Nonlinear chiral models and many-dimensional solitons

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The problem of obtaining exact, finite-energy, static solutions to the equations of motion of nonlinear chiral $SU(2)\times SU(2)$ Lagrangians is investigated in a framework which is independent of the coordinate system for the pion fields. This approach considerably simplifies the derivation of the equations and exhibits clearly the topological nature of the solutions where these are known. Two promising unsolved models are considered and shown to contain (in appropriate limits) previously considered models for which properties of the static solutions have been established.

I. INTRODUCTION

Recently the problem of finding classical, static, finite-energy solutions for field theories in more than one dimension has been considered.[1] From the virial theorem,[2] such theories must include nonzero-spin particles or high powers of derivatives of the type encountered[3] in nonlinear pion Lagrangians. It has been further argued[4] that a realistic three-dimensional field theory with localized soliton solutions should contain a nonlinear chiral field. The equations of motion for such field theories turn out to be complicated linked nonlinear differential equations with Christoffel symbols computed from the internal-symmetry metric. In conventional approaches[3,5] these expressions are evaluated in a specific coordinate system for the pion fields, and the consequent equations are considered directly. However, the resulting equations (when not completely intractable) have been found always to have solutions given effectively by the one-dimensional sine-Gordon equation. In the present work a (previously described[6]) formal and general treatment of nonlinear realizations of chiral algebras will be shown to simplify the approach to such nonlinear field theories and to remove the complications of representation dependence. This simplification is important since many of the attempts[3,4,5] to find confined solutions within a nonlinear realization have invoked unconventional Lagrangians involving four derivative terms or fractional powers of more conventional Lagrange densities. Moreover, the topological nature of the solutions seems to be directly exhibited in the proposed framework, whereas it is frequently obscured by the use[5] of a specific coordinate system for the pion fields.

The exotic Lagrangians which have been introduced arise naturally in certain cases if the (many-dimensional) solitons are to have finite-energy static solutions which remain stable under the scaling transformation $\pi^i(\vec{x})\to\pi^i(\lambda\vec{x})$. The types of nonlinear equations which give rise to the soliton can be characterized by the behavior of the energy E under such scale changes. If E is neutral, that is if E is unaltered under the transformation, then (in all cases known to us) the equations of motion can be transformed by changes of variables until the final effective nonlinear equation is the one-dimensional sine-Gordon equation for which exact solutions exist. Naturally the resulting solutions contain an arbitrary length parameter which does not appear in the energy of the soliton. After appropriate Lagrangians have been established, these solutions are retrieved in the new framework in Sec. III, and this treatment reveals infinite classes of similar models (with exact solutions) in any number of dimensions. If E is not neutral but is stationary with respect to λ, that is $\partial E/\partial\lambda|_{\lambda=1}=0$, only static solutions with a certain size which determines the energy may be expected. Such models are easily found[3] by taking for the Lagrangian density a sum of a term which scales with a higher power of λ than the number of space dimensions and a term which scales with a lower power. No exact solutions to such models have yet been found. In Secs. IV and V the two most natural such models proposed are treated in the new framework. Although neither has been solved exactly, they are shown to contain, in appropriate limits, previously considered models with either exact solutions or upper and lower bounds on the energies.

II. NONLINEAR PION LAGRANGIANS

The specification of chiral $SU(2)\times SU(2)$-invariant Lagrangians in a representation-independent manner has been treated in detail in Ref. 6, and reproduced here are only the basic results needed. If the pion fields are parametrized in the form

$$\pi^i=\phi n^i\,, \qquad (1)$$

where

$$n^i n_i=1\,, \qquad (2)$$

then the Gürsey matrix[7] may be written as

$$\hat{U}=\left(\frac{1+\gamma_5}{2}\right)U^{-2}+\left(\frac{1-\gamma_5}{2}\right)U^{2}, \qquad (3)$$

where

$$U=\exp\left(-in_i\tau^i\frac{\theta}{2}\right), \qquad (4)$$

and θ is any arbitrary function of ϕ subject only to weak-field limits. Then any Lagrangian constructed from $\gamma_\mu\hat{U}$ will be chiral $SU(2)\times SU(2)$ invariant if SU(2) and Lorentz invariance are imposed. Moreover any number of derivatives may be used, and in this way all the invariant Lagrangians are produced. The important feature for the present work is that these Lagrangians do not contain ϕ explicitly, so that any soliton behavior found will be displayed independently of the specific pion realization. In practice this also simplifies the working considerably.

For our present purposes it will be sufficient to find all invariant Lagrangians with either two or four derivatives. In order to impose SU(2) and Lorentz invariances it is simply necessary to trace out over all Dirac and Pauli matrices and to saturate all four-vector indices. The identity

$$\gamma_\mu\hat{U}=\hat{U}^{-1}\gamma_\mu \qquad (5)$$

shows that these explicit Dirac matrices may be moved together in any trace considered, and that the remaining part of the traced expression has U (or derivatives) and its inverse appearing alternately. But the matrices

$$P^{\pm}=\left(\frac{1\pm\gamma_5}{2}\right) \qquad (6)$$

appearing in $\hat{U}$ act as projection operators in the Dirac space, so that there is a nonzero trace only when the product of the other Dirac matrices (previously grouped together) is proportional to either γ_5 or the unit matrix. If γ_5 appears, then so must the Levi-Civita tensor $\epsilon_{\mu\nu\rho\lambda}$ and these indices can only be saturated against ones on derivatives. In this case, not only must four derivatives appear, but (to give antisymmetry) they must each act on separate matrices. However, the identity

$$U^2(\partial_\mu U^{-2})+(\partial_\mu U^2)U^{-2}=0 \qquad (7)$$

shows that no more than a single undifferentiated matrix need be considered in any trace involving otherwise only first derivatives, and indeed none need be considered unless an odd number of differentiated matrices appear in the trace. Thus, since total divergences may be discarded, the term

$$\mathrm{Tr}[(\partial_\mu U^2)U^{-2}]$$

must appear once at least in any such expression. This term can at once be seen to vanish. Hence only the unit matrix results in the products and the problem thus becomes simply one of constructing SU(2) invariants from expressions in which U^2 and its inverse (each differentiated as appropriate) appear alternately, which are, by Eq. (3), even in θ, and in which the vector indices on the derivatives are saturated.

It is now straightforward, if a little tedious, to establish that the only invariant involving two derivatives is

$$2f_\pi^{-2}L_0=(\partial_\mu\theta)(\partial^\mu\theta)+\sin^2\theta(\partial_\mu n^i)(\partial^\mu n_i) \qquad (8)$$

and to eliminate all possibilities except

$$\mathrm{Tr}[(\partial^2U^2)U^{-2}(\partial^2U^2)U^{-2}+(\theta\to-\theta)],$$

$$\mathrm{Tr}[(\partial^2U^2)U^{-2}+(\theta\to-\theta)]\mathrm{Tr}[(\partial^2U^2)U^{-2}+(\theta\to-\theta)],$$

$$\mathrm{Tr}[(\partial_\mu\partial_\nu U^2)U^{-2}+(\theta\to-\theta)]\times\mathrm{Tr}[(\partial^\mu\partial^\nu U^2)U^{-2}+(\theta\to-\theta)],$$

$$\mathrm{Tr}[(\partial^2U^2)(\partial_\mu U^{-2})(\partial^\mu U^2)U^{-2}+(\theta\to-\theta)],$$

and

$$\mathrm{Tr}[(\partial_\mu\partial_\nu U^2)(\partial^\mu U^{-2})(\partial^\nu U^2)U^{-2}+(\mu\leftrightarrow\nu)+(\theta\to-\theta)]$$

as candidates for independent invariants involving four derivatives. To establish the interdependence of the last five candidates, it is necessary to use either completeness of the Pauli matrices or to work directly with the explicit forms given in Eq. (4); in fact this latter approach seems slightly quicker for this simple SU(2) case. The last two are directly shown to be linearly dependent on the first three, and the expressions

$$-f_\pi^{2}L_1=(L_0)^2, \qquad (9)$$

$$-4f_\pi^{-2}L_2=(\partial_\mu\theta)(\partial^\mu\theta)(\partial_\nu\theta)(\partial^\nu\theta)+\sin^4\theta(\partial_\mu n^i)(\partial^\nu n_i)(\partial^\mu n_j)(\partial_\nu n^j)+2\sin^2\theta(\partial_\mu\theta)(\partial^\mu n_i)(\partial^\nu\theta)(\partial_\nu n^i), \qquad (10)$$

and

$$-4f_\pi^{-2}L_3=4f_\pi^{-4}(L_0)^2-(\partial^2\theta)^2+\sin^4\theta[(\partial^\mu n_i)(\partial_\mu n^i)]^2+\sin2\theta(\partial^2\theta)(\partial_\mu n^i)(\partial^\mu n_i)$$
$$-4\cos^2\theta(\partial_\mu\theta)(\partial^\mu n_i)(\partial^\nu\theta)(\partial_\nu n^i)-\sin^2\theta(\partial^2 n_i)(\partial^2 n_i)-2\sin2\theta(\partial^2 n_i)(\partial^\mu\theta)(\partial_\mu n^i) \qquad (11)$$

are established as the only independent forms, where the normalization factors have been arranged for future convenicnce. Here the final expression, L_3, intrinsically involves fields with at least two derivatives. It is recorded simply for the convenience of future workers in the field, and will not be further investigated in the present work.

III. EXACT STATIC SOLUTIONS

Once an appropriate invariant Lagrangian has been picked, the task of finding exact solutions to the static limit of the equations of motion may be undertaken. In the present notation, a Lagrange multiplier η is introduced and a term $\frac{1}{2}\eta(n_i n_i - 1)$ is added to ensure the normalization of n, and this leads to

$$(\delta_{ij} - n_i n_j)\left[\nabla^a \frac{\partial L}{\partial(\nabla_a n^j)}\right] = 0 \tag{12}$$

and

$$\nabla^a \frac{\partial L}{\partial(\nabla_a \theta)} + \frac{\partial L}{\partial \theta} = 0 \tag{13}$$

as the static equations to be solved. All attempts at solution, of which we are aware, introduce at this point some ansatz effectively designed to satisfy Eq. (12) by making the quantity within the large square bracket proportional to n_j and to reduce Eq. (13) to one involving a single space variable only. This latter equation is then the one genuine nonlinear equation yielding the soliton behavior. Naturally these features are not easily identified in the original papers, but emerge only after appropriate changes of variable; the main advantage of the present notation is probably that it exhibits the essence of the idea directly.

The strategy may then be regarded as the search for a simple ansatz for n_i which will (through $n^2 = 1$) remove the n^i dependence from the Lagrangian. In all cases the n^i have been taken to be functions of angular variables, while θ is a function of a radial variable so that terms involving $(\nabla_a\theta)(\nabla^a n_i)$ are eliminated at once. A further ansatz of

$$n^i = \frac{x^i}{r}, \quad n^j = 0, \quad i \le d < j \tag{14}$$

implies that

$$(\nabla^a n_i)(\nabla_a n^i) = \frac{d-1}{r^2}, \tag{15}$$

and

$$(\nabla_a n^i)(\nabla^b n_i)(\nabla^a n_j)(\nabla_b n^j) = \frac{d-1}{r^4}, \tag{16}$$

where d is the number of space dimensions, and thus ensures that any of the Lagrangians considered becomes dependent on θ and the radial variable alone. In two dimensions, the scaling argument suggests the use of L_0; the less restrictive ansatz

$$\vec{n} = (\cos Q\psi, \sin Q\psi, 0), \tag{17}$$

where ψ is the angular variable defined by

$$\tan\psi = \frac{y}{x} \tag{18}$$

and Q is an integer to ensure single valuedness of n^i, is also possible. However, these exhaust the ideas which have been put forward. Moreover, in the cases where solutions have been found, the Lagrangians have been taken to be neutral under scaling. This means that, in the appropriate coordinates, all spatial dependence is effectively intrinsic so that an immediate first integral of the equations is known. Since these solutions are now very simple in the present notation they are reproduced briefly below.

In two dimensions, with the Lagrange density L_0 and the ansatz given in Eq. (17), the form

$$E = \pi f_\pi^2 \int_0^\infty \left(r\theta'^2 + \frac{Q^2 \sin^2\theta}{r}\right) dr, \tag{19}$$

where

$$\theta' = \frac{d\theta}{dr} \tag{20}$$

emerges as the effective energy. The equation of motion is simply

$$r^2 \nabla^2 (2\theta) = Q^2 \sin 2\theta \tag{21}$$

in the static limit, where θ is a function of the radial variable, r, only. But then a change of variable to

$$y = \ln r, \tag{22}$$

with

$$\dot\theta = \frac{d\theta}{dy}, \tag{23}$$

gives the energy as

$$E = \pi f_\pi^2 \int_{-\infty}^\infty (\dot\theta^2 + Q^2 \sin^2\theta)\, dy \tag{24}$$

and expresses Eq. (21) as

$$2\ddot\theta = Q^2 \sin 2\theta, \tag{25}$$

which is recognizable as the one-dimensional sine-Gordon equation. The first integral of this equation is immediately

$$\dot\theta^2 = Q^2 \sin^2\theta, \tag{26}$$

where the additive constant of integration has been equal to zero so that the energy takes on the value $4\pi f_\pi^2 Q$ and remains finite. Of course, the solution may then be taken as

$$\theta = 2\tan^{-1}\left[\left(\frac{r}{r_0}\right)^{\pm Q}\right], \tag{27}$$

where the scaling freedom $r \to \lambda r$ allows the arbitrary constant of integration, r_0, to be picked freely. Finally it should be noted that the current

$$N_\mu = \frac{1}{2\pi}\epsilon_{\mu\rho\lambda}\partial^\rho n_i \partial^\lambda n_j \epsilon^{ij3} \tag{28}$$

is divergence free irrespective of the equations of motion. The associated charge

$$\int d^2\tilde{x} N_0 = Q \tag{29}$$

may therefore be regarded as topologically conserved; the integrand does, of course, vanish everywhere except at the origin.

In three dimensions the exotic Lagrange density

$$3f_\pi L = -(-2L_0)^{3/2}, \tag{30}$$

suggested in Ref. 4, yields the effective energy

$$E = \frac{4\pi f_\pi^2}{3}\int_0^\infty \left(\theta'^2 + \frac{2\sin^2\theta}{r^2}\right)^{3/2} r^2 dr \tag{31}$$

when the ansatz in Eq. (14) is inserted. This may be rewritten as

$$E = \frac{4\pi f_\pi^2}{3}\int_{-\infty}^\infty (\dot\theta^2 + 2\sin^2\theta)^{3/2} dy \tag{32}$$

by using the change of variable in Eq. (22), and once again there is no explicit y dependence because of the scaling neutrality. Thus the first integral is immediately

$$(\dot\theta^2 + 2\sin^2\theta)^{1/2}(\dot\theta^2 - \sin^2\theta) = 0, \tag{33}$$

where the constant of integration has been set equal to zero to ensure finite energy. Once again, therefore, the one dimensional sine-Gordon equation is retrieved and

$$\theta = \pm 2\tan^{-1}\left(\frac{r}{r_0}\right) \tag{34}$$

emerges as the only nontrivial solution with

$$E = 2\pi^2\sqrt{3}\, f_\pi^2 \tag{35}$$

as the energy. This time the topologically divergence-free current is given by

$$N_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\lambda}(\partial^\nu n_i)(\partial^\rho n_j)(\partial^\lambda n_k)\epsilon^{ijk}, \tag{36}$$

and the conserved charge takes on the value unity. It is worth noting that when

$$\phi = f_\pi \sin\theta \tag{37}$$

the original result[3]

$$\pi^i = 2f_\pi \frac{x^i}{r}\left(\frac{r}{r_0}\right)\left[1 + \left(\frac{r}{r_0}\right)^2\right]^{-1} \tag{38}$$

for the pion fields is retrieved. However, whereas θ changes by π between infinity and the origin, the pion fields vanish in both regions, so that the topological nature[5] of the solution is obscured by such a choice of coordinates.

It will be apparent from the above discussion that there are available whole families of Lagrangians in any number of dimensions which are neutral under scaling (and for which first integrals of the equations of motion are immediately known) provided that exotic constructions are allowed. For example, each of the independent Lagrangian pieces of Sec. II can be raised to appropriate powers and added in linear combinations. So far no suggestion has been put forward as to the possible physical significance of such models, and a preliminary investigation of several similar Lagrangians reveals no features not exhibited in the prototypes above. A more promising approach seems to be the investigation of more realistic models (i.e., ones which have Lagrangians with a more conventional interpretation in terms of the fields from which the they are constructed) which are not neutral under scaling but for which the virial theorem allows solutions. Two such models are discussed in the following sections.

IV. A NATURAL THREE-DIMENSIONAL MODEL

The authors of Ref. 3 point out that the theory taking (in the present notation)

$$L = L_0 + \epsilon L_1 + \beta L_2 \tag{39}$$

as a Lagrangian density has both the virtues that scaling allows for static solutions in three space dimensions and that naive power counting suggests that the theory is renormalizable. Obviously the scaling behavior of the first term differs from that of the latter two, so that there will be no translational invariance to give the first equation of motion. Nevertheless, the ansatz of Eq. (14) may be used to give

$$E = \pi f_\pi^2 \int_0^\infty \left[2\left(\theta'^2 + \frac{2\sin^2\theta}{r^2}\right) + \epsilon\left(\theta'^2 + \frac{2\sin^2\theta}{r^2}\right)^2 + \beta\left(\theta'^4 + \frac{2\sin^4\theta}{r^4}\right)\right] r^2 dr \tag{40}$$

as the effective energy, so that

$$\frac{d}{dr}\left\{r^2\theta'\left[1 + (\epsilon+\beta)\theta'^2 + \frac{2\epsilon\sin^2\theta}{r^2}\right]\right\} = \sin 2\theta\left[1 + (2\epsilon+\beta)\frac{\sin^2\theta}{r^2} + \epsilon\theta'^2\right] \tag{41}$$

emerges as the equation of motion. This agrees with the correct method of proceeding from the equations of motions Eqs. (12) and (13) and the ansatz of Eq. (14). This nonlinear equation has not yet been solved; it seems to be closely related to the equation governing synchronous electric motors.[8] The weak-coupling limit of this equation

$$\frac{d}{dr}(r^2\theta') = \sin 2\theta \tag{42}$$

is identifiable at once as that of a damped pendulum if the change of variable in Eq. (22) is used. Clearly this leads always to an infinite-energy solution, in agreement with the observation that neither weak- nor strong-coupling limits by themselves can satisfy the virial theorem.

However, the case with $\epsilon = \beta$ has been previously considered by others,[4,9] and the energy has been shown to lie between $6\pi^2 f_\pi^2\sqrt{\epsilon}$ and $9\pi^2 f_\pi^2\sqrt{\epsilon}$, so that the soliton mass can be made arbitrarily small by choice of ϵ. It seems to the present authors that the full model deserves further consideration, but that numerical investigation of solutions seems an almost inevitable next step.

V. GAUGE-FIELD INTERACTIONS

One possible alternative to introducing higher numbers of derivatives is to modify the basic chiral Lagrangian by the inclusion of vector non-Abelian gauge fields. In this way, the complication of powers of derivatives in the equations of motion may be traded for the penalty of having linked differential equations for an increased number of fields. The existence of the nonlinear term in the covariant curl of the gauge fields V_μ^i,

$$F_{\mu\nu}^i = \partial_\mu V_\nu^i - \partial_\nu V_\mu^i + e\epsilon^{ijk} V_{\mu j} V_{\mu k}, \tag{43}$$

indicates the scaling properties of these extra fields and the second term in the invariant Lagrangian density

$$L = L_0(\theta, D_\mu n^i) - \tfrac{1}{4} F_{\mu\nu}^i F_i^{\mu\nu}, \tag{44}$$

where

$$D_\mu n^i = \partial_\mu n^i + e\epsilon^{ijk} V_{\mu j} n_k, \tag{45}$$

therefore takes over the role previously played by the four derivative terms. In the static case, with the ansatz[10]

$$V_i^l = \epsilon^l{}_{ij} n^j \frac{1-K(r)}{er}, \tag{46}$$

$$V_0^l = 0, \tag{47}$$

the energy is effectively

$$E = \frac{4\pi}{e^2}\int_0^\infty \left[\frac{e^2 f_\pi^2}{2}(r^2\theta'^2 + 2K^2\sin^2\theta) + K'^2 + \frac{(K^2-1)^2}{r^2}\right] dr, \tag{48}$$

and leads to

$$\frac{d}{dr}(r^2\theta') = K^2 \sin 2\theta, \tag{49}$$

$$r^2 K'' = e^2 f_\pi^2 r^2 K \sin^2\theta + K(K^2-1) \tag{50}$$

as equations of motion.

No exact solution to the above equations of motion has been found. However, in the weak-field limit ($\sin\theta \to \theta$), comparison with the work of Prasad and Sommerfield[11] gives the solutions

$$K = \frac{ar}{\sinh ar}, \tag{51}$$

$$\theta = \frac{ar\coth ar - 1}{ef_\pi r}, \tag{52}$$

where a is an arbitrary constant. That a whole family of solutions, parametrized by a, appears is a consequence of this weak-field limit. The energy is easily computed to be a/α so that, since the vacuum sector is reached as $a \to 0$, this is dynamically unstable. Naturally the topological behavior of θ is not now that of Sec. III; indeed, the solution is continuously deformable to zero. However, the presence of two mass parameters in the theory gives some encouragement that a dynamically stable solution may exist.

VI. CONCLUSIONS

A framework has been presented which seems ideally suited for the investigation of soliton-like solutions of nonlinear chiral theories. The previously discovered exact static solutions in two and three dimensions are retrieved immediately, and the framework reveals that there are whole families of similarly solvable analogous models available in any number of dimensions. Moreover, the intrinsic topological nature of the solutions is exhibited in this framework but may often be obscured in terms of specific coordinate choices for the original pion fields. However, all these models are neutral under scaling. The two most natural non-neutral models have been exhibited in a general form in the new notational framework, and some properties of the resulting equations have been indicated by establishing correspondences (in appropriate limits) with previously studied systems. Although these features are encouraging, clearly much further work is required before solutions of fixed size can be established in a sufficiently detailed way that identification with physical systems can be attempted.

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