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Connection between fixed-angle and fixed-momentum-transfer scattering in the framework of the double phase representation

G. Grunberg

Département de Physique Mathématique, *

Université des Sciences et Techniques du Languedoc, 34060 Montpellier Cedex, France

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The double phase representation of Sugawara and Nambu, which assumes some simple analytical properties with respect to momentum transfer for the phase of the amplitude, is applied to the discussion of large-angle scattering. From the assumptions of Regge behavior and finite asymptotic Regge trajectories, one derives the fixed-angle scaling laws $F(s,\theta)\simeq s^{\Delta}F(\theta)$, and the extrapolation to the large-angle region of the Regge formula; the high-momentum-transfer behavior of Regge residues is thereby obtained. As a byproduct, we get the general form of the amplitude (including its phase) in the large-angle region for given values of Δ and of the three asymptotic leading Regge trajectories in the s, t, and u channels. The angular dependence of various scattering processes is determined, using as input the values of Δ and of the asymptotic trajectories given by the constituent-interchange model. A strong correlation between the forward-backward asymmetry of the angular distribution in a given channel and the ratio of the 90° cross sections in the other two channels is shown to be present in a simple case. Applied to $\pi^0\pi^0$ scattering the double phase representation shows, together with positivity and the Froissart bound, that the angular distribution can take only two possible forms for each choice of the fixed-angle power Δ; in this case one also finds, using the Kinoshita-Loeffel-Martin upper bound, that the asymptotic Pomeron trajectory is bounded from above either by 0 or by 2/3.

I. INTRODUCTION

Experiments have shown that large-momentum-transfer exclusive scattering processes obey fixed-angle scaling laws of the form (modulo possible $\ln s$ factors)

$$\frac{d\sigma}{dt}(AB\to CD)\underset{s\to\infty}{\sim} s^{-R}f(\theta)\,,$$

where R is independent of θ. This behavior has been related[1] to the finitely composite nature of hadrons, and has led to the conjecture that Regge trajectories may be asymptotically finite, in contrast with the situation generally assumed in dual models with no fundamental constituents: infinitely raising trajectories and exponential energy dependence at fixed angle. A correlation between the asymptotic behavior of Regge trajectories and the fixed-angle high-energy behavior is of course expected on an intuitive ground: Continuity with the Regge regime seems to imply that fixed-angle scaling laws are consistent only with finite asymptotic Regge trajectories. In fact, a general scheme for deriving the properties of Regge trajectories and residues for large values of the momentum transfer from the knowledge of the amplitude in the fixed-angle regime has been established in Ref. 2, and applied to the constituent-interchange model (CIM). The main purpose of this paper is to show that it is also possible to proceed in the reverse direction in a somewhat model-independent way, i.e., to derive fixed-angle scaling laws from the assumption of finite asymptotic Regge trajectories. To achieve this, we will rely on the Mandelstam analyticity of the amplitude, implemented, however, in the more restrictive form of the double phase representation (DPR) of Sugawara and Nambu.[3] The DPR assumes some simple analytical properties with respect to momentum transfer for the phase of the amplitude, and says essentially that the amplitude can be factorized as the product of a polynomial times a function which contains all the Mandelstam cuts, but has no zeros.

As a by-product of our method, we obtain the general form of the amplitude (including its phase) at large energy and fixed angle, for given values of the fixed-angle power R and of the three leading asymptotic trajectories in the s, t, and u channels. We can thus strongly constrain the angular dependence $f(\theta)$, much in the spirit of the works of Uematsu[4] and Pire.[5]

Further, the DPR leads to an interesting connection between the fixed-angle high-energy behavior and the zeros of the amplitude: One shows that the high-energy fixed-angle amplitude is completely determined, apart from normalization, by the knowledge of the polynomial which takes care of these zeros in the DPR, and that of the leading asymptotic trajectories in the s, t, and u channels. This fact, together with positivity and the Froissart bound, severely restricts the possible forms of the fixed-angle amplitude for $\pi^0\pi^0$ elastic scattering. Essentially, the angular dependence can take in this case only two forms for each choice of R. Moreover, by requiring consistency with the Kinoshita-Loeffel-Martin upper bound[6] for fixed-angle scattering, we find that the asymptotic value of the effective Pomeron trajectory in $\pi^0\pi^0$ scattering cannot be larger than $\frac{2}{3}$. If the intercept of the Pomeron is assumed to be at unity, this result implies, without using analyticity in the angular momentum plane, that the Pomeron cannot be a fixed pole for $\pi^0\pi^0$ scattering.

We are also able to establish in the present framework the asymptotic behavior of Regge residues. Here the basic result is that the fixed-angle amplitude can in general be obtained as the large-momentum-transfer extrapolation of the leading Regge-pole contribution in a given channel (eventually supplemented or replaced by a few "daughters" as defined in Sec. IV). This is similar to what has been found in the CIM,[2,7] with some differences mentioned in Sec. IV and further discussed in the conclusion (Sec. VI).

This paper is organized as follows: The DPR is described in Sec. II. The fixed-angle asymptotic behavior is discussed in Sec. III (an outline of the derivation in a simple case is given in the Appendix), and the extrapolation from the Regge to the fixed-angle regime is considered in Sec. IV. Applications are given in Sec. V: Some Pomeranchuk-type theorems for fixed-angle scattering are discussed in Sec. V A; $\pi^0\pi^0$ scattering is treated in Sec. V B; the angular dependence of some other meson-meson and meson-baryon scattering processes is derived in Sec. V C, taking for the asymptotic trajectories and Δ the values obtained in the CIM.

II. THE DOUBLE PHASE REPRESENTATION

To simplify the notation, we will restrict ourselves to amplitudes with no poles (whose inclusion is straightforward). The DPR of Sugawara and Nambu[3] is essentially a double dispersion relation for the logarithm of the amplitude. It is an extension of the single-phase representation of Sugawara and Tubis,[8] which takes into account the analyticity of the amplitude with respect to momentum transfer. It can be given[9] the explicitly crossing-symmetric form

$$\frac{F(s,t,u)}{F(0,0,a)}=P(s,t,u)\times Q(s,t,u), \tag{1a}$$

where $P(s,t,u)$ is a polynomial in s,t,u with real coefficients which contains all the zeros of F, and

$$\begin{aligned}Q(s,t,u)=\exp\Bigg[&\frac{s}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'}\,\frac{\rho(s')}{s'-s}+\frac{t}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'}\,\frac{\rho(t')}{t'-t}+\frac{u-a}{\pi}\int_{u_0}^{\infty}\frac{du'}{u'-a}\,\frac{\rho(u')}{u'-u}\\&+\frac{st}{\pi^2}\int_{s_0}^{\infty}\int_{t_0}^{\infty}\frac{ds'}{s'}\,\frac{dt'}{t'}\,\frac{\rho(s',t')}{(s'-s)(t'-t)}+\frac{t(u-a)}{\pi^2}\int_{t_0}^{\infty}\int_{u_0}^{\infty}\frac{dt'}{t'}\,\frac{du'}{u'-a}\,\frac{\rho(t',u')}{(t'-t)(u'-u)}\\&+\frac{(u-a)s}{\pi^2}\int_{u_0}^{\infty}\int_{s_0}^{\infty}\frac{du'}{u'-a}\,\frac{ds'}{s'}\,\frac{\rho(u',s')}{(u'-u)(s'-s)}\Bigg].\end{aligned} \tag{1b}$$

In Eq. (1), F is the invariant amplitude which satisfies the usual Mandelstam representation, s,t,u are the Mandelstam variables with $s+t+u=a$, where a is a constant, and we have subtracted at $s=0$, $t=0$, $u=a$, assuming for simplicity $F(0,0,a)\neq 0$. For ease of notation, we have distinguished different functions by the different names of their variables: For instance, $\rho(s')$ and $\rho(t')$ are in general different functions. The DPR (1) can be cast into the form of the single-phase representation at fixed t,

$$F(s,t,u)=\left[\sum_{n=0}^{N}b_n(t)\left(\frac{s}{s_0}\right)^n\right]\exp\left[\frac{s}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'}\,\frac{\delta_t(s')}{s'-s}+\frac{s}{\pi}\int_{u_0}^{\infty}\frac{du'}{a-t-u'}\,\frac{\delta_t(u')}{u'-u}\right], \tag{2a}$$

where we set

$$F(0,0,a)P(s,t,u)Q(0,t,a-t)\equiv\sum_{n=0}^{N}b_n(t)\left(\frac{s}{s_0}\right)^n .$$

$\delta_t(s)$ and $\delta_t(u)$ are respectively the s- and u-channel phases at fixed t, and satisfy dispersion relations in momentum transfer such as

$$\delta_t(s)=\delta_{t=0}(s)+\frac{t}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'}\,\frac{\rho(s,t')}{t'-t}+\frac{t}{\pi}\int_{u_0}^{\infty}\frac{du'}{a-u'-s}\,\frac{\rho(u',s)}{u'-u}\,. \tag{2b}$$

Equation (2b) identifies as usual the double spectral function $\rho(s,t)$ as the discontinuity of the s-channel phase with respect to t for $t>t_0$, and a similar interpretation holds for $\rho(u,s)$ and $\rho(t,u)$. The single spectral functions in (1b) are related to the phases and the double spectral functions through the relations[9]

$$\begin{aligned}\rho(s')&=\delta_{t=0}(s')-\frac{s'}{\pi}\int_{u_0}^{\infty}\frac{du'}{u'-a}\,\frac{\rho(u',s')}{t'}\\&=\delta_{u=a}(s')-\frac{s'}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'}\,\frac{\rho(s',t')}{u'-a}\end{aligned} \tag{3}$$

and similar relations for $\rho(t'), \rho(u')$. To derive the second equality in (3), one uses the fact that $\delta_t(s)$ and $\delta_u(s)$ are the same s-channel phase, but expressed with different variables, i.e., $\delta_u(s) = \delta_{t=a-u-s}(s)$.

The conditions for the validity of the DPR are essentially[3,9]:

(i) that single-phase representations of the form (2a) exist at fixed s, t, and u, and

(ii) that the phases in the s, t, and u channels have the simple analytical properties and rate of growth at infinity with respect to momentum transfer expressed by Eq. (2b) and its analogs. We further assume that the double spectral functions in (1b) are bounded uniformly by constants at infinity, which still allows a logarithmic increase of the phases when their momentum-transfer variable tends to infinity.

The main consequence of (i) and (ii) is that all the zeros of $F(s,t,u)$ are contained in the polynomial $P(s,t,u)$. Concerning condition (i), we merely recall here that it is implied[8] by Regge behavior at fixed s, t, and u. Concerning condition (ii), we remark that it is a stronger requirement than the Mandelstam analyticity of the amplitude. The main reason is that the ratio (deleting the explicit u dependence)

$$\frac{F(s+i\epsilon,t)}{F(s-i\epsilon,t)} = e^{2i\delta_t(s)} \quad (s>s_0)$$

may well vanish (or become infinite) for s fixed above threshold for some values of t [lying of course outside the region where $F(s,t)$ is a real analytic function of s], which will introduce extra branch points in $\delta_t(s)$. The hypothesis that this ratio stays finite for all t's appears to be much more restrictive than the Mandelstam representation, and one can check that it excludes such simple functions as

$$F(s,t) = a_1(-s)^{\alpha_1}(-t)^{\beta_1} + a_2(-s)^{\alpha_2}(-t)^{\beta_2},$$

where a_1, a_2 are real constants, $\alpha_1+\beta_1=\alpha_2+\beta_2$ and $\alpha_2-\alpha_1 \neq$ integer.

III. REGGE BEHAVIOR AND FIXED -ANGLE SCATTERING

We now restrict ourselves to amplitudes with Regge-pole behavior at fixed s, t, and u. It is well known[8] that Regge behavior implies that the phases have definite limits at infinite energy when their momentum transfer variable is kept fixed. More precisely, let us call $\beta(t)$ the t-channel Regge trajectory; we have[8] in an obvious notation

$$\beta(t) = N - \frac{1}{\pi}[\delta_t(u=\infty) + \delta_t(s=\infty)], \tag{4}$$

where N is the number of zeros in the fixed-t amplitude. If the Regge trajectories are asymptotically finite, i.e., $\beta(\infty) \equiv \lim_{t\to-\infty}\beta(t) < \infty$, Eq. (4) then suggests that $\lim_{t\to-\infty}\delta_t(s=\infty)<\infty$ and $\lim_{t\to-\infty}\delta_t(u=\infty)<\infty$. (In fact, these two conditions can be obtained, under physical reasonable assumptions, as a consequence of Regge behavior and the DPR[9]: The finiteness of asymptotic Regge trajectories thus hardly appears as an independent hypothesis here.) One can then derive[9] from these conditions the following result, relevant to the large-angle behavior:

$$Q(s,t,u) \underset{|s|,|t|,|u|\to\infty}{\sim} \text{const}\times\left(-\frac{s}{s_0}\right)^{-(1/\pi)\delta(s=\infty)}\left(-\frac{t}{t_0}\right)^{-(1/\pi)\delta(t=\infty)}\left(-\frac{u}{u_0}\right)^{-(1/\pi)\delta(u=\infty)}[1+o(1)] \tag{5}$$

with

$$\delta(s=\infty) \equiv \lim_{t\to-\infty}\delta_t(s=\infty) = \lim_{u\to-\infty}\delta_u(s=\infty) \tag{6}$$

and similar relations for $\delta(t=\infty)$, $\delta(u=\infty)$. The argument leading to this result, although not completely rigorous, is rather simple, and its main features are most clearly illustrated in the simpler case where the amplitude has no u cut, which is discussed in the Appendix. A treatment of the general case will be given in Ref. 9.

We note that (5) gives the fixed-angle behavior when there are no zeros. Otherwise, we have to multiply (5) by the asymptotically leading part of the polynomial $P(s,t,u)$, i.e., by the terms of maximal degree in $P(s,t,u)$ considered as a function of two independent variables. When this is done[9] one gets the result

$$F(s,t,u) \underset{|s|,|t|,|u|\to\infty}{\sim} \left(-\frac{u}{u_0}\right)^{\Delta-\gamma(\infty)}\left[\sum_{q=0}^{q_{\max}} c_q\left(-\frac{s}{s_0}\right)^{\Delta-\alpha(\infty)+q}\left(-\frac{t}{t_0}\right)^{\alpha(\infty)+\gamma(\infty)-\Delta-q}\right][1+o(1)], \tag{7}$$

where

$$q_{\max} = \alpha(\infty)+\beta(\infty)+\gamma(\infty)-2\Delta \tag{8}$$

and the coefficients c_q are real. Two other equivalent forms can be obtained by circular permutation on s,t,u and $\alpha(\infty)$, $\beta(\infty)$, and $\gamma(\infty)$ (but with different sets of coefficients). $\alpha(\infty)$, $\beta(\infty)$, and $\gamma(\infty)$ are the asymptotic Regge trajectories in the s, t, and u channels At fixed angle, we obviously get in any channel a fixed-angle scaling law of the form

$$F(s,\theta) \underset{s\to\infty}{\sim} s^{\Delta}F(\theta), \tag{9}$$

which gives the interpretation of Δ as the fixed-angle power of the amplitude. We note that Δ is a constant, independent of θ and of the considered channel, and that (8) implies the bound

$$2\Delta \leq \alpha(\infty)+\beta(\infty)+\gamma(\infty). \tag{10}$$

The angular dependence of the amplitude is given by $F(\theta)$, which is channel-dependent. Setting $t/s=-\frac{1}{2}(1-z)$, $u/s=\frac{1}{2}(1+z)$, with $z=\cos\theta$, we obtain for instance from (7) the angular dependence in the s channel as

$$F_s(z)=e^{-i\pi(\Delta-\alpha(\infty))}\left(\frac{1+z}{2}\right)^{\Delta-\gamma(\infty)}\left(\frac{1-z}{2}\right)^{\Delta-\beta(\infty)}P_s(z), \tag{11}$$

where $P_s(z)$ is a polynomial of z of maximal degree q_{max}, with real coefficients [which implies that the phase of the amplitude is given by $-\pi(\Delta-\alpha(\infty))$]. If Δ is smaller than $\beta(\infty)$ and $\gamma(\infty)$, the angular distribution exhibits in general forward and backward peaks [this may not be true if $P_s(z)$ has a zero at $z=\pm 1$]; then it is possible in principle to determine $\Delta-\beta(\infty)$ and $\Delta-\gamma(\infty)$ by looking at the tail of the angular distribution in the forward and backward directions.

We end this section with a few miscellaneous remarks:

(a) An interesting case is obtained when $2\Delta=\alpha(\infty)+\beta(\infty)+\gamma(\infty)$ (which includes the case where the amplitude has no zeros). The amplitude is then completely determined (except for normalization) and is given by

$$F(s,t,u) \underset{|s|,|t|,|u|\to\infty}{\sim} \text{const}\times\left(-\frac{s}{s_0}\right)^{\Delta-\alpha(\infty)} \times\left(-\frac{t}{t_0}\right)^{\Delta-\beta(\infty)}\left(-\frac{u}{u_0}\right)^{\Delta-\gamma(\infty)} \times[1+o(1)], \tag{12}$$

where the normalization constant is real.

(b) We note that $\alpha(\infty)$, $\beta(\infty)$, $\gamma(\infty)$, and Δ need not be integers in the present scheme—this must be the case only for their combination q_{max} [see (8)]. However, if we require the amplitude to be real at large angle in the three channels, then these four parameters must be integers. It is interesting to remark that Uematsu's results[4] concerning the angular dependence of the amplitude for a given negative integer value of Δ turn out to be equivalent to Eq. (7) with $\alpha(\infty)=\beta(\infty)=\gamma(\infty)=-1$, although they were obtained by a different method, which does not assume the DPR. However, Eq. (7), which is stated in a more compact form, taking into account that only two among the three variables s, t, and u are independent, shows that the amplitude really depends only upon $q_{max}+1=-2-2\Delta$ independent combinations of the $-3-3\Delta$ parameters appearing in Ref. 4.

(c) For $\gamma(\infty)=\Delta$, (7) becomes

$$F(s,t) \underset{|s|,|t|\to\infty}{\sim} \left[\sum_{q=0}^{q_{max}} c_q\left(-\frac{s}{s_0}\right)^{\Delta-\alpha(\infty)+q}\left(-\frac{t}{t_0}\right)^{\alpha(\infty)-q}\right] \times[1+o(1)]. \tag{13}$$

One can check that this is exactly the result one would get under the stronger assumption that the u-channel discontinuity vanishes ("exotic" u channel); in the latter case one expects in general $\gamma(u)=\gamma(\infty)=\Delta$ for any u, since backward scattering then appears as a special case of fixed-angle scattering for $\theta=\pi$ (such a behavior has been first conjectured in the CIM[7]). The condition $\gamma(\infty)=\Delta$ could thus give a more general definition of exoticity in the u channel for large-momentum-transfer scattering, compatible with a nonvanishing u-channel discontinuity for finite u. In particular, $\gamma(\infty)=\Delta$ implies no backward peak, as can be seen from (11).

(d) Finally, we note that the knowledge of the polynomial $P(s,t,u)$ and of the asymptotic Regge trajectories completely determines the fixed-angle amplitude at high energy, apart from normalization. This will be particularly useful for $\pi^0\pi^0$ scattering.

IV. CONNECTION BETWEEN THE REGGE AND THE FIXED-ANGLE REGIMES

It is interesting to note that in the present framework there exists a smooth extrapolation from the Regge to the fixed-angle regime, similar to what has been first observed in the model amplitudes of the CIM[2,7] (although there are some differences which will be mentioned below). The extrapolation procedure takes it simplest form in the case where the u cut is absent and there are no zeros (or more generally $q_{max}=0$). Then we can go to the fixed-angle region in a two-step process[9] [see (13)]: First we take $|s|\gg|t|$, which leads to the Regge form $F(s,t)\simeq b(t)(-s/s_0)^{\beta(t)}$; then we let $|t|\to\infty$ and obtain for the Regge residue $b(t)\simeq\text{const}\times(-t/t_0)^{\alpha(\infty)}$, hence $F(s,t)\simeq\text{const}\times(-t/t_0)^{\alpha(\infty)}\times(-s/s_0)^{\beta(\infty)}$ (one can also check that one gets the correct normalization this way). The situation is in general less simple when there are zeros: Then we usually have several terms in (13), which can be interpreted as the contribution of a superposition of the leading Regge pole $\beta(t)$ and of $q_{\max} = \alpha(\infty)+\beta(\infty)-\Delta$ "daughter" trajectories $\beta(t)-1$, $\beta(t)-2, \ldots, \beta(t)-q_{\max}$. The residues of these secondary poles fall off sufficiently slowly at large momentum transfer to compensate for the lower values of the pole positions, thus allowing them to contribute at fixed angle [if the c_q's do not vanish in (13)]. It is remarkable that the DPR does not allow other kinds of "genuine" secondary poles, other than those particular daughters of the leading one, to contribute to the fixed-angle amplitude. This is rather different from what is obtained in the CIM,[2] and may indicate that the present framework is too restrictive: Indeed in the CIM the fixed-angle amplitude is generally built up from pairs of asymptotically degenerate Regge trajectories, which therefore cannot be "daughters" in our sense.

In the general case where the three cuts are present, a similar extrapolation procedure exists, but one must pay attention to writing the Regge form of the amplitude in the appropriate way. For instance, if the amplitude has no zeros, the t-channel Regge-pole contribution must be written in the multiplicative form

$$b(t)\left(-\frac{s}{s_0}\right)^{\beta_1(t)}\left(-\frac{u}{u_0}\right)^{\beta_2(t)}$$

with $\beta_1(t)=-(1/\pi)\delta_t(s=\infty)$ and $\beta_2(t)=-(1/\pi)\delta_t(u=\infty)$ [one can check from (4), with $N=0$, that the overall power at fixed t is indeed $\beta(t)=\beta_1(t)+\beta_2(t)$]. Instead, the CIM suggests[2] the use of the additive form

$$b_1(t)\left(-\frac{s}{s_0}\right)^{\beta(t)}+b_2(t)\left(-\frac{u}{u_0}\right)^{\beta(t)}.$$

Although these two forms are equivalent in the Regge region, where $u \simeq -s$, they are not in general in the fixed-angle region. We note in this connection that the simplest Regge-pole fit to elastic scattering, including the Pomeron contribution, favors the multiplicative form over the additive one: The amplitude $F(s,t)$ at $t=0$ is proportional to is in this case, which can easily be represented in the multiplicative form as $\sqrt{-s}\times\sqrt{-u}$, but not in the additive one, which would give $(-s)+(-u)=0$ for $s=-u$, and is real anyway, if one uses only poles (this is also the basic reason why the Pomeron cannot be represented by a Veneziano formula,[10] which takes care of signatured Regge poles in the additive form, and should rather correspond to a multiplicative Virasoro amplitude[11] in the framework of dual models). When there are zeros, Eq. (7) can be viewed again as a superposition of the leading t-channel Regge pole and its $q_{\max}$ daughters. In particular, it suggests that the large-t behavior of the leading t-channel Regge-pole residue is given by (provided $c_{q_{\max}} \neq 0$)

$$b(t) \underset{|t|\to\infty}{\sim} \text{const}\times\left(-\frac{t}{t_0}\right)^{\Delta-\beta(\infty)}[1+o(1)]\,. \tag{14}$$

Since Δ is the same in the $s, t,$ and u channels, (14) gives relations of the type mentioned in Ref. 2 between asymptotic Regge poles and residues in these three channels. If the leading t-channel Regge pole has a definite signature, (7) also suggests one must have

$$\frac{\beta(\infty)}{2}+\gamma(\infty)-\Delta=\text{integer or half-integer}$$

[and a similar relation replacing $\gamma(\infty)$ by $\alpha(\infty)$], depending upon whether the signature is even or odd.

V. APPLICATIONS

A. Pomeranchuk-type theorems

It can be immediately realized from the previous results that the ratio of the differential cross sections at fixed angle in two different channels related by crossing tends to a finite value [except eventually in a few exceptional directions which correspond to zeros of the polynomial $P_s(z)$ in (11)], since the fixed-angle power Δ is the same for the s, t, and u channels. In general, one cannot predict this value, which is angular dependent, without knowing the polynomial $P(s,t,u)$ which contains the zeros of the amplitude. However, in the particular case where $2\Delta=\alpha(\infty)+\beta(\infty)+\gamma(\infty)$, there is no arbitrary parameter except an overall normalization constant. Taking for instance the ratio of the moduli of the 90° amplitudes in the s and u channels, one immediately gets from (12)

$$\lim_{s\to\infty}\left|\frac{F(s,\theta_s=90°)}{F(s,\theta_u=90°)}\right|=(\tfrac{1}{2})^{\alpha(\infty)-\gamma(\infty)}\,. \tag{15}$$

As an application of this formula, let us assume that the u channel carries exotic quantum numbers. Then it is natural to expect $\gamma(\infty)<\alpha(\infty)$, and therefore the ratio in (15) to be smaller than 1. One thus expects the 90° cross section in a nonexotic channel to be appreciably smaller than the corresponding cross section in a crossing-related exotic channel. This may explain the large value of the 90° pp elastic cross section, compared to that of the $p\bar{p}$ elastic cross section. More quantitatively, it is easy to check that (12) gives in this case (we neglect spin in the present argument)

$$\left.\frac{d\sigma}{dt}\right|_{pp} \underset{s\to\infty}{\sim} \text{const}\times s^{2\Delta-2}(1-z^2)^{\alpha(\infty)}\,, \tag{16}$$

where $\alpha(\infty)$ is the asymptotic trajectory in the pp channel, and also

$$\left(\frac{d\sigma}{dt}\bigg|_{pp} \Big/ \frac{d\sigma}{dt}\bigg|_{p\bar{p}}\right)_{90^\circ} \underset{s\to\infty}{\simeq} 2^{[2\Delta - 3\alpha(\infty)]}. \tag{17}$$

A good fit to experiment seems to be[1]

$$\frac{d\sigma}{dt}\bigg|_{pp} \underset{s\to\infty}{\simeq} \text{const} \times s^{-12}(1-z^2)^{-6} ,$$

which is also one of the forms predicted by the CIM,[1] and corresponds to $\Delta = -5$, $\alpha(\infty) = -6$. Putting these values in (17) then leads to the prediction

$$\left(\frac{d\sigma}{dt}\bigg|_{pp} \Big/ \frac{d\sigma}{dt}\bigg|_{p\bar{p}}\right)_{90^\circ} \underset{s\to\infty}{\simeq} 2^8 = 256$$

(the experimental value[12] at 5 GeV/c is about 100). The value of this ratio is very sensitive to the input parameters: Had we taken the dimensional-counting result[13] $\Delta = -4$, together with $\alpha(\infty) = -6$, we would have got the value $2^{10} = 1024$ (also predicted in Ref. 14).

A slightly different way to express the content of Eqs. (12) and (15) is to note that they predict a strong correlation between the forward-backward asymmetry of the angular distribution in a given channel and the ratio of the 90° cross sections in the other two channels. Indeed, (12) shows that the angular distribution in the t channel is proportional to $(1-z_t)^{\Delta-\alpha(\infty)}(1+z_t)^{\Delta-\gamma(\infty)}$; a strong forward-backward asymmetry in this channel therefore means a large difference $|\alpha(\infty) - \gamma(\infty)|$, hence a very large (or small) value for the ratio (15). This is what happens in the pp case, since $p\bar{p}$ elastic scattering shows a very sharp peak [which behaves as $(1-z)^{-6}$] in the forward direction, and no peak in the backward direction (at least away from the region where peripheral Regge peaks may dominate). A similar phenomenon occurs in K^-p elastic scattering; hence one could expect a rather large ratio

$$\left(\frac{d\sigma}{dt}\bigg|_{K^+p\to K^+p} \Big/ \frac{d\sigma}{dt}\bigg|_{p\bar{p}\to K^+K^-}\right)_{90^\circ}$$

at sufficiently high energy.

B. $\pi^0\pi^0$ elastic scattering

This case is of particular interest, since it lends itself to a typical application of our results: The high symmetry of this reaction makes it possible to strongly restrict the form of the polynomial $P(s,t,u)$, hence to make some rather model-independent statements about the angular dependence of the amplitude. We note in this connection that the same symmetry makes a straightforward application of the dimensional-counting rules[13] fail: It is easy to check that, with the values $\Delta = -2$, $\alpha(\infty) = \beta(\infty) = \gamma(\infty) = -1$ suggested by dimensional counting and the CIM,[1] (7) cannot lead to a completely s, t, u symmetric amplitude. In fact, an individual (st) interchange graph for meson-meson scattering is expected to give a contribution c/st in the CIM[1]; adding symmetrically the (su) and (tu) contributions then gives

$$F(s,t,u) \simeq \frac{\text{const}}{stu} , \tag{18}$$

which corresponds to $\Delta = -3$. On the other hand, if the Landshoff contributions[15] are important, one expects still another form, namely

$$F(s,t,u) \simeq \frac{ic}{(stu)^{1/2}}$$

and $\Delta = -\frac{3}{2}$. Both of these forms will be obtained as special cases below.

(1) Possible forms of the amplitude. One can show[3] that positivity, the Froissart bound, and the s, t, u symmetry of the $\pi^0\pi^0$ amplitude imply that the most general form of $P(s,t,u)$ is given by

$$P(s,t,u) = c_0 + c_2(s^2+t^2+u^2) + c_3(s^3+t^3+u^3) , \tag{19}$$

where c_0, c_2, and c_3 are real. One can also show[16] that the amplitude has no zeros, i.e., $c_2 = c_3 = 0$, if its S-wave scattering length is negative. Furthermore, putting $\delta(s=\infty) = \delta(t=\infty) = \delta(u=\infty) \equiv \delta$ in (5), we get

$$Q(s,t,u) \sim \text{const} \times (-stu)^{-(1/\pi)\delta} . \tag{20}$$

Combining (19) and (20), we obtain only two possible forms of the fixed-angle amplitude for a given value of the fixed-angle power Δ [note also we have $\alpha(\infty) = \beta(\infty) = \gamma(\infty)$].

(a) If $c_3 \neq 0$, then

$$F(s,t,u) \sim \text{const} \times (-stu)^{\Delta/3} \tag{21a}$$

and

$$2\Delta = 3\alpha(\infty) \tag{21b}$$

(this same form is obtained when the amplitude has no zeros).

(b) If $c_3 = 0$, $c_2 \neq 0$, then

$$F(s,t,u) \sim \text{const} \times (s^2+t^2+u^2)(-stu)^{(\Delta-2)/3} \tag{22a}$$

and

$$2\Delta = 3\alpha(\infty) - 2 . \tag{22b}$$

We note that the Landshoff amplitude[15] corresponds to case (a) with $\alpha(\infty) = -1$, whereas the CIM amplitude (34) corresponds to case (a) with $\alpha(\infty) = -2$.

(2) Upper bound on the asympotic Pomeron tra-

jectory. An interesting result is obtained if one combines the relations (21b) and (22b) with the fixed-angle upper bound of Kinoshita, Loeffel, and Martin[6] (which can be derived from unitarity and the Mandelstam representation)

$$|F(s,\cos\theta)| < \text{const} \times \frac{(\ln s)^{\frac{3}{2}}}{\sin^2\theta}. \tag{23}$$

This bound implies $\Delta \leq 0$ in (21b) and (22b). We therefore get either $\alpha(\infty) \leq 0$ for case (a) or $\alpha(\infty) \leq \frac{2}{3}$ for case (b). In both cases, we must have $\alpha(\infty) \leq \frac{2}{3}$. The possibility $\alpha(\infty) = 1$, sometimes expected in theories with vector-gluon exchange,[1] is therefore excluded in the present scheme. If one further assumes that $\alpha(0) = 1$, this result shows that the Pomeron cannot behave as a fixed pole for $t \leq 0$. We therefore have here an independent argument for the necessity of including some t dependence in the Pomeron trajectory ("shrinking"), at least for $\pi^0\pi^0$ scattering, which is perhaps more direct than Gribov's argument,[17] since it does not make use of the analyticity in the angular momentum plane. Most importantly, the present argument deals with the behavior at large negative t, in contrast with Gribov's argument, which is concerned with the behavior at small positive t.

C. Applications to meson-meson and meson-baryon scattering

The formula (7) can be used to determine the angular dependence of the amplitude for given values of $\alpha(\infty)$, $\beta(\infty)$, $\gamma(\infty)$, and Δ, consistent with the requirement that $\alpha(\infty) + \beta(\infty) + \gamma(\infty) - 2\Delta$ be a positive (or null) integer. In the following, $\alpha(\infty)$, $\beta(\infty)$, and $\gamma(\infty)$ are not necessarily the true leading Regge-pole trajectories, but the "effective" leading trajectories, i.e., the highest-lying trajectories which effectively contribute to the fixed-angle amplitude, and which can well be "daughters" of the leading one (in the sense of Sec. IV) if coefficients such as $c_{q_{max}}$ in (7) vanish; one can check that with this more general interpretation the formula (7) is still correct. We shall consider a few examples, using for the trajectories and Δ the values obtained in the CIM,[7,18] although some features of the CIM are sometimes in conflict with the DPR, as noted in Sec. IV. The apparent inconsistency of this procedure is removed by the observation that the following applications do not really depend on the validity of the DPR. This is due to the circumstance that in the CIM, the asymptotic trajectories take negative integer values, and in this case, as stated in Sec. III, remark (b), the formula (7) is equivalent to Uematsu's relations.[4] The latter are obtained from a $1/s$ expansion in the Mandelstam representation, and do not assume the DPR (however, the DPR may give some justification to this procedure; see the concluding remarks in Sec. VI).

In a recent paper,[5] Pire has derived the angular distribution of some exclusive scattering processes using Uematsu's relations and the values of the trajectories abstracted from the CIM, together with some additional physical assumptions. We show here that the latter are unnecessary, and that Pire's results can be obtained from the first two ingredients only. We also refer to Pire's paper for a comparison between the results obtained in this way and in other models.

(1) $\pi^- K^+$ elastic scattering. In this case, the CIM gives, using for the trajectories the notations of Ref. 7,

$$\Delta = -2, \quad \alpha(\infty) = \alpha_{K^-\pi^-} = -1,$$
$$\beta(\infty) = \alpha_{\pi\pi} = -1, \quad \gamma(\infty) = \alpha_{\pi^- K^+} = -2\,.$$

We note that $\alpha(\infty) + \beta(\infty) + \gamma(\infty) - 2\Delta = 0$; therefore, applying (12), we get

$$F(s,t,u) \sim \frac{\text{const}}{st}, \tag{24}$$

which is the result given by Uematsu[4] and Pire.[5] However, we do not have to use the extra assumption of a vanishing u-channel discontinuity ("exoticity"). The fact that the CIM predicts $\gamma(\infty) = \Delta$ in this case has the same effect, as far as the angular dependence is concerned, as the exoticity assumption (which is therefore redundant), in accordance with remark (c) of Sec. III.

(2) $\pi^{\pm}p$ elastic scattering. Assuming γ_5 invariance at high energies, the spin-average differential cross section for meson-baryon elastic scattering has the form[19]

$$\frac{d\sigma}{dt} \sim \left(-\frac{u}{s}\right)|B|^2, \tag{25}$$

where B is an invariant amplitude which has Mandelstam analyticity. The CIM gives[18]

$$R = 8, \quad \alpha_{\pi^{\pm}p} = -\tfrac{3}{2}, \quad \alpha_{\pi\pi} = -1\,,$$

where we recall that R is defined by $d\sigma/dt \simeq s^{-R} f(\theta)$. The above values of the fermionic trajectories differ by a spin-flip factor of $\frac{1}{2}$ unit [related to the kinematical factor in (25)] from those given in Ref. 7, and agree with those suggested by Pire.[5] We deduce the values of the corresponding parameters for the amplitude B, taking into account the kinematical factor in (25):

$$\Delta = -4, \quad \alpha(\infty) = \alpha_{\pi^- p} - \tfrac{1}{2} = -2,$$
$$\beta(\infty) = \alpha_{\pi\pi} - 1 = -2, \quad \gamma(\infty) = \alpha_{\pi^+ p} - \tfrac{1}{2} = -2\,.$$

We note that $\alpha(\infty) + \beta(\infty) + \gamma(\infty) - 2\Delta = 2$. From (7) we then obtain

$$B(s,t,u) \sim \frac{1}{u^2}\left(\frac{c_0}{s^2} + \frac{c_1}{st} + \frac{c_2}{t^2}\right);$$

hence, by circular permutation on s,t,u we get the equivalent formula

$$B(s,t,u) \simeq \frac{1}{t^2}\left(\frac{a_0}{u^2}+\frac{a_1}{su}+\frac{a_2}{s^2}\right).$$

Since

$$\frac{1}{t^2su} \sim -\frac{1}{t^3s}-\frac{1}{t^3u}$$

for $s,t,u \to \infty$, this gives

$$B(s,t,u) \sim \frac{a_0}{t^2u^2} - a_1\left(\frac{1}{t^3s}+\frac{1}{t^3u}\right)+\frac{a_2}{t^2s^2}, \tag{26}$$

which agrees with Pire's result, but is obtained without any extra dynamical assumption. As noted by Pire,[5] (26) and isospin invariance lead to a unique prediction for the charge-exchange reaction $\pi^- p \to \pi^0 n$.

(3) *$K^\pm p$ elastic scattering*. The CIM gives in this case[18]

$$R=8, \quad \alpha_{K^-p}=-\tfrac{7}{2}, \quad \alpha_{K^+p}=-\tfrac{3}{2}, \quad \alpha_{KK}=-1.$$

We deduce, proceeding as above,

$$\Delta=-4, \quad \alpha(\infty)=\alpha_{K^-p}-\tfrac{1}{2}=-4,$$

$$\beta(\infty)=\alpha_{KK}-1=-2, \quad \gamma(\infty)=\alpha_{K^+p}-\tfrac{1}{2}=-2.$$

Since we have $\alpha(\infty)+\beta(\infty)+\gamma(\infty)-2\Delta=0$, we get from (12)

$$B(s,t,u) \sim \frac{\text{const}}{t^2u^2}, \tag{27}$$

which agrees with Pire's result. Here again we note the relation $\alpha(\infty)=\Delta$, which replaces the exoticity assumption for the s channel.

VI. CONCLUSION

The results of this paper show the usefulness of the DPR for questions dealing with large-angle scattering. The DPR was originally introduced and used by Sugawara and Nambu[3] to discuss the shape of the forward peak at all values of t, assuming that it does not shrink. We have extended their analysis to the case of a shrinking peak (moving Regge pole), and to the study of the scattering amplitude in the fixed-angle, high-energy regime; we thus obtained the extrapolation at large momentum transfer of the Regge formula, together with fixed-angle scaling laws, from the assumption of finite asymptotic Regge trajectories. One can note that the latter assumption is necessary if one requires the amplitude to have Regge behavior and to satisfy the Mandelstam representation with a finite number of subtractions. However, as pointed out in Sec. II, the DPR is more restrictive than the Mandelstam representation plus Regge behavior. One must stress that the Mandelstam representation by itself gives very little information on the large-angle behavior of the amplitude, even if the values of the fixed-angle power and of the leading asymptotic trajectories are given. For instance, amplitudes of the form

$$F(s,t)=\sum_{i=0}^{N} c_i(-s)^{\beta-\epsilon_i}(-t)^{\Delta-\beta+\epsilon_i} \tag{28}$$

with

$$0=\epsilon_0<\epsilon_1<\cdots<\epsilon_i<\cdots<\epsilon_N=\alpha+\beta-\Delta$$

obviously satisfy the Mandelstam representation, a fixed-angle scaling law with the fixed-angle powe Δ, and Regge behavior in the s and t channels with constant leading trajectories α and β, but the number N of terms in the summation above, as well as the values of the ϵ_i (which do not need to be integers), is largely arbitrary. The DPR gives a way to considerably reduce the nature and the number of terms in expansions of the type (28) by relating them to the polynomial of the zeros of the amplitude. We note in this respect that Uematsu[4] was able to achieve a similar goal for Δ an integer, starting from the Mandelstam representation, but assuming that an $1/s$ expansion is possible under the double dispersion integrals; now this appears to be a rather restrictive hypothesis in view of the above example, even when Δ is an integer.

To better understand the restrictive nature of the DPR, and the connection it may have with Uematsu's procedure, we note that an amplitude obtained in superposing several amplitudes, which separately obey the DPR, will not in general satisfy it. For instance, if the superposed amplitudes have the same fixed-angle power, but are characterized by asymptotic trajectories which differ by noninteger values, the resultant amplitude will behave like (28) at large angle (in the case of no u cut), with $\epsilon_i \neq$ integer, a result clearly incompatible with the DPR. Another possibility, which may well occur in the CIM (and explain the discrepancies between this model and the DPR discussed in Sec. IV), is that the amplitude can be split into two (or more) amplitudes, each satisfying the DPR with the same fixed-angle power and asymptotic values of the trajectories, but with distinct trajectories at finite-momentum transfer. The resultant amplitude then possesses asymptotically degenerate trajectories which contribute at fixed angle, a feature again forbidden by the DPR (see Sec. IV), but often realized in the CIM.[2] However, the form of the angular distribution, which depends only on the values of Δ and of the asymptotic trajectories, will be the same in such a case as if the amplitude would satisfy the DPR; indeed, this kind of possibility could give an alternative rationale to Uematsu's relations,[4] which are valid more generally for amplitudes which can be written as a sum of a few DPR amplitudes admitting negative-integer asymptotic trajectories. As another aspect of this lack of stability of the DPR against superposition of amplitudes, we note that an amplitude which satisfies the DPR and has discontinuities in the s, t, and u channels cannot in general be decomposed into a sum of three amplitudes, each satisfying the DPR but having only the (st), (tu), and (us) discontinuities respectively—a similar fact exists in the realm of simple Born dual models, where it is well known that a Virasoro amplitude[11] cannot in general be split as a sum of three (st), (tu), and (us) Veneziano terms.[10]

Finally, we mention that the methods used here can be extended in a simple way[9] to the case of logarithmic Regge trajectories [which correspond to $\lim_{s,t\to\infty}\rho(s,t)\neq 0$] when the amplitude has no u cut. Then one finds that the amplitude at large s,t has the form of the right-hand side of Eq. (13), multiplied by a factor $(-s)^{\beta(t)}$, where

$$\beta(t)\sim\rho\ln(-t)+\text{const}\quad\text{for } t\to-\infty$$

(this is the same factor as in the logarithmic trajectoried dual model of Baker and Coon[20]). At fixed angle, we then get an exponential high-energy behavior in $(-s)^{\rho\ln s+\Delta(\theta)}$, where

$$\Delta(\theta)=\rho\ln(1-\cos\theta)+\Delta(\theta=90°)\,.$$

The effective fixed-angle power thus shows a correlated dependence in s and θ. Note that at fixed u, we get the exponential energy dependence corresponding to $\theta=\pi$: This is the basic reason why unbounded Regge trajectories are not in general compatible with Regge behavior at fixed s, t, and u in the framework of the DPR, as mentioned in Sec. III. If one insists on having Regge behavior at fixed u in this case, one must relax the DPR, and use rather a superposition of DPR amplitudes of the (st), (tu), and (us) type: Regge behavior at fixed u is then implemented by the (tu) and (us) terms (a situation closely similar to that encountered in the Veneziano model[10]).

APPENDIX

We derive here the fixed-angle behavior in the case where the amplitude has no u cut. Then (1b) becomes

$$Q(s,t)=\exp\left(\frac{s}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'}\,\frac{\delta_{t=0}(s')}{s'-s}\right)\times\exp\left(\frac{t}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'}\,\frac{\delta_{s=0}(t')}{t'-t}\right)\exp G(s,t)\,,\quad\text{(A1)}$$

where

$$G(s,t)\equiv\frac{st}{\pi^2}\int_{s_0}^{\infty}\int_{t_0}^{\infty}\frac{ds'}{s'}\,\frac{dt'}{t'}\,\frac{\rho(s',t')}{(s'-s)(t'-t)}\quad\text{(A2)}$$

and (2b) becomes

$$\delta_t(s)=\delta_{t=0}(s)+\frac{t}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'}\,\frac{\rho(s,t')}{t'-t}\,.\quad\text{(A3)}$$

From (A3) we see that Regge behavior at fixed t implies

$$\lim_{s\to\infty}\rho(s,t')\equiv\rho(\infty,t')=\frac{1}{2i}[\delta_{t'+i\epsilon}(\infty)-\delta_{t'-i\epsilon}(\infty)]<\infty\quad\text{(A4)}$$

and

$$\delta_t(s=\infty)=\delta_{t=0}(s=\infty)+\frac{t}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'}\,\frac{\rho(\infty,t')}{t'-t}\,.\quad\text{(A5)}$$

The assumption of finite asymptotic Regge trajectory in the t channel then leads to the relation

$$\delta(s=\infty)\equiv\lim_{t\to-\infty}\delta_t(s=\infty)=\delta_{t=0}(s=\infty)-\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{t'}\,\rho(\infty,t')\,,\quad\text{(A6)}$$

where we have taken the limit under the integral in (A5); this is legitimate when[21] $\rho(\infty,t')$ is bounded as $t'\to\infty$, which is assumed to be the case here [see (A4)]. Equation (A6) implies in particular

$$\int_{t_0}^{\infty}\frac{dt'}{t'}\,\rho(\infty,t')<\infty\,.\quad\text{(A7)}$$

Similarly, we have, from the asymptotic behavior of the s-channel Regge trajectory,

$$\int_{s_0}^{\infty}\frac{ds'}{s'}\,\rho(s',\infty)<\infty\,.\quad\text{(A8)}$$

Furthermore, the hypothesis of Regge-*pole* behavior at fixed t is known[8] to imply the following constraint (which eliminates eventual $\ln s$ factors in the s dependence at fixed t):

$$\int_{s_0}^{\infty}\frac{ds'}{s'}[\delta_t(s')-\delta_t(s'=\infty)]<\infty\,.\quad\text{(A9)}$$

Hence, using (A3) and (A5)

$$D(t')\equiv\int_{s_0}^{\infty}\frac{ds'}{s'}[\rho(s',t')-\rho(\infty,t')]<\infty\quad\text{(A10)}$$

with a similar equation from the behavior at fixed s. Comparison of (A10) with (A8) suggests the important relation

$$D(\infty)\equiv\lim_{t'\to\infty}D(t')=\int_{s_0}^{\infty}\frac{ds'}{s'}\,\rho(s',\infty),\quad\text{(A11)}$$

where we used that $\lim_{t'\to\infty}\rho(\infty,t')=0$ from (A7). One can show[9] that (A11) has a simple physical interpretation: It means that the t-channel Regge-pole residue has a generalized power-law behavior at large $|t|$, in the sense that its large t dependence is given by a power, eventually multiplied by some logarithmic-like function.

We are now ready to establish the asymptotic behavior of $G(s,t)$ defined in (A2). For this purpose, we introduce

$$\tilde{G}(s,t) \equiv \frac{st}{\pi^2}\int_{s_0}^{\infty}\int_{t_0}^{\infty}\frac{ds'}{s'}\,\frac{dt'}{t'}\,\frac{\tilde{\rho}(s',t')}{(s'-s)(t'-t)}\,, \tag{A12}$$

where

$$\tilde{\rho}(s',t') \equiv \rho(s',t') - \rho(s',\infty) - \rho(\infty,t') \tag{A13}$$

and take the limit $(|s|,|t|)\to\infty$ under the integral in (A12), thus getting

$$\lim_{|s|,|t|\to\infty}\tilde{G}(s,t) = \frac{1}{\pi^2}\int_{s_0}^{\infty}\int_{t_0}^{\infty}\frac{ds'}{s'}\,\frac{dt'}{t'}\,\tilde{\rho}(s',t'). \tag{A14}$$

This step is justified, at least away from the cuts, provided[21] the integral on the right-hand side of (A14) effectively converges (on the cuts, oscillations may occur in the principal-part integrals, but we assume, in the spirit of the Regge model, that there is smooth behavior, so that the previous limit holds also there). We therefore consider

$$I(s,t) \equiv \int_{s_0}^{s}\int_{t_0}^{t}\frac{ds'}{s'}\,\frac{dt'}{t'}\,\tilde{\rho}(s',t') \tag{A15}$$

and show that (A11) "nearly" implies

$$I(\infty,\infty) \equiv \int_{s_0}^{\infty}\int_{t_0}^{\infty}\frac{ds'}{s'}\,\frac{dt'}{t'}\,\tilde{\rho}(s',t') < \infty \tag{A16}$$

(the exact meaning of "nearly" will be clarified below). We first observe that $I(\infty,t)$ converges. Indeed we can write, using (A10) and (A11),

$$I(\infty,t) = \int_{t_0}^{t}\frac{dt'}{t'}\,[D(t') - D(\infty)], \tag{A17}$$

which shows also that $I(\infty,t)$, if it diverges as $t\to\infty$, must do so less fast than $\ln t$. The simplest possibility is therefore that $I(\infty,t)$ in fact converges toward $I(\infty,\infty)$ as $t\to\infty$, which will be assumed here [a similar statement holds of course for $I(s,\infty)$]. One can show[9] that this assumption simply eliminates eventual logarithmic factors in the large-momentum-transfer dependence of the t-channel Regge-pole residue, exactly in the same way as condition (A9) ensures pure Regge-pole behavior for the fixed-t amplitude. It is therefore physically reasonable to expect that any violation of the condition $I(\infty,\infty)<\infty$ will give rise only to logarithmic modifications of the basic fixed-angle scaling laws we are going to derive. [Alternatively, one can obtain[9] (A16) from the hypothesis of Regge-pole behavior in the u channel (or more precisely the conditions

$$\int_{s_0}^{\infty}\frac{ds'}{s'}\,[\delta_u(s') - \delta_u(s'=\infty)] < \infty$$

and

$$\int_{t_0}^{\infty}\frac{dt'}{t'}\,[\delta_u(t') - \delta_u(t'=\infty)] < \infty),$$

together with the assumption (which will also be used below) that the s- and t-channel Regge trajectories reach their asymptotes sufficiently fast; this approach turns out to be particularly useful in the general case where the three cuts are present.] From (A2), (A12), (A13), and (A14), we now easily obtain

$$\begin{aligned} G(s,t) \underset{|s|,|t|\to\infty}{\sim} & \frac{1}{\pi^2}I(\infty,\infty) \\ & - [\delta_s(t=\infty) - \delta_{s=0}(t=\infty)]\,\frac{1}{\pi}\ln\left(-\frac{t}{t_0}\right) \\ & - [\delta_t(s=\infty) - \delta_{t=0}(s=\infty)]\,\frac{1}{\pi}\ln\left(-\frac{s}{s_0}\right) \\ & + o(1), \end{aligned} \tag{A18}$$

where we also used (A5) and its s-channel analog. At fixed angle, $|s|$ and $|t|$ tend to infinity at the same rate; therefore, provided the Regge trajectories reach their asymptotes fast enough, we can replace $\delta_s(t=\infty)$ and $\delta_t(s=\infty)$ by their limits $\delta(t=\infty)$ and $\delta(s=\infty)$ in (A18) (again, if this condition is not satisfied, this introduces only logarithmic modifications to the fixed-angle scaling laws).

On the other hand, the asymptotic behavior of the single integrals in (A1) are well known.[8] We have for instance

$$\frac{s}{\pi}\int_{s_0}^{\infty}\frac{ds'}{s'}\,\frac{\delta_{t=0}(s')}{s'-s} \underset{|s|\to\infty}{\sim} -\frac{1}{\pi}\,\delta_{t=0}(\infty)\ln\left(-\frac{s}{s_0}\right) + \text{const} + o(1) \tag{A19}$$

with a similar expression for the other integral. Combining (A19) with (A18) in (A1), we get finally

$$Q(s,t) \underset{|s|,|t|\to\infty}{\sim} \text{const}\times\left(-\frac{s}{s_0}\right)^{-(1/\pi)\delta(s=\infty)} \times\left(-\frac{t}{t_0}\right)^{-(1/\pi)\delta(s=\infty)}[1+o(1)]. \tag{A20}$$

The result (5) given in the text thus appears as a particularly simple generalization of (A20).

*Physique Mathematique et Théorique, Equipe de Recherche associée au C.N.R.S.

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