more relevant to the description of the phenomena at hand. This is akin to choosing one model over another in that one has less free parameters than the other.

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APPENDIX

Here is a summary of formulas used for the linear-regression analysis and their relation to scattering-length and effective-range parameters.

Assuming normal distribution of errors throughout, and $\sigma_i^2 = \sigma^2 \ \forall i$, the mean is

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y(W_i)\,, \tag{A1}$$

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X(W_i)\,, \tag{A2}$$

the variance about the mean is

$$S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2 , \tag{A3}$$

$$S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 , \tag{A4}$$

and the linear regression is

$$\hat{Y} = B_0 + B_1 X = \overline{Y} + B_1(X - \overline{X}) \tag{A5}$$

with coefficients

$$B_0 = \overline{Y} - B_1\overline{X} , \tag{A6}$$

$$B_1 = \sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y}) \Big/ \sum_{i=1}^{n}(X_i - \overline{X})^2 = S_Y R_{XY}/S_X , \tag{A7}$$

where R_{XY} is the correlation coefficient.

The residual mean square is

$$s^2 = \frac{1}{n-2}\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 = (n-1)S_Y^2(1 - R_{XY}^2)/(n-2) , \tag{A8}$$

and the estimates of variance are

$$V(B_1)_{\text{est}} = s^2/[(n-1)S_X^2] , \tag{A9}$$

$$V(\hat{Y})_{\text{est}} = s^2\left[\frac{1}{n} + \frac{(X - \overline{X})^2}{(n-1)S_X^2}\right] , \tag{A10}$$

$$V(B_0)_{\text{est}} = s^2\left[\frac{1}{n} + \frac{\overline{X}^2}{(n-1)S_X^2}\right] . \tag{A11}$$

At threshold we have

$$Y = (q/\mu)^{2l+1}\cot\delta_l \simeq a_l^{-1} + \tfrac{1}{2}r_l(q/\mu)^2 , \tag{A12}$$

$$X \equiv \mathrm{Re}f_l^{(4)}/\mathrm{Re}f_l^{(2)} \simeq x_0 + x_1(q/\mu)^2 , \tag{A13}$$

$$f_1 \equiv \mathrm{Re}\, f_l^{(2)} \simeq (q/\mu)^{2l+1}[\phi_0 + \phi_1(q/\mu)^2] . \tag{A14}$$

Thus the approximation gives for the scattering length

$$a_l^{-1} = [\overline{Y} + B_1(x_0 - \overline{X})]/\phi_0 \tag{A15}$$

where the estimated standard error of a_l^{-1} is

$$\left[\frac{1}{n} + \frac{(x_0 - \overline{X})^2}{(n-1)S_X^2}\right]^{1/2} s \Big/ \phi_0 \tag{A16}$$

and for the effective range

$$\tfrac{1}{2}r_l = (\phi_1/\phi_0^2)[(x_1\phi_0/\phi_1 - x_0 + \overline{X})B_1 - \overline{Y}] , \tag{A17}$$

and, since B_1 and $\overline{Y}$ can be shown to be uncorrelated random variables, it follows that the estimated standard error of $\frac{1}{2}r_l$ is

$$\left[\frac{(x_1\phi_0/\phi_1 - x_0 + \overline{X})^2}{(n-1)S_X^2} + \frac{1}{n}\right]^{1/2} s\phi_1 \Big/ \phi_0^2 . \tag{A18}$$

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