Baryon spectrum in a linear string model*

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Results are presented on the spectrum of baryon $SU_6 \otimes O_3$ supermultiplets based on a relativistic string model with linear string-length potentials. Calculations are carried out for a four-constituent baryon model as well as the normal three-quark model. Deviations from the conventional harmonic-oscillator spectrum, lack of quark-diquark structure, and the possibility of a low-lying $(56, 1^-)$ supermultiplet are discussed.

I. INTRODUCTION

A recent πN partial-wave analysis[1] has shown evidence for the existence of a D_{35} resonance at 1925 MeV. This state was tentatively identified as a member of a previously unseen $SU_6 \otimes O_3$ supermultiplet,[2] the $(56, 1^-)$. Since a low-lying $(56, 1^-)$ multiplet is not easily accommodated by conventional harmonic-oscillator baryon models,[3] a new model was proposed to account for the $(56, 1^-)$ multiplet.

This new phenomenological baryon model was suggested partly by dual resonance and string models.[4] The model depicts ordinary baryons as three light quarks bound together by "strings" which represent gluon fields and nonvalence quarks. The leading term in the potential energy of the baryon system is proportional to the minimum string length needed to connect the three quarks in a given configuration. The three quarks need not lie along a single string; a three-string vertex is allowed in this string-length potential (SLP) model. To represent phenomenologically the kinetic energy and momentum carried by the strings we add a fourth constituent to the baryon: the monad, a colorless, massless, neutral scalar particle. The monad, being colorless, is connected to the three quarks by at least two strings. We refer to the baryon composed of three quarks and a monad bound by SLP interactions as the 4C model, the more conventional baryon composed of three quarks with SLP interactions as the 3C model.

De Rujula, Georgi, and Glashow have used a variant of the 3C model to calculate splittings of the s-wave and p-wave baryon multiplets.[5] Gunion and Willey use a linear potential between a quark and diquark pair to calculate SU_6 baryon multiplet levels.[6] These calculations assume nonrelativistic Hamiltonians involving massive quarks. Chodos and Thorn, Bars and Hansen, and Bardeen *et al.* discuss theoretical aspects of relativistic string models, such as the problem of covariant quantization.[4]

The purpose of this paper is to describe calculational techniques used in the phenomenological SLP models and to discuss the qualitative features of the resulting baryon spectrum. The formalism described here will serve as the groundwork for precise numerical calculations of the SLP baryon spectrum to be reported later.

Several qualitative features of the SLP model are common to the 3C and 4C versions. The well-known Regge result of trajectories rising linearly with the square of the baryon mass results from the choice of our Hamiltonian. The degeneracies of the harmonic-oscillator spectrum[3] are broken in the SLP model. Radially excited states are displaced below orbitally excited states with the same principal quantum number. Orbitally excited multiplets degenerate in the harmonic-oscillator spectrum are themselves split in the SLP model in a characteristic way.

In addition, the SLP models favor a linear configuration of the constituents, with two quarks relatively close together. However, the resulting SLP spectrum does not resemble the minimal spectrum of $SU_6 \otimes O_3$ supermultiplets[7] resulting from quark-diquark interaction models. Only with the addition of explicit exchange forces can the SLP model shift toward the minimal spectrum.

Finally, we wish to determine whether the monad, representing the extra degrees of freedom of the strings, is necessary for agreement with baryon spectroscopy. It was initially thought that a low-lying $(56, 1^-)$ required the monad. The monad could be in an $L = 1$ state relative to the three quarks in a symmetric ground state at relatively low energies, in much the same way that the bag can be in an $L = 1$ state relative to the symmetric three-quark state to give a low-lying $(56, 1^-)$ in the MIT bag model.[8] However, our present calculations show that in the SLP model, at the $n = 3$ level, the $(56, 1^-)$ may be sufficiently low to account for the $D_{35}(1925)$ in a 3C version of the model. Detailed calculations of the splitting at the $n = 3$ level will be necessary to determine whether the additional dynamical degrees of freedom of the

strings are an essential feature of the model.

For numerical calculations of the spectrum in either the 3C or 4C versions of the model, it is convenient to use a set of coordinates similar to those of the harmonic-oscillator model. In addition, harmonic-oscillator wave functions are used as a set of basis functions. We review these elements of the calculations in Sec. II. Hamiltonians in the 3C and 4C versions of the SLP model, along with a description of the calculational techniques, are given in Sec. III. The SLP spectrum as a perturbation of the harmonic-oscillator spectrum is treated in Sec. IV, the effect of exchange forces is discussed in Sec. V, and a more complete discussion of our results is given in Sec. VI.

II. COORDINATES AND BASIS FUNCTIONS

In the 4C model we introduce, in the center-of-mass system, three independent (vector) coordinates (η, ζ, ξ) and the respective conjugate momenta (ϵ, σ, χ). The momentum of the monad is $p_0 = \chi$, and the momenta of the quarks are

$$\begin{aligned} p_1 &= -\tfrac{1}{3}\chi - \tfrac{1}{3}\sqrt{3}\,\sigma + \epsilon, \\ p_2 &= -\tfrac{1}{3}\chi - \tfrac{1}{3}\sqrt{3}\,\sigma - \epsilon, \\ p_3 &= -\tfrac{1}{3}\chi + \tfrac{2}{3}\sqrt{3}\,\sigma. \end{aligned} \tag{1}$$

The displacements of the quarks from the monad are

$$\begin{aligned} r_1 &= \tfrac{1}{2}\eta - \tfrac{1}{6}\sqrt{3}\,\zeta - \xi, \\ r_2 &= \tfrac{1}{2}\eta - \tfrac{1}{6}\sqrt{3}\,\zeta - \xi, \\ r_3 &= \tfrac{1}{3}\sqrt{3}\,\zeta - \xi. \end{aligned} \tag{2}$$

The relative displacements of quarks are $(r_{ij} = r_i - r_j)$

$$\begin{aligned} r_{12} &= \eta, \\ r_{23} &= -\tfrac{1}{2}\eta - \tfrac{1}{2}\sqrt{3}\zeta, \\ r_{31} &= -\tfrac{1}{2}\eta + \tfrac{1}{2}\sqrt{3}\zeta. \end{aligned} \tag{3}$$

In the 3C model, we omit χ from Eqs. (1), and omit Eqs. (2). We also define

$$\rho = (\eta^2 + \zeta^2)^{1/2}, \quad \lambda = (\epsilon^2 + \sigma^2)^{1/2}, \tag{4}$$

which are the magnitudes of conjugate 6-dimensional vectors.

For ease of calculation, and also to simplify comparisons with the standard harmonic-oscillator model, we use harmonic-oscillator wave functions as a basis. The η and ζ oscillators have the same radii, the ξ oscillator has a different radius (these radii might be variationally adjusted).

Horgan[9] has shown how to construct the states of the conventional model, using a "dynamical SU_6" which is the symmetry group of the 6-dimensional harmonic oscillator. This dynamical SU_6 is distinct from the usual SU_6, which is the covering group for spin and flavor transformations. The dynamical SU_6 group includes O_3, the group of rotations in three dimensions, and S_3, the group of permutations of three objects, according to the decomposition $SU_6 \subset O_6 \subset S_3 \otimes O_3$. Table I gives the decomposition of the lowest-energy SU_6 multiplets into O_6, S_3, and O_3 multiplets. The SU_6 quantum number is n, the O_6 quantum number is Λ. In general, $\Lambda = n, n-2, n-4, \ldots$, etc. The S_3 quantum number is $1^{\pm}$ for singlets, 2 for doublets, the $\pm$ referring to parity under permutations. The $(1^+, 1^-, 2)$ S_3 states give rise, respectively, to the (56, 20, 70)-dimensional representations of "flavor-spin" SU_6. The N values in the table describe the degeneracies at each value of n, Λ, L, and S_3. For example, the single state at $n = \Lambda = L = 0$ is the ground-state $(56, 0^+)$ supermultiplet. The six states at $n = \Lambda = L = 1$ correspond to the lowest $(70, 1^-)$ supermultiplet, which transforms like a doublet under S_3 and a triplet under O_3. States with $\Lambda = n-2, n-4, \ldots$ are "radial excitation" states; wave functions with the same Λ and different n differ only in their dependence on the radial variable ρ. In the 4C model, these 6-dimensional harmonic-oscillator states are combined with states of the ξ oscillator. Excitations of the ξ oscillator do not affect the symmetry under quark interchange, but can contribute to the total orbital angular momentum.

Angular wave functions for the $\Lambda = 2$ and $\Lambda = 3$ states (with m_L maximum) are given in Tables II and III. The subscripts on η and ζ in the $L \neq 0$ cases refer to the action on the m_L quantum number: + raises m_L, 0 leaves m_L unchanged. In the case of S_3 doublets two orbital wave functions

TABLE I. Three-quark harmonic-oscillator states.

n	Λ	L	S_3	$N(S_3 \otimes O_3)$	$N(O_6)$	$N(SU_6)$
0	0	0	1^+	1	1	1
1	1	1	2	6	6	6
2	0	0	1^+	1	1	
	2	0	2	2		
		1	1^-	3		
		2	1^+	5		
		2	2	10	20	21
3	1	1	2	6	6	
	3	1	1^+	3		
		1	1^-	3		
		1	2	6		
		2	2	10		
		3	1^+	7		
		3	1^-	7		
		3	2	14	50	56

TABLE II. Orbital wave functions for $\Lambda=2$.

(S_3,L)	$\psi(M=L)$	Norm.
$(2,0)$	$\eta^2-\zeta^2$ $2\zeta\cdot\eta$	4
$(1^-,1)$	$\eta_+\zeta_0-\eta_0\zeta_+$	24
$(1^+,2)$	$\zeta_+^2+\eta_+^2$	12
$(2,2)$	$\eta_+^2-\zeta_+^2$ $2\zeta_+\eta_+$	12

TABLE III. Orbital wave functions for $\Lambda=3$.

(S_3,L)	$\psi(M=L)$	Norm.
$(1^+,1)$	$\zeta_+(\eta^2-\zeta^2)+2\eta_+(\eta\cdot\zeta)$	12
$(1^-,1)$	$\eta_+(\eta^2-\zeta^2)-2\zeta_+(\eta\cdot\zeta)$	12
$(2,1)$	$4\eta_+(\zeta\cdot\eta)+\zeta_+(\zeta^2-3\eta^2)$	6
	$4\zeta_+(\zeta\cdot\eta)+\eta_+(\eta^2-3\zeta^2)$	
$(2,2)$	$\eta_+^2\zeta_0-\eta_0\eta_+\zeta_+$	40
	$\zeta_+^2\eta_0-\zeta_0\zeta_+\eta_+$	
$(1^+,3)$	$3\zeta_+\eta_+^2-\zeta_+^3$	20
$(1^-,3)$	$3\eta_+\zeta_+^2-\eta_+^3$	20
$(2,3)$	$\zeta_+(\zeta_+^2+\eta_+^2)$	60
	$\eta_+(\zeta_+^2+\eta_+^2)$	

are given, corresponding to the two eigenstates of the quark-interchange operator P_{12}. The normalization factors have been chosen to give a unit average over a unit 6-dimensional sphere. The square of the wave function is to be multiplied by the factor listed.

III. KINETIC AND POTENTIAL ENERGIES

For simplicity, we take all the constituents to be massless; the kinetic energy is $\sum|p_i|$. With this assumption, our calculations apply most directly to N and Δ states.

The leading term in the potential energy is assumed to be proportional to the minimum total lengths of "strings" which are needed to join the constituents. At least one string is attached to each quark, at least two are attached to the monad, and a vertex with three intersecting strings is allowed. In the 3C model, the minimum-length configuration has three strings meeting at 120°, provided all interior angles of the quark triangle are less than 120° [Fig. 1(a)]. Otherwise there are two strings meeting at the obtuse vertex [Fig. 1(b)]. To calculate S, we first define

$$A=[\eta^2\zeta^2-(\eta\cdot\zeta)^2]^{1/2} \tag{5}$$

and

$$X=\tfrac{1}{2}\sqrt{3}\,(\rho^2+2A)^{1/2}. \tag{6}$$

Then S is the largest of the four quantities

$$\begin{aligned} &S_{123}=X,\\ &S_{ij}=\tfrac{2}{3}X+\frac{1}{X}(r_{ij}^2-\tfrac{1}{2}\rho^2). \end{aligned} \tag{7}$$

The expressions on the right-hand side of (7) are the sums of either two or three string segments. The subscripts on the left-hand side indicate which quarks are attached to only one segment. In the 4C model, the minimum-length configuration always has two strings attached to the monad. One of these strings joins a single quark, the ith one, to the monad. The other string is part of a minimum-length string system which joins the monad and the other two quarks [Fig. 1(c) or (d)]. This configuration has a string length S_i; the minimum possible length is

$$S=\min(S_1,S_2,S_3). \tag{8}$$

To calculate S_i, we first define vectors η_i and ζ_i. For $i=3$, for example, we have

$$\begin{aligned} &\eta_3=r_{12}=\eta,\\ &\zeta_3=-\tfrac{1}{3}\sqrt{3}(r_1+r_2)=\tfrac{1}{3}\zeta+\tfrac{2}{3}\sqrt{3}\xi. \end{aligned} \tag{9}$$

The other vectors are obtained by interchange of the quarks. In analogy with Eqs. (4)–(6) we introduce the quantities

$$\begin{aligned} &\rho_i^2=\eta_i^2+\zeta_i^2,\\ &A_i^2=\eta_i^2\zeta_i^2-(\eta_i\cdot\zeta_i)^2,\\ &X_i^2=\tfrac{3}{4}(\rho_i^2+2A_i). \end{aligned} \tag{10}$$

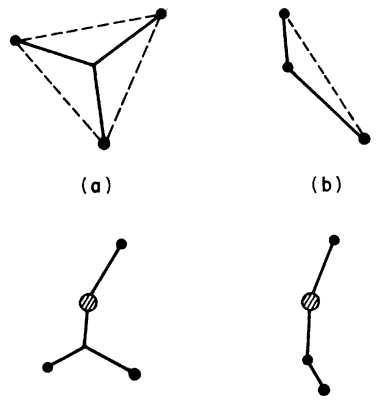

FIG. 1. Minimum-string-length configurations in the 3C model [(a) and (b)], and in the 4C model [(c) and (d)]. Solid lines represent the gluon "strings."

Using Eqs. (7), but with η and ζ replaced throughout by η_i and ζ_i, we have

$$S_i = r_i + \max[s_{123}(i), s_{12}(i), s_{13}(i), s_{23}(i)]. \quad (11)$$

The SLP given by Eq. (7) or Eqs. (8), (11) is not convenient for explicit calculations because it is nonanalytic; different expressions hold in different configurations. For explicit calculations we replace it by a sum of analytic terms which are homogeneous (of degree one) in the coordinates. The terms are required to be simple enough that their matrix elements in the harmonic-oscillator basis are easy to calculate. In the 3C model we may include the following quantities:

$$\rho, \quad b = \sum |r_{ij}|, \qquad a = A^2/\rho^3. \quad (12)$$

Note that b is the perimeter of the quark triangle, and that a, for fixed ρ, is proportional to the squared area of the quark triangle. In the 4C model we may include also the quantities

$$\xi, \quad h = \sum |r_i|,$$
$$f = [(\xi \circ \zeta)^2 + (\xi \cdot \eta)^2 - \tfrac{1}{3}\xi^2\rho^2]/(\xi^2\rho), \quad (13)$$
$$g = [2\xi \circ \eta\zeta \cdot \eta + \xi \circ \zeta(\eta^2 - \zeta^2)]/(\xi\rho^2).$$

The last two terms (f and g) would contribute only to matrix elements in which the ξ oscillator does not remain in its ground state.

A characteristic of the string potential is that it favors positions in which the constituents lie along a line, so that a is small. In fact, the string length S in the 3C model is always greater than $\frac{1}{2}b$, unless $a = 0$. This is illustrated in Table IV for isosceles-triangle configurations ($\eta \circ \zeta = 0$). Thus, in the 3C model we might approximate S by $\frac{1}{2}b$, provided we add a term proportional to a. Similar qualitative considerations apply to the 4C model, although numerical values would differ.

In addition to the dominant SLP term, the potential energy might contain terms with different quark-exchange properties as well as terms with a different dependence on the distances between constituents.

TABLE IV. String length S versus half-perimeter $\frac{1}{2}b$ for isosceles triangles with $\rho = 1$.

	S	$\frac{1}{2}b$
$\eta \approx 0$	$\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{3}\,\eta - \frac{1}{4}\sqrt{3}\,\eta^2 + \cdots$	$\frac{1}{2}\sqrt{3} + \frac{1}{2}\eta - \frac{1}{6}\sqrt{3}\,\eta^2$
$\eta \approx 1$	$1 + 2(1-\eta) + \cdots$	$1 + \frac{1}{2}(1-\eta)$
$\eta = \frac{1}{2}\sqrt{2}$ (equilateral)	$(\frac{3}{2})^{1/2}$	$\frac{3}{4}\sqrt{2}$
$\eta = \frac{3}{10}\sqrt{10}$ (120°)	$(\frac{6}{5})^{1/2}$	$(\frac{1}{2} + \frac{1}{4}\sqrt{3})\,(\frac{6}{5})^{1/2}$

IV. PERTURBATION OF HARMONIC-OSCILLATOR STATES

We treat the replacement of the conventional harmonic-oscillator Hamiltonian by the Hamiltonian for a string model with relativistic constituents as a perturbation which shifts the energy levels of the conventional model. (In the 4C model, the ξ oscillator is taken to be in its ground state.) The most significant effect of this perturbation is to strongly depress the energies of radial excitation states ($\Lambda < n$) below the energies of orbital excitation states ($\Lambda = n$). When ρ or λ is such that energy terms linear in displacements or momenta exceed terms which are quadratic, we observe that the radial wave functions for $K = 1$ have nodes, while those for $K = 0$ have maxima, where $2K = n - \Lambda$. It can also be shown that if the potential and kinetic energies are, respectively, proportional to ρ and λ, the energy levels are proportional to

$$E_{\Lambda,K} = \Lambda + \sqrt{2}K + \text{constant} \quad (14)$$

for large Λ (a similar formula is given in Ref. 1).

Perturbations of the various $S_3 \otimes O_3$ states with the same Λ depend on more complicated properties of the SLP. To discuss these perturbations, we examine the density functions $D_\Lambda(\zeta, \eta)$ which can be constructed from squares of wave functions. Here the subscript Λ refers to the O_6 classification of the density function. These density functions are scalar ($L = 0$) and are symmetric in the quarks (1^+). For $\Lambda \leq 6$ the following density functions occur:

$$D_0 = 1,$$
$$D_4 = 8[\eta^2\zeta^2 - (\eta \cdot \zeta)^2] - (\eta^2 + \zeta^2)^2, \quad (15)$$
$$D_6 = (\zeta^2 - \eta^2)^3 - 12(\eta \cdot \zeta)^2(\zeta^2 - \eta^2).$$

By considering the reduction of $\Lambda \otimes \Lambda$ (in O_6) we see that the first-order splitting of $\Lambda = 2$ states is given entirely by the D_4 density function, while the splitting of $\Lambda = 3$ states depends on both D_4 and D_6. Table V gives the coefficients d_4 and d_6 in the expression for the averaged density (for $\rho = 1$)

$$\sum_i \psi_i^2 = N(1 + d_4 D_4 + d_6 D_6). \quad (16)$$

The sum is extended over all N members of a given $(S_3 \otimes O_3)$ multiplet.

The first-order energy shifts are proportional to the averages of the D_Λ multiplied by the kinetic energy and by the various potential terms from Eqs. (12), (13). The first term in D_4 is proportional to a. Thus, on averaging over O_6 rotations, $\langle aD_4 \rangle > 0$. Next, consider the effect of a term in

TABLE V. Coefficients of $(1^+,0)$ densities.

Λ	(S_3,O_3)	d_4	d_6
2	(2,0)	−1	
	$(1^+,2)$	$-\frac{3}{5}$	
	(2,2)	$\frac{1}{5}$	
	$(1^-,1)$	1	
3	$(1^+,1)$	−1	2
	$(1^-,1)$	−1	−2
	(2,3)	$-\frac{5}{7}$	0
	(2,1)	0	0
	$(1^+,3)$	$\frac{3}{7}$	$\frac{4}{7}$
	$(1^-,3)$	$\frac{3}{7}$	$-\frac{4}{7}$
	(2,2)	1	0

the potential proportional to b. Using quark-interchange symmetry, it is sufficient to consider (for $\rho=1$)

$$\langle r_{12}D_4\rangle=\langle \eta D_4\rangle=\tfrac{1}{63}. \tag{17}$$

For the kinetic energy, we calculate in momentum space (for $\lambda=1$) using $D_4(\sigma,\epsilon)$. In the 3C model we have

$$\tfrac{1}{2}\sqrt{3}\langle p_3 D_4\rangle=\langle \sigma D_4\rangle=\tfrac{1}{63}. \tag{18}$$

In the 4C model the numbers are somewhat different, but the signs remain the same: Both potential and kinetic energies give positive contributions to an average with D_4.

Since a is a linear combination of D_0 and D_4,

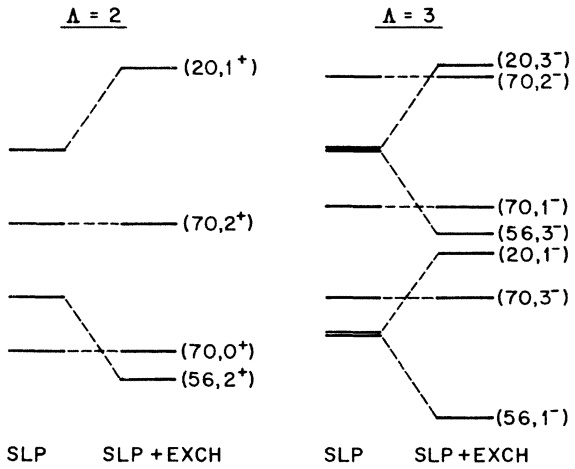

FIG. 2. Energy levels for $\Lambda=2$ and $\Lambda=3$ states from the string-length potential and as they might be altered by a simple exchange potential. The energy scales for $\Lambda=2$ and $\Lambda=3$ are not related, and the magnitude of the splitting due to exchange forces is arbitrary.

$\langle aD_6\rangle\equiv 0$. The averages of η and σ times D_6 are

$$\langle \eta D_6(\zeta,\eta)\rangle=-\tfrac{1}{231}\,,\qquad \langle \sigma D_6(\sigma,\epsilon)\rangle=+\tfrac{1}{231}\,. \tag{19}$$

The opposite signs in (12) occur because D_6 changes sign when η and ζ are interchanged. Since averages with D_6 are small and since there is a cancellation between potential and kinetic energy terms, the splitting of $n=3$ states with symmetry 1^+ and 1^- produced by the D_6 term are expected to be very small.

For both $\Lambda=2$ and $\Lambda=3$ states, therefore, the influence of the D_4 density function will be dominant, and the direction and relative magnitudes of the first-order energy shifts are given by the coefficients d_4 from Table V. The resulting spectra are shown in Fig. 2. In particular, note that for $\Lambda=2$ the (2,0) state has the lowest energy, followed by the $(1^+,2)$ state. This means that among orbital excitations at the $n=2$ level, the $(70,0^+)$ lies lowest, followed by the $(56,2^+)$. For $\Lambda=3$ the $(1^\pm,1)$ states are lowest followed by the (2,3), so the $(56,1^-)$, $(20,1^-)$, and then the $(70,3^-)$ are the lowest orbital excitations at the $n=3$ level.

V. EXCHANGE FORCES

The inclusion of additional potential energy terms containing various quark-interchange operators may be required in order to provide agreement with empirical baryon spectroscopy. For completeness, we discuss here the additional density functions required for calculating the influence of exchange operators. A quark-exchange operator gives an extra factor (±1) when applied to a state with symmetry $(1^\pm)$. For these states the D_Λ density functions are sufficient, provided the sign factor is taken into account. For doublets under S_3, additional density functions transforming as (2,0) and as $(1^-,0)$ under $S_3\otimes O_3$ are required.

For $\Lambda\leq 6$ we define the following (2,0) density functions $E_\Lambda(\zeta,\eta)$:

$$E_2=\begin{pmatrix}\zeta^2-\eta^2\\ -2\zeta\cdot\eta\end{pmatrix},$$

$$E_4=\begin{pmatrix}(\zeta^2-\eta^2)^2-4(\zeta\cdot\eta)^2\\ 4(\zeta^2-\eta^2)(\zeta\cdot\eta)\end{pmatrix}, \tag{20}$$

$$E_6=[\zeta^4-10\zeta^2\eta^2+\eta^4+12(\zeta\cdot\eta)^2]\begin{pmatrix}\zeta^2-\eta^2\\ -2\zeta\cdot\eta\end{pmatrix}.$$

In analogy with the definition of the coefficients d_Λ in Eq. (16), we define coefficients e_Λ by

$$\sum_i \psi_i P_{12}\psi_i=N\sum_\Lambda e_\Lambda E^u_\Lambda, \tag{21}$$

where E^u_Λ is the upper component in (20). The e_Λ

TABLE VI. Coefficients of (2,0) densitities.

Λ	(S_3, O_3)	e_2	e_4	e_6
1	(2,1)	1		
2	(2,0)	0	2	
	(2,2)	0	$\frac{4}{5}$	
3	(2,1)	$-\frac{1}{3}$	0	$\frac{4}{3}$
	(2,2)	$-\frac{1}{6}$	0	$\frac{1}{6}$
	(2,3)	$\frac{4}{3}$	0	$\frac{8}{21}$

are listed in Table VI. Densities involving other elements of S_3 can be obtained by considering the transformation properties of the E_Λ.

There is also a $(1^-,0)$ density function for $\Lambda = 6$:

$$F_6 = \zeta \circ \eta[6(\eta^2 - \xi^2)^2 - 8(\zeta \cdot \eta)^2]. \quad (22)$$

However, F_6 does not occur in any diagonal matrix element involving $\Lambda = 2$ or $\Lambda = 3$ wave functions.

The pattern of first-order energy shifts produced by exchange forces would depend in a very complicated way on the manner in which the exchange forces involve the quark coordinates. This is indicated by the complicated pattern of the coefficients d_Λ and e_Λ given in Tables V and VI. However, a simple statement can be made about an exchange potential of the form

$$V_{\text{exch}} = (P_{12} + P_{23} + P_{31})v(\rho) \quad (23)$$

because all density functions D_Λ or E_Λ with $\Lambda \geq 2$ vanish on averaging over a 6-dimensional sphere of radius ρ. The potential (23) therefore does not shift any states of symmetry (2), while states of symmetry (1^+) and (1^-) within a given O_6 multiplet are shifted by equal amounts but in opposite directions. This is indicated schematically in Fig. 2.

VI. DISCUSSION

The Hamiltonian for relativistic constituents interacting via the SLP leads to several characteristic alterations of the three-quark harmonic-oscillator spectrum. First, radially excited states are displaced below orbitally excited states. The orbitally excited multiplets are also split in a characteristic way determined by their O_6 transformation properties.

As noted in Ref. 1, the string-length potential favors a linear configuration for the constituents, with two quarks always relatively close together. It was conjectures on this basis that the minimal quark-diquark spectrum[7] would be favored. This minimal spectrum contains only $(56, L^+)_{L=\text{even}}$ and $(70, L^-)_{L=\text{odd}}$ multiplets of $SU_6 \otimes O_3$. The calculations presented in this paper show that the kinematical diquark effect from the string-length potential is not sufficient to give the minimal spectrum. In fact, among $\Lambda = 2$, states, the $(70, 0^+)$ multiplet is lowest, with the $(56, 2^+)$ multiplet next. Among $\Lambda = 3$ states, the $(56, 1^-)$ and $(20, 1^-)$ multiplets are lowest (and nearly degenerate), followed by the $(70, 3^-)$ multiplet.

To reproduce the minimal spectrum one would need to add explicit exchange forces having a complicated dependence on the quark coordinates. A simple exchange force such as Eq. (23) could lower the $(56, 2^+)$ energy below the $(70, 0^+)$ energy. It would also raise the $(20, 1^\pm)$ energies, but would leave the $(56, 1^-)$ multiplet well below the $(70, 3^-)$ multiplet. A limit to the strength of such a force is provided by the $(70, 1^-)$ multiplet ($\Lambda = 1$), which is observed to lie below the $(56, 2^+)$ multiplet.

In Ref. 1 evidence was put forth for the existence of a $(56, 1^-)$ multiplet with about the same energy as the $(56, 2^+)$ multiplet. The 4C model was proposed as a natural way to obtain such a relatively low-lying $(56, 1^-)$ (in the notation of this paper, as an excitation of the degrees of freedom associated with the variable ξ). The qualitative calculations presented in this paper show that the 3C model is also able to give a $(56, 1^-)$ multiplet which lies below the $(70, 3^-)$ multiplet. It is noteworthy that perturbation terms for the conventional harmonic-oscillator model, as empirically determined by Horgan,[10] lead to a similar order for the $\Lambda = 3$ states. Detailed numerical calculations will be needed to determine whether the observed $(56, 1^-)$ can be accommodated by the 3C model, or whether the extra degrees of freedom of the 4C model are required.

In either the 3C or 4C model, we find that $(70, 0^+)$ and $(70, 2^+)$ multiplets are expected at energies close to that of the $(56, 2^+)$ multiplet, with the $(70, 2^+)$ somewhat higher than the $(70, 0^+)$. There is no evidence for the existence of $(70, 0^+)$ states, and only very meager evidence for existence of a state attributable to the $(70, 2^+)$ multiplet.[11] The question of these multiplets' existence, and their energies if they do exist, are important outstanding problems of empirical baryon spectroscopy.

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