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Dynamical equations for a Regge theory with crossing symmetry and unitarity. III. Crossing-symmetric representation with explicit Regge-pole terms*

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In parts I and II of this series, a system of partial-wave equations for construction of a crossing-symmetric unitary Regge theory of meson-meson scattering was described. Here we show that the sum of the partial waves of a solution has a representation in which crossing symmetry is apparent, all integrals converge without subtractions, double-spectral funcions have the correct support, and the contributions of Regge poles in all three channels are displayed simultaneously. We obtain the Regge asymptotic limit for $s \rightarrow \infty$ at arbitrary fixed t by a method which avoids a difficulty in the usual heuristic argument. We also discuss the consequences at high energy of a new method of avoiding ghost poles at $l = 0$ on even-signature trajectories.

I. INTRODUCTION

The scheme for the construction of a relativistic Regge theory,[1] described in parts I and II, works with N/D equations for partial-wave quantities. Although crossing symmetry and proper support of double-spectral functions were essential ingredients in the derivation of the equations, it is not obvious that the total amplitude, formed by summing the partial waves of the given solution, will in fact have those properties. Here we show that the total amplitude does have the proper symmetry and support properties. Moreover, we find that the amplitude has an elegant crossing-symmetric representation, which may be of interest beyond the domain of our particular theory. Except for modifications to incorporate the ghost-elimination scheme of part II, and a trivial rearrangement of terms, the representation is the same as that given in Ref. 2, Eqs. (2.20)–(2.27). The representation is similar in spirit to those of Khuri[3] and Chew and Jones,[4] except that it comes about rather more naturally and incorporates correct support of double-spectral functions.

In Sec. II we state the crossing-symmetric representation, and then show that its partial waves are identical to those obtained from the N/D system. In the course of the argument we show that the N/D partial wave, $a(l. s)$, is given by a Froissart-Gribov formula. We also obtain bounds on $a(l, s)$ at large s, uniformly for all directions in the cut s plane.

In Sec. III we obtain the Regge asymptote of the total amplitude $A(s, t)$ for large s and fixed t. Arbitrary complex values of t are allowed. We point out that the usual derivation of the Regge asymptote from the t channel Watson-Sommerfeld representation is not justified in the main case of interest, namely, for s and t in the s-channel physical region. There is the difficulty that the background integral is not known to converge uniformly, so there is no basis for the everyday contention that the integral behaves in the same way as its integrand (namely, as $s^{\mathrm{Re}\, l}$) at large s. In spite of the failure of the standard argument, we find that the usual Regge asymptote is valid in the theory under consideration.

In part II, a plan for elimination of ghost poles at $l = 0$ on even-signature trajectories was proposed. It involved making the physical s-wave $a_0(s)$ different from the l-analytic amplitude $a(l, s)$ at $l = 0$. We show in Sec. II that the proposal is consistent with crossing symmetry and that $A(s, t)$ has single-spectral terms and possibly a constant term as well. In Sec. IV we consider the question of how the ghost-elimination scheme could be tested experimentally. We also make some speculative remarks about the role of negative values of l in Regge phenomenology.

In the theory as developed in part II, there are no Regge poles in the inelastic function

$$\frac{1 - \hat{\eta}^2(l, s)}{4r(l, s)}, \tag{1.1}$$

as defined in (I2.36). It should be possible to allow Regge poles in this function, but certain technical problems stand in the way at present. In Sec. V of this paper, we argue that such poles do not change the form of the crossing-symmetric representation of $A(s, t)$. The Appendix contains details about the work of Sec. II.

We suppose that the reader is acquainted with the notation and general approach of parts I and II, but full familiarity with those papers is hardly necessary. The most important background is in part I, Sec. II, in part II, Sec. II, and in the first few equations of part II, Sec. III. A summary of notation is to be found in part II, Sec. V.

II. CROSSING-SYMMETRIC REPRESENTATION OF TOTAL AMPLITUDE

To be definite, we shall employ the ghost-elimination scheme of part II, Sec. IV. Then a solution of the equations for neutral meson-meson scattering which were proposed in II consists of the partial wave $a(l,s)$ meromorphic for $\mathrm{Re}\,l>-\epsilon$, $-\frac{1}{2}<\epsilon<0$, and the physical s-wave $a_0(s)$. From $a(l,s)$ and $a_0(s)$, for $4\leq s<\infty$, we may construct the total amplitude

$$A(s,t)=\frac{1}{\pi^2}\int_0^\infty dx\int_0^\infty dy\,\hat\rho(x,y)\left(\frac{1}{x-s}\,\frac{1}{y-t}+\frac{1}{x-u}\,\frac{1}{y-s}+\frac{1}{x-t}\,\frac{1}{y-u}\right)+\psi(s,t)+\psi(t,u)+\psi(u,s)+\psi(t,s)$$
$$+\psi(u,t)+\psi(s,u)+\frac{1}{\pi}\int_4^\infty dx\,\rho(x)\left(\frac{1}{x-s}+\frac{1}{x-t}+\frac{1}{x-u}\right)+a_0(\infty)\,. \tag{2.1}$$

The function $\hat\rho$ is constructed from the externally assigned central spectral function $v(x,y)$ and the parts of the elastic spectral functions that arise from Watson-Sommerfeld background integrals:

$$\hat\rho(x,y)=\hat\rho^{\mathrm{el}}(x,y)+\hat\rho^{\mathrm{el}}(y,x)+v(x,y)\,, \tag{2.2}$$

where, as in (II2.32),

$$\hat\rho^{\mathrm{el}}(x,y)=\frac{\theta(x-4)}{4i}\int_{-\epsilon} dl(2l+1)q(x)h(x)\times a(l,x_+)a(l,x_-)P_l(z_{xy})\,. \tag{2.3}$$

The integral follows the line $\mathrm{Re}\,l=-\epsilon$, $0<\epsilon<\frac{1}{2}$. The trajectory $\alpha(s)$ and residue $\beta(s)$ are determined from the partial-wave amplitude $a(l,s)$. The function ψ is defined in terms of α and β:

$$\psi(s,t)=-\tfrac{1}{2}\int_4^{s_1}\frac{dx}{x-s}\,\Delta\left(\frac{[2\alpha(x)+1]\beta(x)}{\sin\pi\alpha(x)}P_{\alpha(x)}(-z_{xt})\right). \tag{2.4}$$

For notational convenience we suppose that there is only one trajectory $\alpha(s)$; it leaves the right half plane $\mathrm{Re}\,l>-\epsilon$ at $s=s_1$. The symbol Δ denotes the discontinuity over the real x axis, as in (II2.25). The last two terms in (2.1) are present only when the ghost-extinction scheme of part II, Sec. IV is invoked. The constant $a_0(\infty)$ is the value of the physical s wave at infinity (which may be zero), and the single-spectral function $\rho(x)$ is the absorptive part of the difference between the physical s wave and the l-analytic wave at $l=0$:

$$\rho(x)=\mathrm{Im}[a_0(x_+)-a(0,x_+)]\,. \tag{2.5}$$

We shall prove that the Legendre projections of (2.1) are identical with the partial waves obtained from the N/D equations of II; i.e.,

$$a(l,s)+\delta_{l0}[a_0(s)-a(0,s)]=\tfrac{1}{2}\int_{-1}^{1}dz\,P_l(z)A(s,t)\,,$$
$$t=(4-s)(1-z)/2,\quad l=0,2,4,\ldots\,. \tag{2.6}$$

Since these partial waves are unitary (in the sense described in I), and since (2.1) is clearly crossing-symmetric, we see that a solution of the dynamical scheme does indeed give a crossing-symmetric, unitary amplitude. Furthermore, $a(l,s)$ has the proper exponential decrease at large $\mathrm{Re}\,l$, so that the Legendre series converges in the Lehmann-Martin ellipse, and the double-spectral functions of $A(s,t)$ have the proper support. The function $\hat\rho(x,y)$ does not have the support appropriate to the complete double-spectral function $\rho(x,y)$; notice that the integrals over $\hat\rho$ begin at zero. There is an additional spectral function from the six ψ functions. It combines with $\hat\rho$ to give the complete double-spectral function with proper support, as is shown at the end of this section.

The representation (2.1) may be rewritten in standard Mandelstam form with a finite number of subtractions. The form (2.1) never requires explicit subtractions, thanks to the extraction of the asymptotically dominant Regge terms through the ψ functions.

We shall argue that $a(l,s)$, the solution of the N/D system, has the following Froissart-Gribov representation:

$$a(l,s)=\frac{4}{\pi(s-4)}\int_4^\infty dy\,Q_l(z_{sy})A_t(s,y)\,, \tag{2.7}$$

$$A_t(s,t)=\sum_{i=1}^{4}A_t^{(i)}(s,t)$$
$$=\mathrm{Im}[a_0(t_+)-a(0,t_+)]\theta(t-4)+\frac{1}{\pi}\int_0^\infty dx\,\hat\rho(x,t)\left(\frac{1}{x-s}+\frac{1}{x-u}\right)-\pi\theta(s_1-t)\theta(t-4)\Delta g(t,s)$$
$$+\tfrac{1}{2}\int_4^{s_1}dx\,\Delta f(x,t)\left(\frac{1}{x-s}+\frac{1}{x-u}\right)\,. \tag{2.8}$$

The functions f and g are determined by α and β [see (II2.24), (II2.26)]:

$$f(s,t)=[2\alpha(s)+1]\beta(s)P_{\alpha(s)}(z_{st}), \tag{2.9}$$

$$g(t,s)=\frac{[2\alpha(t)+1]\beta(t)}{\sin\pi\alpha(t)}P^{(e)}_{\alpha(t)}(z_{ts}). \tag{2.10}$$

Let us postpone the proof that the N/D amplitude does have the form (2.7), and proceed to show that the Froissart-Gribov integral (2.7) is identical with the Legendre projection (2.6) of $A(s,t)$. Henceforth, we shall take the lower limit in the integral (2.7) to be zero. Since $A_t(s,y)=0$ for $y<4$, the change is innocuous; the point of the change is that the separate terms in A_t are not all zero for $y<4$.

The integral (2.7) is absolutely convergent and defines an analytic function of l if Rel is sufficiently large. That function has an analytic continuation to the half-plane Re$l>-\epsilon$, if s is sufficiently large. (Recall that there is no Regge pole in the half-plane at large s). We can find an explicit expression for the continued function, by the method of part II, Sec. III. The first three terms in (2.8) give absolutely convergent contributions to the y integral (2.7) for Re$l>-\epsilon$. We may dispose of these terms immediately, casting them into the form of Legendre projections. For l a non-negative even integer,

$$Q_l(\zeta)=\tfrac{1}{4}\int_{-1}^{1}dz\,P_l(z)\left(\frac{1}{\zeta-z}+\frac{1}{\zeta+z}\right). \tag{2.11}$$

When this formula is substituted in (2.7), one easily finds that the first three $A_t^{(i)}$ give contributions to $a(l,s)$ as follows:

$$a^{(i)}(l,s)=\tfrac{1}{2}\int_{-1}^{1}dz\,P_l(z)A^{(i)}(s,t),\quad i=1,2,3, \tag{2.12}$$

$$A^{(1)}(s,t)=\frac{1}{\pi}\int_4^\infty dy\,\mathrm{Im}[a_0(y_+)-a(0,y_+)]\left(\frac{1}{y-t}+\frac{1}{y-u}\right), \tag{2.13}$$

$$A^{(2)}(s,t)=\frac{1}{\pi^2}\int_0^\infty dx\int_0^\infty dy\,\hat{\rho}(x,y)\left(\frac{1}{x-s}\frac{1}{y-t}+\frac{1}{x-t}\frac{1}{y-u}+\frac{1}{x-u}\frac{1}{y-s}\right), \tag{2.14}$$

$$A^{(3)}(s,t)=\psi(t,s)+\psi(u,s)-\tfrac{1}{2}\int_4^{s_1}dx\Delta\left(\frac{f(x,s)}{\sin\pi\alpha(x)}\right)\left(\frac{1}{x-t}+\frac{1}{x-u}\right). \tag{2.15}$$

In the derivation of $A^{(2)}$ we used the symmetry of $\hat{\rho}$ and separated terms through partial fractions.

The transformation of the remaining term $a^{(4)}(l,s)$ into a Legendre projection is a more involved matter. We first deform the contour of the x integral, in the manner of part II, Sec. III. The deformed integral is

$$A_t^{(4)}(s,t)=\frac{1}{4i}\int_{\omega(\Gamma)}dx\left(\frac{1}{x-s}+\frac{1}{x-u}\right)f(x,t)$$
$$+\frac{1}{4i}\int_4^{-s_0}dx\left(\frac{1}{x-s}+\frac{1}{x-u}\right)\left(\frac{(2\alpha+1)\tilde{\beta}(x)}{p(x)^\alpha}[(x_+-4)^\alpha P_\alpha(z_{x_+t})-(x_--4)^\alpha P_\alpha(z_{x_-t})]\right)_{\alpha=\alpha(x)} \tag{2.16}$$

We have extracted the threshold factor $(x-4)^\alpha$ from $\beta(x)$ as in (II2.19), and have changed the contours of the x_+ and x_- terms so that they go to the left from $x=4$ to $x=-s_0$, and then back to s_1 along complex paths $\omega(\Gamma_+)$ and $\omega(\Gamma_-)$, respectively; see Fig. (II4). Since the contours are finite, and $t>0$, we may choose s so that the contours do not cross zeros of the denominators, $x-s$ and $x-u$, in the course of the deformation. Let us take s positive and large; then $x-s$ and $x-(4-s-t)$ will never vanish. The discontinuity of $(x-4)^\alpha P_\alpha$ may be evaluated through standard identities (II2.49):

$$(x_+-4)^\alpha P_\alpha(z_{x_+t})-(x_--4)^\alpha P_\alpha(z_{x_-t})=\begin{cases}2i(4-x)^\alpha\sin\pi\alpha\,P_\alpha(z_{xt}), & -s_0<x<4-t,\\ \dfrac{4}{\pi i}(4-x)^\alpha\sin^2\pi\alpha\,Q_\alpha(-z_{xt}), & 4-t<x<4.\end{cases} \tag{2.17}$$

Because of the rapid decrease of $Q_\alpha(-z_{xt})$ at large t, the second term in $A_t^{(4)}$ will vanish at large t; it is $O(t^{-1-\alpha(-s_0)})$, and the corresponding part of the integral (2.7) converges absolutely for Re$l\geq-\epsilon$. The first term in $A_t^{(4)}$ is $O(t^{\alpha(-s_0)})$, so it will give a convergent contribution to (2.7) for $l=0$ if $\alpha(-s_0)<0$. We proceed under the condition $\alpha(-s_0)<0$, although this entails a restriction on α which may not be desirable.[5] In the Appendix we show that the condition $\alpha(-s_0)<1$ is sufficient for the work of this section, even if this weaker condition is not necessarily sufficient for the developments of part II. We next introduce (2.17) in

(2.16), and use (2.11) and (2.7). By reversing the order of z and y integrals we obtain

$$a^{(4)}(l,s)=\tfrac{1}{2}\int_{-1}^{1}dzP_l(z)\frac{1}{\pi}\int_0^\infty dyA_t^{(4)}(s,y)\left(\frac{1}{y-t}+\frac{1}{y-u}\right)$$

$$=\tfrac{1}{2}\int_{-1}^{1}dzP_l(z)\Bigg[\frac{1}{\pi}\int_0^\infty dy\left(\frac{1}{y-t}+\frac{1}{y-u}\right)\frac{1}{4i}\int_{\omega(\Gamma)}dx\left(\frac{1}{x-s}+\frac{1}{x-(4-s-y)}\right)[(2\alpha+1)\beta(x)P_\alpha(z_{xy})]_{\alpha=\alpha(x)}$$

$$-\frac{1}{2\pi}\int_0^{4+s_0}dy\int_{-s_0}^{4-y}dx\left(\frac{1}{y-t}+\frac{1}{y-u}\right)\left(\frac{1}{x-s}+\frac{1}{x-(4-s-y)}\right)$$

$$\times\left(\frac{(2\alpha+1)\bar\beta(x)}{p(x)^\alpha}(4-x)^\alpha\sin\pi\alpha\,P_\alpha(z_{xy})\right)_{\alpha=\alpha(x)}$$

$$+\frac{1}{\pi}\left(\int_0^{4+s_0}dy\int_{4-y}^{4}dx+\int_{4+s_0}^{\infty}dy\int_{-s_0}^{4}dx\right)\left(\frac{1}{y-t}+\frac{1}{y-u}\right)\left(\frac{1}{x-s}+\frac{1}{x-(4-s-y)}\right)$$

$$\times\frac{1}{\pi}\left(\frac{(2\alpha+1)\bar\beta(x)}{p(x)^\alpha}(4-x)^\alpha\sin^2\pi\alpha\,Q_\alpha(-z_{xy})\right)_{\alpha=\alpha(x)}\Bigg]. \tag{2.18}$$

To reduce this unpleasant expression, we first make all the denominators linear in y by partial fractions, and then reverse the order of the x and y integrals. In the integral over $\omega(\Gamma)$ on the right-hand side of (2.18) we may then recognize Cauchy representations of P_α; for instance,

$$\frac{1}{\pi}\int_0^\infty\frac{dy}{y-t}P_\alpha(z_{xy})=-\frac{1}{\sin\pi\alpha}P_\alpha(-z_{xt}). \tag{2.19}$$

The equation (2.19) is valid at complex x. It is obtained by rotating the contour of the usual Cauchy representation of P_α, in which the argument of the integrated Legendre function is real. In rotating the contour one encounters no zero of the Cauchy denominator $y-t$, except on the part of $\omega(\Gamma)$ where x is real. For x real an "$i\epsilon$ limit" arises: The contour just touches the pole as it comes to its final position. The result for the integral on $\omega(\Gamma)$ in (2.18) is then

$$\tfrac{1}{2}\int_{-1}^{1}dzP_l(z)\frac{i}{4}\int_{\omega(\Gamma)}dx\left\{\frac{(2\alpha+1)\beta(x)}{\sin\pi\alpha}\left[\frac{1}{x-s}[P_\alpha(-z_{xt})+P_\alpha(-z_{xu})]+\frac{1}{x-u}P_\alpha(-z_{xt})+\frac{1}{x-t}P_\alpha(-z_{xu})\right.\right.$$

$$\left.\left.-\left(\frac{1}{x-t}+\frac{1}{x-u}\right)P_\alpha(z_{xs})\right]\right\}_{\alpha=\alpha(x)}. \tag{2.20}$$

The integrand of (2.20) does not have poles at $x=t$ and $x=u$, since the separate terms with poles cancel. In the sum of the remaining terms in (2.18) one recognizes the Cauchy representation of Q_α; namely,

$$Q_\alpha(z_{xt})=\tfrac{1}{2}\int_0^{4-x}\frac{dy}{y-t}P_\alpha(z_{xy})-\frac{1}{\pi}\int_{4-x}^{\infty}\frac{dy}{y-t}\sin\pi\alpha\,Q_\alpha(-z_{xy}). \tag{2.21}$$

The remaining terms in (2.18) then yield

$$-\tfrac{1}{2}\int_{-1}^{1}dzP_l(z)\frac{1}{\pi}\int_{-s_0}^{4}dx\left\{\frac{(2\alpha+1)\bar\beta(x)}{p(x)^\alpha}(4-x)^\alpha\sin\pi\alpha\right.$$

$$\left.\times\left[\frac{1}{x-s}[Q_\alpha(z_{xt})+Q_\alpha(z_{xu})]+\frac{1}{x-u}Q_\alpha(z_{xt})+\frac{1}{x-t}Q_\alpha(z_{xu})-\left(\frac{1}{x-t}+\frac{1}{x-u}\right)Q_\alpha(-z_{xs})\right]\right\}_{\alpha=\alpha(x)}. \tag{2.22}$$

One can relate the function $a^{(4)}(l,s)$, given by the sum of (2.20) and (2.22), to the function

$$\tfrac{1}{2}\int_{-1}^{1}dzP_l(z)\left[\psi(s,t)+\psi(s,u)+\psi(u,t)+\psi(t,u)+\tfrac{1}{2}\int_4^{s_1}dx\Delta\left(\frac{f(x,s)}{\sin\pi\alpha(x)}\right)\left(\frac{1}{x-t}+\frac{1}{x-u}\right)\right]. \tag{2.23}$$

In fact, (2.23) is equal to $a^{(4)}$, unless $\alpha(s_*)=0$ for some $s_*\in(-s_0,4)$. That is seen by deforming the x contours in (2.23), in the same way as in the derivation of (2.16). On the part of the deformed path from 4 to $-s_0$, the P_α's are replaced by Q_α's, through the discontinuity relation (2.17). If $\alpha(s_*)=0$ for $-s_0<s_*<4$, one gets an additional term from integration around the pole of $(\sin\pi\alpha)^{-1}$. In that case, $a^{(4)}(l,s)$ is equal to (2.23) plus the s-wave ghost pole term

$$-\frac{1}{s-s_*}\frac{\beta(s_*)}{\alpha'(s_*)}\delta_{l0}. \tag{2.24}$$

The amplitude $a(l,s)$ may now be assembled from (2.12), (2.23), and (2.24). In the usual case in which the ghost term (2.24) appears we have

$$a(l,s)=\tfrac{1}{2}\int_{-1}^{1}dz\,P_l(z)A(s,t)-\delta_{l0}\left\{\frac{1}{\pi}\int_4^\infty\frac{dy}{y-s}\mathrm{Im}[a_0(y_+)-a(0,y_+)]+a_0(\infty)+\frac{1}{s-s_*}\frac{\beta(s_*)}{\alpha'(s_*)}\right\}. \tag{2.25}$$

We finish the proof of (2.6) by noting that the expression in curly brackets in (2.25) is just the Cauchy representation of $a_0(s)-a(0,s)$. The functions $a_0(s)$ and $a(0,s)$ were constructed in II so as to have the same left-cut discontinuity, and so that $a(0,s)$ and $a_0(s)-a_0(\infty)$ vanish at large s, uniformly in direction. Since $a_0(s)$ has no poles, and $a(0,s)$ has a pole at $s=s_*$ with residue $-\beta(s_*)/\alpha'(s_*)$, the quantity in braces in (2.25) is indeed $a_0(s)-a(0,s)$. This establishes relation (2.6) for large positive s [recall that we made the restriction to large s to avoid Regge poles in $a(l,s)$]. Since both sides of Eq. (2.6) are analytic in the cut s plane, the equation must in fact hold for all s in the cut plane. Since $A(s,t)$ clearly has no pole at $s=s_*$, there is no ghost in the physical amplitude, even though $a(l,s)$ does in general have a ghost pole.

To verify the contention that $A(s,t)$, as defined by (2.1), has the correct double-spectral regions, we first apply the discontinuity relations of Legendre functions to calculate its t discontinuity $A_t(s,t)$. We find that the latter is given by (2.8). Next we calculate $\rho(s,t)$ from (2.8):

$$\begin{aligned}\rho(s,t)&=\frac{1}{2i}[A_t(s_+,t)-A_t(s_-,t)]\\&=\rho^{el}(s,t)+\rho^{el}(t,s)+v(s,t),\\ \rho^{el}(s,t)&=\frac{\theta(s-4)}{4i}\int_{-6}dl(2l+1)q(s)h(s)\\&\qquad\times a(l,s_+)a(l,s_-)P_l(z_{st})\\&\quad+\tfrac{1}{2}\pi\theta(s_1-s)\Delta f(s,t).\end{aligned} \tag{2.27}$$

As was shown in Eq. (II2.20)ff, one may move the Watson-Sommerfeld contour in (2.27) to the line $\mathrm{Re}\,l=L_0$, where $L_0>\max(\mathrm{Re}\,\alpha)$, to obtain

$$\rho^{el}(s,t)=\frac{\theta(s-4)}{4i}\int_{L_0}dl(2l+1)q(s)h(s)\times a(l,s_+)a(l,s_-)P_l(z_{st}). \tag{2.28}$$

The large-$|l|$ bound of $a(l,s_\pm)$, stated in (I2.20), now guarantees that $\rho^{el}(s,t)$ is zero for $t<16s/(s-4)$: when the latter inequality is met we can close the l contour in (2.28) and conclude from Cauchy's theorem that $\rho^{el}=0$; see (I2.20)ff.

We have yet to settle the deferred problem of showing that the N/D amplitude of II has the Froissart-Gribov (FG) representation (2.7). It is sufficient to make the proof for $l=0,2,4,\ldots$. We shall show that the N/D amplitude, $a(l,s)$ in (II5.5), and the FG amplitude, (2.7), have identical discontinuities over their s-plane branch cuts, and the same ghost pole (if there is any) at $l=0$. Since we also argue that both amplitudes vanish at large $|s|$, uniformly in direction, it will follow that the amplitudes are equal. The left-cut discontinuity of the N/D amplitude agrees with that of (2.7), since the left-cut input term of the N/D equation was in fact calculated from the FG integral; see II, Sec. II.[6] The right-cut discontinuity of the N/D amplitude is given by

$$q(s)h(s)a(l,s_+)a(l,s_-)+\frac{1-\hat{\eta}^2(l,s)}{4q(s)h(s)}, \tag{2.29}$$

$$\frac{1-\hat{\eta}^2(l,s)}{4q(s)h(s)}=\frac{4}{\pi(s-4)}\int_4^\infty dt\,Q_l(z_{st})[\rho^{el}(t,s)+v(s,t)], \tag{2.30}$$

whereas the corresponding quantity for the FG amplitude is obtained from the discontinuity of $A_t(s,t)$ as

$$\frac{4}{\pi(s-4)}\int_4^\infty dt\,Q_l(z_{st})[\rho^{el}(s,t)+\rho^{el}(t,s)+v(s,t)]. \tag{2.31}$$

One can show that the term in (2.31) involving $\rho^{el}(s,t)$ is identical to the first term of (2.29). Substitute formula (2.28) for $\rho^{el}(s,t)$, and reverse the order of t and l integrals, taking $l>L_0$. Put in the value of the t integral stated in (I2.32), and close the l contour by an infinite semicircle in the right half plane. The contour encloses one pole, and the evaluation of the residue by (I2.33) gives the desired first term of (2.29). Because of analyticity in l, the result is good also for $0\le l<L_0$. For the matter of ghost poles, we have already shown that the FG amplitude $a(0,s)$ has a pole (if any) as in (2.24). But α and β in (2.24) are by definition the Regge parameters of the pole in the N/D amplitude. It remains only to show that the amplitudes vanish uniformly at large $|s|$. Consider the most doubtful case, in which there is a Pomeron trajectory with $\alpha(0)=1$. According to (I2.51) and (I2.27), the N/D amplitude is dominated at large $|s|$ by $C(l,s)$; the other term in (I2.51) is $O(s^{-1})$, thanks to the cutoff in $r(l,s)$. If we look through the various terms in $C(l,s)$, as given in (II2.46), we find that the term c_1, which contains the contribution of the t-channel Regge

poles, is the dominant dynamical contribution at large $|s|$. The input term $V(l, s)$ may be of a similar order, but it vanishes at infinity by assumption. By contour deformations similar to that of (II3.2)ff, and applications of the identities (II2.49), one finds that $c_1(l, s) = O(|\ln s|^{-1})$ as s tends to $\pm\infty$ on the real axis. The calculation for $s \to -\infty$ requires careful attention to the cuts of Legendre functions; some of the Legendre functions are continued onto second sheets in the course of contour deformation. By an application of the Phragmén-Lindelöf theorem we can infer that the N/D amplitude is bounded in the cut plane as

$$|a(l, s)| \leq \kappa(\ln|s|)^{-1}, \tag{2.32}$$

for sufficiently large $|s|$. By (II2.29), (II2.37) we see that the FG amplitude is also dominated by $c_1(l, s)$, so that it satisfies (2.32) as well. The bound (2.32) holds for any fixed complex l with $\text{Re}\, l \geq -\epsilon$. The constant κ depends on l, and in fact tends to infinity at large $\text{Im}\, l$.

III. REGGE ASYMPTOTIC BEHAVIOR

We wish to point out first a shortcoming in the usual derivation of the Regge asymptote. We begin as usual with the Watson-Sommerfeld representation of $A(s, t)$ with (s, t) in the t-channel physical region:

$$\begin{aligned} A(s, t_+) &= A(t_+, s) \\ &= \sum_{l=0}^{\infty} (2l+1) a(l, t_+) P_l^{(e)}(z_{ts}) \\ &= \frac{i}{2} \int_{-\epsilon} dl \frac{(2l+1)}{\sin\pi l} a(l, t_+) P_l^{(e)}(z_{ts}) \\ &\quad - \pi \frac{[2\alpha(t_+)+1]\beta(t_+)}{\sin\pi\alpha(t_+)} P^{(e)}_{\alpha(t_+)}(z_{ts}), \\ &\qquad t \geq 4, \quad 4 - t \leq s \leq 0. \end{aligned} \tag{3.1}$$

In the t-channel physical region, where $-1 \leq z_{ts} \leq 1$, the integral in (3.1) converges exponentially, provided $z_{ts} \neq \pm 1$. The convergence follows from the bound

$$|P_l(\cos\theta)| \leq \kappa e^{\theta\, \text{Im}\, l}[|\ln(\pi - \theta)| + 1], \qquad 0 \leq \theta < \pi, \quad \text{Im}\, l > 0,$$

which may be proved from the Mehler integral representation.[7]

Let us attempt the continuation of (3.1) to the s-channel physical region, $s \geq 4$, $4 - s \leq t \leq 0$. Considering first the Regge pole term, we take s to a value much greater than 4, say on the upper side of the s cut, while keeping t fixed at a value greater than 4. Then expression (3.1) for $A(t_+, s_+)$ entails

$$P_{\alpha(t_+)}(-z_{ts_+}) = P_{\alpha(t_+)}\left(-1 - \frac{2(s+i0)}{t-4}\right), \qquad s > 4, \quad t > 4; \tag{3.2}$$

that is, a Legendre function evaluated on the lower side of its cut. Next we let t go to a point less than 4, by a path through the upper t plane. Then the argument of (3.2) goes into the upper half plane, which is to say onto the second sheet of P_α, and returns to the real axis at a point greater than -1. The second-sheet continuation of (3.2), evaluated at the final point where $s \gg 4$, $t < 4$, is

$$P_{\alpha(t)}(-z_{ts}) - 2i \sin\pi\alpha(t) P_{\alpha(t)}(z_{t_+s}). \tag{3.3}$$

The other term in $P_\alpha^{(e)}$ starts at a point off its cut, and ends on the cut with an evaluation

$$P_{\alpha(t)}(z_{t_+s}). \tag{3.4}$$

The residue $\beta(t_+)$ is continued to $t < 4$ and acquires the value

$$\frac{\tilde{\beta}(t)}{p(t)^{\alpha(t)}} (4-t)^{\alpha(t)} e^{i\pi\alpha(t)}, \tag{3.5}$$

according to (II2.19). Next, to take advantage of the asymptote of $P_\alpha(z)$ for large positive z, we use the identity

$$P_\alpha(-z_\pm) = e^{\mp i\pi\alpha} P_\alpha(z) - \frac{2}{\pi} \sin\pi\alpha\, Q_\alpha(z) \tag{3.6}$$

to eliminate functions with negative argument in (3.3) and (3.4). The continued Regge pole term is seen to have the form

$$-\frac{\pi}{2}\left[\frac{(2\alpha+1)\tilde{\beta}(t)}{\sin\pi\alpha}\left(\frac{4-t}{p(t)}\right)^{\alpha}\left((1+e^{-i\pi\alpha})P_\alpha(-z_{ts}) - \frac{2}{\pi} e^{i\pi\alpha} \sin\pi\alpha\,(1 - 2i\sin\pi\alpha) Q_\alpha(-z_{ts})\right)\right]_{\alpha=\alpha(t)}. \tag{3.7}$$

Now for $z \to +\infty$,

$$P_\alpha(z) \sim \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)} \frac{(2z)^\alpha}{\pi^{1/2}}, \tag{3.8}$$

$$Q_\alpha(z) = O(z^{-1-\text{Re}\,\alpha}), \tag{3.9}$$

so that the asymptote of (3.7) is

$$-\frac{\pi^{1/2}}{2}\left[\frac{(2\alpha+1)\tilde{\beta}(t)}{\sin\pi\alpha} \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)} (1+e^{-i\pi\alpha})\left(\frac{4s}{p(t)}\right)^{\alpha}\right]_{\alpha=\alpha(t)}. \tag{3.10}$$

The factor $p(t)^{-\alpha(t)}$ in (3.10) takes the part of the scale factor $s_c^{-\alpha(t)}$ which is usually introduced in an *ad hoc* way in phenomenological studies.[8] Correspondingly, the reduced residue $\tilde{\beta}(t)$ has a clearer theoretical meaning than the (various) residue functions used in phenomenology.

If we attempt the continuation of the integral in (3.1) by simply continuing its integrand, the steps followed above lead to

$$\frac{i}{4}\int_{-\epsilon} dl \frac{2l+1}{\sin\pi l} c(l, t_+)\left(\frac{4-t}{p(t)}\right)^l \left((1+e^{-i\pi l})P_l(-z_{ts}) - \frac{2}{\pi} e^{i\pi l}\sin\pi l\,(1-2i\sin\pi l)Q_l(-z_{ts})\right). \tag{3.11}$$

This integral is very far from absolute convergence, and even some weaker form of convergence seems extremely unlikely: The coefficient of Q_l has a factor $\exp(2\pi\,\mathrm{Im}l)$, Q_l itself vanishes only as $(\mathrm{Im}l)^{-1/2}$, and the best bound we have on $c(l, t_+)$ behaves as $(\mathrm{Im}l)^{-3/2}$; see (I3.5). It appears that the analytic continuation of the Watson-Sommerfeld background integral (a continuation which certainly exists) cannot be obtained by the simple process of continuing the integrand. Thus, we are in no position to make the usual assertion that the background integral is $O(s^{-\epsilon})$ at large s.

Our theory makes no use of the Watson-Sommerfeld integral in (3.1). We avoid the convergence problem by writing Watson-Sommerfeld integrals only for absorptive parts, which are quadratic in the partial-wave amplitudes and converge absolutely. To derive the Regge asymptote (3.10) we appeal directly to the representation (2.1). For the limit of large s at fixed t the only significant terms are

$$\psi(t, s)+\psi(t, u) = -\tfrac{1}{2}\int_4^{s_1}\frac{dx}{x-t}\Delta\left(\frac{(2\alpha+1)\beta(x)}{\sin\pi\alpha}[P_\alpha(-z_{xu})+P_\alpha(-z_{xs})]\right)_{\alpha=\alpha(x)}. \tag{3.12}$$

The sum of the other terms is bounded by a constant. Initially we suppose that there is no zero of $\alpha(x)$ for $-s_0 \leq x < 4$; later we shall account for a zero. We take large positive s, with $-s_0 < t < 4$, and consider $\psi(t, s_+)+\psi(t, u)$. To extract the dominant term at large s, we must deform the x contour, just as we did in (2.16). A new feature is that the functions

$$P_{\alpha(x_+)}(-z_{xs_+}),\quad P_{\alpha(x_-)}(-z_{xs_+}) \tag{3.13}$$

are evaluated on the lower sides of their cuts prior to contour distortion. Consequently, $P_{\alpha(x_+)}$ goes onto its second sheet when x goes into the upper half plane, while $P_{\alpha(x_-)}$ stays on its first sheet when x goes into the lower half plane. The functions

$$P_{\alpha(x_+)}(-z_{xu}),\quad P_{\alpha(x_-)}(-z_{xu}) \tag{3.14}$$

are evaluated off their cuts initially, but are finally on the lower and upper sides of their cuts, respectively. The paths between 4 and $-s_0$ strike the zero of the denominator $x-t$, giving principal-value integrals and δ-function terms in the limit in which the paths follow the real axis. The discontinutiy over the real axis is calculated as in (2.17). We apply (3.3) for the second-sheet continuation of P_α, and (3.6) to effect rearrangements of the δ-function terms. After some calculation we find that

$$\begin{aligned}\psi(t,u)+\psi(t,s_+) = &-\frac{\pi}{2}\left[\frac{(2\alpha+1)\tilde{\beta}(t)}{\sin\pi\alpha}\left(\frac{4-t}{p(t)}\right)^\alpha[e^{-i\pi\alpha}P_\alpha(-z_{ts})+P_\alpha(z_{tu})]\right]_{\alpha=\alpha(t)}\\ &+\frac{\mathrm{P}}{\pi}\int_4^{-s_0}\frac{dx}{x-t}\left[(2\alpha+1)\tilde{\beta}(x)\sin\pi\alpha\left(\frac{4-x}{p(x)}\right)^\alpha[Q_\alpha(z_{xu})+Q_\alpha(z_{x_+s})]\right]_{\alpha=\alpha(x)}\\ &+\left[(2\alpha+1)\tilde{\beta}(t)\left(\frac{4-t}{p(t)}\right)^\alpha[\cos\pi\alpha\,Q_\alpha(z_{tu})+i\sin\pi\alpha\,Q_\alpha(z_{t_+s})]\right]_{\alpha=\alpha(t)}\\ &-\frac{1}{4i}\int_{\omega(\Gamma)}\frac{dx}{x-t}\left(\frac{(2\alpha+1)\beta(x)}{\sin\pi\alpha}[P_\alpha(-z_{xu})+P_\alpha(-z_{xs})]\right)_{\alpha=\alpha(x)}\\ &+\frac{1}{2}\int_{\omega(\Gamma_+)}\frac{dx}{x-t}[2\alpha(x)+1]\beta(x)P_{\alpha(x)}(z_{x_+s}).\end{aligned} \tag{3.15}$$

The asymptote of the first term agrees precisely with (3.10). The second and third terms are $O(s^{-1-\alpha(-s_0)})$, and the fourth and fifth terms are $O(s^{\alpha(-s_0)})$. Since we require that $\alpha(-s_0)<0$, the amplitude at large s is given by (3.10) plus terms which are bounded by a constant.

We have yet to account for a zero of $\alpha(x)$ at $x = s_* \in (-s_0, 4)$. In the integrals over the upper and lower sides of the line segment $[-s_0, 4]$, which are obtained when the integration path in (3.12) is deformed, we introduce small semicircular detours around the pole at $x = s_*$, in the upper and lower half planes, respectively. The coefficients of $(\sin\pi\alpha)^{-1}$ in the upper and lower integrals become equal at $\alpha=0$, so the two semicircles contribute just $2\pi i$ times the pole residue, which is to say the term

$$\frac{1}{s_*-t}\,\frac{\tilde{\beta}(s_*)}{\alpha'(s_*)}. \qquad (3.16)$$

Since (3.16) is to be added to (3.15), there is, of course, no pole at $t=s_*$ in $\psi(t,u)+\psi(t,s_+)$. The first term on the right-hand side of (3.15) has a pole which cancels (3.16).

We conclude this section by stating the asymptote of $A(s,t)$ for large s and fixed complex t. Consider the closed curve $\omega(\hat{\Gamma})$ consisting of the parts of $\omega(\Gamma_+)$ and $\omega(\Gamma_-)$ connecting s_1 to s_2; i.e., $\hat{\Gamma}$ is the straight line between $\alpha(s_{1-})$ and $\alpha(s_{1+})$ in the l plane. If t lies outside $\omega(\hat{\Gamma})$, then

$$A(s,t)=O(1), \qquad (3.17)$$

since we may deform the x contour to obtain a formula like (3.15), but without the first term on the right-hand side. If t is complex and inside $\omega(\hat{\Gamma})$, we pick up an additional contribution from the pole at $x=t$. The leading part of the pole contribution has the same form as the first term in (3.15), provided that $(4-t)^\alpha$ is defined as follows:

$$(4-t)^\alpha = \exp\{[\mathrm{Re}\ln(t-4)+i\arg(4-t)]\alpha\},$$
$$-\pi<\arg(4-t)<\pi. \qquad (3.18)$$

The nonleading part of the pole term involves Q_α, and is $O(s^{-1-\mathrm{Re}\alpha})=O(s^{-1+\epsilon})$. Thus, the familiar Regge asymptote holds at complex t, provided t is inside $\omega(\hat{\Gamma})$, $(4-t)^\alpha$ is interpreted properly, and $\mathrm{Re}\,\alpha(t)>0$.

IV. POSSIBLE EXPERIMENTAL TEST OF A GHOST-EXTINCTION SCHEME

We are concerned solely with ghosts at $l=0$ on even-signature trajectories. As was explained in II, Sec. IV, our theory can have no other ghosts. According to the work of the previous section, the large-s behavior of the amplitude at fixed t is given by

$$A(s,t)\sim\frac{-\pi^{1/2}}{2}\left[\frac{(2\alpha+1)\tilde{\beta}(t)}{\sin\pi\alpha}\,\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}(1+e^{-i\pi\alpha})\left(\frac{4s}{p(t)}\right)^\alpha\right]_{\alpha=\alpha(t)}-\frac{1}{s_*-t}\,\frac{\tilde{\beta}(s_*)}{\alpha'(s_*)}+a_0(\infty)+\frac{1}{\pi}\int_4^\infty\frac{dx\,\rho(x)}{x-t}, \quad s_2<t\leq 0. \qquad (4.1)$$

The last three terms in (4.1) are included if and only if the ghost-extinction scheme of II, Sec. IV is employed. By contrast, in the conventional approach to ghost extinction[8] there is a zero of $\tilde{\beta}(t)$ at the location $t=s_*$ of the ghost pole, and the background to the Regge asymptote is supposed to vanish at large s for all t. It is clear from (4.1) that our scheme leads to a different kind of high-energy prediction, for t such that $-\epsilon\leq\alpha(t)\leq 0$. The last three terms in (4.1) are energy independent and dominate the Regge term for t in that range.

A test of this prediction should properly await an elaboration of our theory to cover processes more accessible to experiment than π-π scattering. For elastic π-π scattering we would in any case have the Pomeron as the leading even-signature trajectory. Since the Pomeron may not behave like an ordinary trajectory, or indeed may not correspond at all to a simple Regge pole, it is presumably safer to test our prediction on an even-signature trajectory arising in inelastic scattering. A good possibility is the A_2 trajectory, which may be isolated in the reaction

$$\pi^-+p\to\eta+n. \qquad (4.2)$$

Data on this reaction up to $p_{\mathrm{lab}}=200$ GeV/c have been fitted in terms of an "effective" A_2 trajectory.[9] If an expression analogous to (4.1) were valid for (4.2), one would expect $\alpha_{\mathrm{eff}}(t)$ to flatten out and acquire zero slope at the t for which $\alpha_{\mathrm{eff}}(t)=0$. The observed behavior seems consistent with this prediction, although the errors are fairly big in the crucial region of large $-t$. The analogous effective ρ trajectory, from

$$\pi^-+p\to\pi^0+n \qquad (4.3)$$

does not show a flattening at $\alpha_{\mathrm{eff}}=0$.[10] Again, that is expected in our scheme, since s-independent terms like those in (4.1) are not associated with odd-signature trajectories.

Since our ghost-extinction scheme entails a difference between the physical s wave and the l-analytic amplitude at $l=0$, it is in conflict with the philosophy of "maximal analyticity of the second degree."[11] It appears to us that the argument for such maximal analyticity is based mostly on aesthetics, and has little support from experiment. Our theory should help to put the experimental and theoretical issues into relief, as soon as it is extended to allow for spin and several coupled channels.

Another feature of our theory with possible experimental consequences is that only the right half plane is involved ($\mathrm{Re}\, l \geq -\epsilon$, $0 < \epsilon < \frac{1}{2}$). Although we regard it as something of a victory to have formulated equations in which no knowledge of amplitudes in the left half plane is necessary, it would still be interesting to perform a continuation into the left half plane. We are unable to make such a continuation, however, since our integral equations become singular at $\mathrm{Re}\, l = -\frac{1}{2}$. In the present state of knowledge we must allow for the possibility of singularities just to the left of the line $\mathrm{Re}\, l = -\epsilon$, or even for a natural boundary preventing continuation into all or part of the left half plane. Nearby singularities or a natural boundary could have an important role in Regge phenomenology. In general, Regge fits of data seem to be more successful when α is positive, and at least some of the difficulties for α negative or near zero may be due to nearby left-half-plane singularities different from usual Regge poles.

V. REGGE POLES IN THE INELASTIC FUNCTION

The inelastic function

$$I(l,s) = \frac{1-\hat{\eta}^2(l,s)}{4r(l,s)}, \tag{5.1}$$

defined in (I2.36), is part of the overlap function. It represents a sum over inelastic partial-wave amplitudes which are generally expected to possess the same Regge poles as the elastic amplitude. Heretofore we have artificially restricted the central spectral function $v(s,t)$ so that $I(l,s)$ could not have poles. We now remove the restriction and consider a v of the form

$$v(s,t) = \rho^{qe}(s,t) + \rho^{qe}(t,s) + \hat{v}(s,t), \tag{5.2}$$

where the "quasielastic" spectral function ρ^{qe} has the form

$$\rho^{qe}(s,t) = \frac{\theta(s-4\mu^2)}{4i}\int_{L_0} dl(2l+1)k(s)t(l,s_+) \times t(l,s_-)P_l(z_{st}), \tag{5.3}$$

and $\hat{v}(s,t) = \hat{v}(t,s)$ is a smooth function of the class to which our former $v(s,t)$ belonged. In (5.3) the Froissart-Gribov amplitude $t(l,s)$ corresponds to a process $\pi\pi \to M\bar{M}$, where M is a meson of mass $\mu > 1$, and $k(s)$ is the corresponding phase-space factor. We suppose that $t(l,s)$ is analytic in l for $\mathrm{Re} l > -\epsilon$, except for a pole at $\alpha(s_\pm)$, where α is the same Regge trajectory that appears in $a(l,s)$:

$$t(l,s_\pm) = \theta(s_1 - s)\frac{b_\pm(s)}{l-\alpha(s_\pm)} + \bar{t}(l,s_\pm), \quad s \geq 4\mu^2. \tag{5.4}$$

The contour in (5.3) is the line $\mathrm{Re} l = L_0 > \max[\mathrm{Re}\alpha(s_\pm)]$. We define $\gamma_\pm$ by

$$\gamma_\pm(s) = \pm 2ik(s)b_\pm(s)t(\alpha(s_\pm), s_\mp), \tag{5.5}$$

and move the contour in (5.3) to $\mathrm{Re} l = -\epsilon$. In analogy to (II2.23) we obtain

$$\rho^{qe}(s,t) = \hat{\rho}^{qe}(s,t) + \tfrac{1}{2}\pi\,\theta(s_1 - s)\theta(s-\mu^2)\Delta w(s,t), \tag{5.6}$$

$$\hat{\rho}^{qe}(s,t) = \frac{\theta(s-4\mu^2)}{4i}\int_{-\epsilon} dl(2l+1)k(s)t(l,s_+) \times t(l,s_-)P_l(z_{st}), \tag{5.7}$$

$$\Delta w(s,t) = \frac{1}{2i}\{[2\alpha(s_+)+1]\gamma_+(s)P_{\alpha(s_+)}(z_{st}) - [2\alpha(s_-)+1]\gamma_-(s)P_{\alpha(s_-)}(z_{st})\}. \tag{5.8}$$

The contribution of $v(s,t)$ to the inelastic function $I(l,s)$ now contains Regge poles at $l = \alpha(s_\pm)$, which come from the term $\Delta w(s,t)$ in $\rho^{qe}(s,t)$. The term in $I(l,s)$ from $\Delta w(s,t)$ may be evaluated with the help of (I2.36) and (I2.32).

$$\frac{4p(s)^l}{\pi(s-4)^{l+1}}\int_{\Sigma(s)}^\infty dt Q_l(z_{st})\tfrac{1}{2}\pi\theta(s_1-s)\Delta w(s,t) = \left(\frac{p(s)}{s-4}\right)^l \theta(s_1-s)\theta(s-4\mu^2)\frac{W(l,\alpha(s_+),z_{s\Sigma(s)})[2\alpha(s_+)+1]\gamma_+(s)}{2i[\alpha(s_+)-l][\alpha(s_+)+l+1]} - (+\to -). \tag{5.9}$$

Here $\Sigma(s)$ is the boundary of the support of $\rho^{qe}(s,t)$; it depends on the rate of decrease of $t(l,s)$ at large $\mathrm{Re} l$ [cf. (I2.20)ff]. Since $W(l,l,x) = -1$, the pole terms in $I(l,s)$ are

$$\left(\frac{p(s)}{s-4}\right)^l \frac{\theta(s_1-s)\theta(s-4\mu^2)}{2i}\left(\frac{\gamma_+(s)}{l-\alpha(s_+)} - \frac{\gamma_-(s)}{l-\alpha(s_-)}\right). \tag{5.10}$$

We must now reexamine the behavior of the unitarity condition near the Regge poles. By the unitarity condition (II2.20), the definition (II2.1) of the reduced amplitude $c(l,s)$, and the result (5.10), we find that

$$a(\alpha(s_\pm), s_\mp) = \pm\frac{\beta(s_\mp) - \gamma_\pm(s)\theta(s-4\mu^2)}{2iq(s)h(s)}. \tag{5.11}$$

This relation replaces our former equation (II2.22). The elastic double-spectral function (II2.8) is modified through (5.11). When the l contour is moved from $\mathrm{Re} l = L_0$ to $\mathrm{Re} l = -\epsilon$ we obtain instead of (2.27) the result

$$\rho^{el}(s,t)=\hat{\rho}^{el}(s,t)+\tfrac{1}{2}\pi\theta(s_1-s)\times[\Delta f(s,t)-\theta(s-4\mu^2)\Delta w(s,t)], \quad (5.12)$$

where $\hat{\rho}^{el}$ is defined in (2.3).

By (5.6) and (5.12) we see that the term involving Δw cancels in the sum

$$\rho^{el}(s,t)+\rho^{qe}(s,t)=\hat{\rho}^{el}(s,t)+\hat{\rho}^{qe}(s,t)+\tfrac{1}{2}\pi\theta(s_1-s)\Delta f(s,t). \quad (5.13)$$

The result (5.13) implies that the crossing-symmetric representation (2.1) of $A(s,t)$ retains its original form, if we redefine $\hat{\rho}$ to be

$$\hat{\rho}(x,y)=\hat{\rho}^{el}(x,y)+\hat{\rho}^{qe}(x,y)+\hat{\rho}^{el}(y,x)+\hat{\rho}^{qe}(y,x)+\hat{v}(x,y). \quad (5.14)$$

With this new definition, A_t is still given by (2.8). It is fortunate that the residue functions $\gamma_\pm(s)$ are not required to be analytic; in fact, we do not expect these functions to have simple analyticity properties.

The cancellation of Δw in the derivation of (5.13) at first seems remarkable. The cancellation is required, however, for meromorphy in l of $a(l,s)$. One sees that from the partial-wave dispersion relation (I2.41):

$$c(l,s)=c_L(l,s)+\frac{1}{\pi}\int_4^\infty\frac{r(l,s')c(l,s'_+)c(l,s'_-)}{s'-s}ds'+\frac{1}{\pi}\int_{16}^\infty\frac{1-\hat{\eta}^2(l,s')}{4r(l,s')}\frac{ds'}{s'-s}. \quad (5.15)$$

The third term on the right-hand side has cuts in the l plane, because of the pole terms (5.10) of the inelastic function. The cuts follow the trajectories $\alpha(s_\pm)$ from $\alpha(4\mu_\pm^2)$ to $\alpha(s_{1\pm})$. Since $c(l,s)$ should be meromorphic in l, the second term must have similar cuts but with opposite discontinuities. By applying (5.11), one can show that the cuts of the second term indeed cancel those of the third, so that the sum is meromorphic with poles only at $l=\alpha(s_\pm)$. This cancellation is the same phenomenon as the elimination of Δw in $\hat{\rho}$, in a slightly different guise.

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APPENDIX

We wish to remove the restriction $\alpha(-s_0)<0$, which was imposed for the discussion of Sec. II. The weaker condition $\alpha(-s_0)<1$ is assumed in the following. We discuss the contribution of $A_t^{(4)}$, defined in (2.8), to the integral (2.7) (the lower limit of the latter is now $y=0$). The second term in $A_t^{(4)}$ may be treated by the method of Sec. II, since it has an inverse power of t from the denominator $x-u$ that is not present in the first term. We need analyze only the first term, call it S_t:

$$S_t(s,t)=\frac{1}{2}\int_4^{s_1}\frac{dx}{x-s}\Delta f(x,t). \quad (A1)$$

We subtract and add a term in the numerator; it consists of $\Delta f(x,t)$ with $(x-4)^\alpha P_\alpha(z_{xt})$ replaced by $(s-4)^\alpha P_\alpha(z_{st})$. After the contour deformation of Sec. II, the integral takes the form

$$S_t(s,t)=\sum_{i=1}^{3}S_t^{(i)}(s,t)$$
$$=\frac{1}{4i}\int_{\omega(\Gamma)}\frac{dx}{x-s}\left(\frac{(2\alpha+1)\tilde{\beta}(x)}{p(x)^\alpha}[(x-4)^\alpha P_\alpha(z_{xt})-(s-4)^\alpha P_\alpha(z_{st})]\right)_{\alpha=\alpha(x)}$$
$$+\frac{1}{4i}\int_4^{-s_0}\frac{dx}{x-s}\left(\frac{(2\alpha+1)\tilde{\beta}(x)}{p(x)^\alpha}[(x_+-4)^\alpha P_\alpha(z_{x_+t})-(x_--4)^\alpha P_\alpha(z_{x_-t})]\right)_{\alpha=\alpha(x)}$$
$$+\frac{1}{4i}\int_{\omega(\Gamma)}\frac{dx}{x-s}\left(\frac{(2\alpha+1)\tilde{\beta}(x)}{p(x)^\alpha}(s-4)^\alpha P_\alpha(z_{st})\right)_{\alpha=\alpha(x)}. \quad (A2)$$

The subtracted term has no x discontinuity between 4 and $-s_0$. We may carry out the Froissart-Gribov integral (2.7) over $S_t^{(3)}$ for $l>1$. Since $S_t^{(3)}(s,t)$ is $O(t^{\alpha(-s_0)})$, the integral converges absolutely and has the value (Ref. 7, formula 7.114)

$$s^{(3)}(l,s)=\frac{1}{2\pi i}\int_{\omega(\Gamma)}\frac{dx}{x-s}\left(\frac{(2\alpha+1)\tilde{\beta}(x)(s-4)^\alpha}{p(x)^\alpha}\frac{1}{l-\alpha}\frac{1}{l+\alpha+1}\right)_{\alpha=\alpha(x)}. \quad (A3)$$

The Froissart-Gribov integral over $S_t^{(1)}$ converges absolutely at $l=0$, since the two terms in square brackets in $S_t^{(1)}$ cancel at large t to yield a sum which is $O(t^{\alpha(-s_0)-1})$. For the integral of $S_t^{(1)}$ at $l=0,2,4,\ldots$ we may then introduce

$$Q_l(\zeta)=\tfrac{1}{2}\int_{-1}^{1}\frac{dz\,P_l(z)}{\zeta-z}\,, \tag{A4}$$

and reverse the order of integrations to obtain

$$s^{(1)}(l,s)=\int_{-1}^{1}dz\,P_l(z)\frac{1}{4i}\int_{\omega(\Gamma)}\frac{dx}{x-s}\left(\frac{(2\alpha+1)\tilde{\beta}(x)}{p(x)^{\alpha}}\,\frac{1}{\pi}\int_0^{\infty}\frac{dy}{y-t}[(x-4)^{\alpha}P_{\alpha}(z_{xy})-(s-4)^{\alpha}P_{\alpha}(z_{sy})]\right)_{\alpha=\alpha(x)},$$
$$t=\tfrac{1}{2}(s-4)(z-1)\,. \tag{A5}$$

Now we assert that the y integral in (A5) is a slightly disguised Cauchy representation of the function

$$\varphi(t,\alpha)=-\frac{1}{\sin\pi\alpha}[(x-4)^{\alpha}P_{\alpha}(-z_{xt})-(s-4)^{\alpha}P_{\alpha}(-z_{st})]\,. \tag{A6}$$

Postponing the proof of this assertion, we see that

$$s^{(1)}(l,s)=\tfrac{1}{2}\int_{-1}^{1}dz\,P_l(z)\frac{i}{4}\int_{\omega(\Gamma)}\frac{dx}{x-s}\left(\frac{(2\alpha+1)\tilde{\beta}(x)(x-4)^{\alpha}}{p(x)^{\alpha}\sin\pi\alpha}[P_{\alpha}(-z_{xt})+P_{\alpha}(-z_{xu})]\right)_{\alpha=\alpha(x)}$$
$$+\frac{1}{4i}\int_{\omega(\Gamma)}\frac{dx}{x-s}\left(\frac{(2\alpha+1)\tilde{\beta}(x)(s-4)^{\alpha}}{p(x)^{\alpha}\sin\pi\alpha}\int_{-1}^{1}d\zeta\,P_l(\zeta)P_{\alpha}(\zeta)\right)_{\alpha=\alpha(x)}. \tag{A7}$$

Upon evaluating the ζ integral (by formula 7.112.3, Ref. 7) we find that the last term in (A7) is equal to $-s^{(3)}(l,s)$. Thus, the Froissart-Gribov integral of S_t consists of the first term in (A7), plus the integral over $S_t^{(2)}$. In other words, the integral of S_t is exactly the same as it was in Sec. II. We obtain our previous expression for $a^{(4)}(l,s)$ as given in (2.18). Notice that we had to use analytic continuation from $l>1$ to $l=0$, since the Froissart-Gribov integral is not known to converge at $l=0$. The continuation from $l>1$ is in fact what we want, since it is identical with the N/D amplitude, the latter being meromorphic for $\mathrm{Re}\,l>-\epsilon$.

We still have to show that the y integral in (A5) is equal to the function φ of (A6). The function $\sin\pi\alpha\;\varphi(t,\alpha)$ is entire in α for $x\neq 4$, $s\neq 4$. We shall obtain an integral representation first for $\mathrm{Re}\,\alpha<0$, and then show that it is valid for $\mathrm{Re}\,\alpha<1$. Let

$$x-4=|x-4|e^{i\theta},\quad -\pi<\theta<\pi\,. \tag{A8}$$

Then for $-\frac{1}{2}<\alpha<0$ and $t<0$,

$$\sin\pi\alpha\;\varphi(t,\alpha)=-(x-4)^{\alpha}\lim_{R\to\infty}\frac{1}{\pi}\int_0^{R\exp(i\theta)}\frac{dy}{y-t}P_{\alpha}(z_{xy})+(s-4)^{\alpha}\frac{1}{\pi}\int_0^{\infty}\frac{dy}{y-t}P_{\alpha}(z_{sy})\,. \tag{A9}$$

The path of integration in the first term is a straight line segment, following the cut of $P_{\alpha}(-z_{xy})$. We may rotate this path so as to put it on the real axis:

$$\lim_{R\to\infty}\int_0^{R\exp(i\theta)}\frac{dy}{y-t}P_{\alpha}(z_{xy})=\lim_{R\to\infty}\left[\int_0^{R}+\int_R^{R\exp(i\theta)}\right]\frac{dy}{y-t}P_{\alpha}(z_{xy})$$
$$=\int_0^{\infty}\frac{dy}{y-t}P_{\alpha}(z_{xy})\,. \tag{A10}$$

The integral from R to $Re^{i\theta}$ follows the arc of a circle of radius R, and tends to zero at large R, since

$$P_{\alpha}(z_{xy})=O(|y|^{\alpha}),\quad |y|\to\infty\,, \tag{A11}$$

uniformly for $|\arg(z_{xy})|\leq\pi-\delta<\pi$. We have established that

$$\sin\pi\alpha\;\varphi(t,\alpha)=-\frac{1}{\pi}\int_0^{\infty}\frac{dy}{y-t}[(x-4)^{\alpha}P_{\alpha}(z_{xy})-(s-4)^{\alpha}P_{\alpha}(z_{sy})]\,, \tag{A12}$$

for $-\frac{1}{2}<\alpha<0$ and $-\pi<\theta<\pi$, but the integral converges uniformly in α in a region

$$\mathfrak{D}=\{\alpha:\ -\tfrac{1}{2}+\delta\leq\mathrm{Re}\,\alpha\leq 1-\delta,\ \mathrm{Im}\,\alpha=0(1)\}\,, \tag{A13}$$

since for $\alpha\in\mathfrak{D}$ one has the bound

$$(x-4)^{\alpha}P_{\alpha}(z_{xy})-(s-4)^{\alpha}P_{\alpha}(z_{sy})=O(|y|^{|\mathrm{Re}\,\alpha|-1})\,, \tag{A14}$$

uniformly for $|\arg(x-4)|\leq\pi-\delta<\pi$. [The bound (A14) may be proved by estimates based on the Laplace integral representation of P_{α}.] By analyticity in α it follows that (A12) is true in the region $\mathfrak{D}$.

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[5]The condition $\alpha(-s_0)<0$ was already assumed in II. It would be an innocuous condition if it were not for the fact that our argument requires $s_0<a^2$, where $s=a^2$ is the location of the branch point of $p(s)$ [Eq. (II2.2)]. Technical requirements in the analysis of II forced a limit on a^2: $a_{max}{}^2\approx 50$. Thus, $\alpha(-s_0)<0$ implies $\alpha(t)<0$ for some $t>-a_{max}{}^2\approx -1$ $(\mathrm{GeV}/c)^2$. This requirement would be violated by the Pomeranchuk trajectory if the latter were as straight and "flat" as is sometimes assumed [say $\alpha(t)=1+0.3t$, $-1<t<0$, with t in $(\mathrm{GeV}/c)^2$].

[6]The expression for A_t used in II, Sec. II is (after modification of the s-wave part according to II, Sec. IV) equivalent to (2.8), although different in appearance. In II, Sec. II we used a Watson-Sommerfeld representation of the t-channel elastic background terms; i.e., we wrote

$$\frac{i}{2}\int_{-\epsilon} dl\,\frac{2l+1}{\sin\pi l}\,g(t)h(t)a(l,t_+)a(l,t_-)P_l^{(e)}(z_{ts})$$

for the term in (2.8) represented by

$$\frac{1}{\pi}\int_0^\infty dx\,\rho^{\mathrm{el}}(t,x)\left(\frac{1}{x-s}+\frac{1}{x-u}\right).$$

These functions are identical, as one sees from (2.3) and (2.19).

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