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Trace and dilatation anomalies in gauge theories

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The form of the anomaly in the trace of the energy-momentum tensor in a general theory of interacting fermions and non-Abelian gauge bosons is derived. The result is shown to involve precisely those gauge-variant operators which are known to mix with the naive trace under renormalization. The trace is shown to be soft on the mass shell if and only if the theory is at an eigenvalue of the Callan-Symanzik β function. The dilatation anomaly in the matrix element of $\theta^{\lambda}{}_{\lambda}$ with two electromagnetic currents (to lowest order in electromagnetism, but including all orders of the strong gauge interaction) is derived and shown to be infinitely renormalized in finite orders of strong perturbation theory. This anomaly is then shown to be canonical and given precisely by the lowest-order result provided the strong interactions are summed to all orders before going to the limit of physical space-time dimensions.

I. INTRODUCTION

The behavior of nontrivial, renormalizable quantum field theories under dilatations has been the subject of considerable attention for quite some time.[1] Stimulated mainly by the existence of various scaling phenomena, the interest has centered on the asymptotic behavior of Green's functions. This behavior exhibits departures from that suggested by naive dimensional analysis, and can be analyzed on the basis of Callan-Symanzik[2] or renormalization-group[3] equations (RGE's), which take explicit account of the anomalies in the dilatation Ward identities caused by the inevitable presence of regulator contributions. The information expressed by the RGE's can also be obtained, less directly, by studying zero-momentum insertions of the trace θ^{μ}_{μ} of the energy-momentum tensor in the general Green's function.[4] There have also been attempts[5,6] to obtain predictions about low-energy phenomena by studying θ^{μ}_{μ} such as the partial conservation of dilatation current (PCDC) calculation of the $(\epsilon \to 2\gamma)/(\epsilon \to 2\pi)$ branching ratio. Here the energy-momentum trace enters directly, as the operator putatively dominated by an ϵ ("dilaton") pole.

In this paper, we will study the trace of the energy-momentum tensor directly in theories containing fermions and non-Abelian gauge vector particles (but *no scalars*).[7] The exact form of the anomaly in the trace was recently derived[8] for QED: For non-Abelian theories, the derivation is somewhat complicated by the mixing[9] of gauge-invariant with gauge-noninvariant operators under renormalization. In Sec. II, we fix our notation and write down the expression for the trace in terms of bare fields, in the dimensionally regulated version of the theory. In Sec. III, we derive an exact expression for the anomalous part of the trace *at zero momentum* in terms of dimensionally subtracted normal-product operators.[10] In Sec. IV, the extension of the result of Sec. III to arbitrary momentum is proved. We note here that one result of the computation is that the energy-momentum trace, on-shell and at nonzero momentum, is soft if and only if $\beta(g)=0$, that is to say, at a Gell-Mann–Low eigenvalue. In the general Green's functions, a hard (and manifestly gauge-noninvariant) operator survives in the trace, even at an eigenvalue. In Sec. V, we study the dilatation anomaly in matrix elements of the energy-momentum trace with the hadronic electromagnetic current, in a theory with non-Abelian strong interactions. It is shown that the "canonical trace anomalies" of Chanowitz and Ellis[6] are in fact infinitely renormalized in higher orders of strong perturbation theory. It is then demonstrated that, by summing to all orders in the strong coupling, *before* passing to physical dimensions ($n=4$), the dilatation anomaly is exactly computable and given precisely by the canonical (lowest-order) result. Some technical details relevant to the analysis in Secs. III and IV are relegated to the Appendix.

II. COMPUTATION OF THE TRACE

A. Preliminary

We shall be considering a non-Abelian gauge theory with fermions.[11] The unrenormalized but dimensionally regularized Green's functions of such a theory can be computed from the effective action (in linear gauges)

$$S_{\text{eff}}[A,\omega,\overline{\omega},\psi,\overline{\psi};g_0,m_0,\xi_0,n]=\int d^nx\Big\{-\tfrac{1}{4}F_{\alpha\mu\nu}(x)F_\alpha^{\mu\nu}(x)+\overline{\psi}(x)(i\partial\!\!\!/+g_0t^\alpha A\!\!\!/_\alpha-m_0)\psi(x)-\frac{1}{2\xi_0}[\partial^\nu A_\nu^\alpha(x)]^2$$
$$+\partial^\mu\overline{\omega}_\alpha(x)[\partial_\mu\omega_\alpha(x)-g_0c_{\alpha\beta\gamma}A_\mu^\gamma(x)\omega_\beta(x)]\Big\}, \tag{2.1}$$

where

$$F_{\mu\nu}^\alpha(x)=\partial_\mu A_\nu^\alpha-\partial_\nu A_\mu^\alpha+g_0c_{\alpha\beta\gamma}A_\mu^\beta A_\nu^\gamma, \tag{2.2}$$

ω_α ($\overline{\omega}_\alpha$) are the Faddeev-Popov ghosts (and antighosts), and g_0, m_0, ξ_0 are the bare coupling, bare mass, and bare gauge parameter, respectively. We have chosen the gauge group to be simple; the t^α are its generators in the fermion representation, and the $c_{\alpha\beta\gamma}$ are its (totally antisymmetric) structure constants.

The unrenormalized Green's functions of the theory are generated by

$$W_0[J_{\alpha\mu}^0,\chi_\alpha^0,\overline{\chi}_\alpha^0,\eta^0,\overline{\eta}^0;g_0,m_0,\xi_0,n,a]$$
$$=\int(dA\,d\omega\,d\overline{\omega}\,d\psi\,d\overline{\psi})\exp\left[\frac{i}{a}\left(S_{\text{eff}}+\int d^nx[J_{\alpha\mu}^0(x)A^{\alpha\mu}(x)+\overline{\chi}_\alpha^0(x)\omega_\alpha(x)+\overline{\omega}_\alpha(x)\chi_\alpha^0(x)+\overline{\eta}^0(x)\psi(x)+\overline{\psi}(x)\eta^0(x)]\right)\right]. \tag{2.3}$$

Here a is a loop expansion parameter (to be set equal to 1 in the end). The unrenormalized connected Green's functions are generated by $Z_0=-i\ln W_0$, and the corresponding proper vertices by Γ_0, obtained from Z_0 by a Legendre transformation:

$$\Gamma_0[a^0,\Omega^0,\overline{\Omega}^0,\zeta^0,\overline{\zeta}^0;g_0,m_0,\xi_0,n,a]=Z_0-\int d^nx[J_{\alpha\mu}^0(x)a_{\alpha\mu}^0(x)+\overline{\Omega}_\alpha^0(x)\chi_\alpha^0(x)+\overline{\chi}_\alpha^0(x)\Omega_\alpha^0(x)+\overline{\zeta}^0(x)\eta^0(x)+\overline{\eta}^0(x)\zeta^0(x)], \tag{2.4}$$

where we have defined

$$a_{\alpha\mu}^0(x)=\frac{\delta Z_0}{\delta J_{\alpha\mu}^0(x)},\quad \overline{\Omega}_\alpha^0=-\frac{\delta Z_0}{\delta\chi_\alpha^0(x)},\quad \Omega_\alpha^0=\frac{\delta Z_0}{\delta\overline{\chi}_\alpha^0(x)},\qquad \zeta^0(x)=\frac{\delta Z_0}{\delta\overline{\eta}^0(x)},\quad \overline{\zeta}^0(x)=-\frac{\delta Z_0}{\delta\eta^0(x)}. \tag{2.5}$$

We have the relations

$$J_{\alpha\mu}^0(x)=-\frac{\delta\Gamma_0}{\delta a_{\alpha\mu}^0(x)},\quad \chi_\alpha^0(x)=-\frac{\delta\Gamma_0}{\delta\overline{\Omega}_\alpha^0(x)},\quad \overline{\eta}^0(x)=\frac{\delta\Gamma_0}{\delta\zeta_\alpha^0(x)},\ \text{etc.} \tag{2.6}$$

Further, one may generate Green's functions with an arbitrary number of insertions of a local composite operator $O(x)$ simply by modifying the exponent in Eq. (2.3) by $(i/a)\int M(x)O(x)d^nx$, where $M(x)$ is a source. Then if we define Z_0 and Γ_0 in an analogous manner (Γ_0 and Z_0 depend on M and M does not appear in the Legendre transform), we have

$$\frac{\delta Z_0}{\delta M(x)}=\frac{\delta\Gamma_0}{\delta M(x)}. \tag{2.7}$$

We further have the Ward-Takahashi (WT) identity:

$$0=\int(dA\cdots)\Big(-\frac{1}{\xi_0}[\partial_\nu A_\alpha^\nu(x)]-i\overline{\omega}_\alpha(x)\int d^ny\{J_{\beta\mu}^0(y)[\partial_\mu\omega_\beta(y)+g_0c_{\beta\gamma\delta}A_\mu^\gamma(y)\omega_\delta(y)]$$
$$+ig_0[\overline{\eta}(y)t^\beta\psi(y)-\overline{\psi}(y)t^\beta\eta(y)]\,\omega_\beta(y)\}\Big)$$
$$\times\exp\left[\frac{i}{a}\left(S_{\text{eff}}+\int d^nx[J^0(x)A_{\alpha\mu}(x)+\overline{\eta}^0(x)\psi(x)+\overline{\psi}(x)\eta^0(x)]\right)\right]. \tag{2.8}$$

We have the equation of motion for the gauge field,

$$0=\int (dA\cdots)\left[\frac{\delta S_{\rm eff}}{\delta A^\alpha_\mu(x)}+J^0_{\alpha\mu}(x)\right]\exp\left[\frac{i}{a}\left(S_{\rm eff}+\cdots\right)\right], \tag{2.9}$$

and its analog for the fermions. We also have the equation of motion for the antighost field,

$$0=\int (dA\cdots)\{\partial_\mu[\partial^\mu\omega_\alpha(x)-g_0c_{\alpha\beta\gamma}A^\gamma_\mu(x)\omega_\beta(x)]+\chi^0_\alpha(x)\}\exp\left[\frac{i}{a}\left(S_{\rm eff}+\cdots\right)\right], \tag{2.10}$$

where, in Eqs. (2.9) and (2.10), the exponents are identical to those in Eq. (2.3).

The theory is made finite by the multiplicative renormalizations given below. We shall determine the renormalization constants by the minimal subtraction scheme,[12] [i.e., counterterms subtract just the poles in $(n-4)$]. Thus,

$$\begin{aligned}
&J^R_{\alpha\mu}=Z_3^{1/2}J^0_{\alpha\mu},\\
&\chi^R_\alpha=\tilde Z^{1/2}\chi^0_\alpha,\quad \overline{\chi}^R=\tilde Z^{1/2}\overline{\chi}^0_\alpha,\\
&\eta^R=Z_2^{1/2}\eta^0,\quad \overline{\eta}^R=Z_2^{1/2}\overline{\eta}^0,\\
&g_R=g_0\mu^{(n-4)/2}Z_1^{-1}Z_2Z_3^{1/2}\equiv g_0\mu^{(n-4)/2}Z_g,\\
&m_R=Z_m^{-1}m_0,\quad \xi_R=Z_3^{-1}\xi_0.
\end{aligned} \tag{2.11}$$

Then the generating functional of renormalized Green's function is ($\mathcal{J}=\{J_{\alpha\mu},\chi_\alpha,\ldots\}$, $\lambda=\{g,m,\xi\}$)

$$W_R[\mathcal{J}^R;\lambda^R,\mu,n,a]\equiv W_0[\mathcal{J}^0;\lambda^0,n,a] \tag{2.12}$$

and is a finite functional of its arguments at $n=4$.

We shall use the notation

$$\langle F[A,\psi,\overline{\psi},\omega,\overline{\omega}]\rangle_{\mathcal{J}^0}=\frac{1}{W_0}\int (dA\,d\omega\,d\overline{\omega}\,d\psi\,d\overline{\psi})F[A,\psi,\overline{\psi},\omega,\overline{\omega}]\exp\left[\frac{i}{a}\left(S_{\rm eff}+\cdots\right)\right]. \tag{2.13}$$

B. Trace of the energy-momentum tensor

Consider the energy-momentum tensor as constructed by Freedman, Muzinich, and Weinberg,[13] modified to include fermions. It reads

$$\begin{aligned}
\theta_{\mu\nu}(x)=&-g_{\mu\nu}\mathcal{L}_{\rm eff}-F^\alpha_{\mu\sigma}F^\alpha_{\nu\sigma}-g_{\mu\nu}\xi_0^{-1}\partial^\rho(A^\alpha_\rho\partial^\sigma A^\alpha_\sigma)+\xi_0^{-1}[A^\alpha_\nu\partial_\mu(\partial^\sigma A^\alpha_\sigma)+(\mu\leftrightarrow\nu)]\\
&+\tfrac{1}{4}i[\overline{\psi}\gamma_\mu(\overrightarrow{\partial}_\nu-ig_0t^\alpha A^\alpha_\nu)\psi-\overline{\psi}(\overleftarrow{\partial}_\nu+ig_0t^\alpha A^\alpha_\nu)\gamma_\mu\psi+(\mu\leftrightarrow\nu)]+[\partial_\mu\overline{\omega}_\alpha(\partial_\nu\omega_\alpha-g_0c_{\alpha\beta\gamma}A^\gamma_\nu\omega_\beta)+(\mu\leftrightarrow\nu)].
\end{aligned} \tag{2.14}$$

Here $\mathcal{L}_{\rm eff}(x)$ is defined to be the expression in the curly brackets in Eq. (2.1). The trace of $\theta_{\mu\nu}$ in n dimensions can be expressed (without use of equations of motion) in the form

$$\begin{aligned}
\theta^\mu_\mu(x)=&\frac{1}{2}\left(\overline{\psi}\frac{\delta S_{\rm eff}}{\delta\overline{\psi}}+\frac{\delta S_{\rm eff}}{\delta\psi}\psi\right)+m_0\overline{\psi}\psi-2\{\partial^\rho[\xi_0^{-1}A^\alpha_\rho\partial^\nu A^\alpha_\nu+\overline{\omega}_\alpha(\partial_\rho\omega_\alpha-g_0c_{\alpha\beta\gamma}A^\gamma_\rho\omega_\beta)]\}\\
&-\overline{\omega}_\alpha\partial^\rho(\partial_\rho\omega_\alpha-g_0c_{\alpha\beta\gamma}A^\gamma_\rho\omega_\beta)-(n-4)[\mathcal{L}_{\rm eff}+\xi_0^{-1}\partial^\rho(A^\alpha_\rho\partial^\nu A^\alpha_\nu)].
\end{aligned} \tag{2.15}$$

(i) As we are using a mass-independent renormalization scheme, we have[14]

$$m_0\overline{\psi}\psi=m_RN(\overline{\psi}\psi)=\text{a finite quantity}. \tag{2.16}$$

(ii) Using the equations of motion for ψ and $\overline{\psi}$,

$$\begin{aligned}
\left\langle\overline{\psi}(x)\frac{\delta S_{\rm eff}}{\delta\overline{\psi}(x)}+\frac{\delta S_{\rm eff}}{\delta\psi(x)}\psi(x)\right\rangle&=-\langle\overline{\psi}\eta^0+\overline{\eta}^0\psi\rangle\\
&=\frac{1}{W_0}i\left[\overline{\eta}^0(x)\frac{\partial}{\partial\overline{\eta}^0(x)}-\eta^0(x)\frac{\partial}{\partial\eta^0(x)}\right]W_0\\
&=\frac{1}{W_R}i\left[\overline{\eta}^R\frac{\partial}{\partial\overline{\eta}^R(x)}-\eta^R(x)\frac{\partial}{\partial\eta^R(x)}\right]W_R\equiv iN_\psi.
\end{aligned} \tag{2.17}$$

The truncated on-shell Green's functions of N_ψ vanish at $q\neq0$ and at $q=0$ they equal the number of external fermions (antifermions included) of the Green's function; in particular, they are finite.

(iii) Further, by using the equation of motion for the antighost, viz., Eq. (2.10), we obtain

$$\langle 2\bar{\omega}_\alpha(x)\partial^\rho[\partial_\rho\omega_\alpha(x) - g_0 c_{\alpha\beta\gamma}A^\gamma_\rho(x)\omega_\beta(x)]\rangle = -2\langle\bar{\omega}_\alpha(x)\chi^0_\alpha(x)\rangle$$
$$= +\frac{2i}{W_0}\chi^0_\alpha(x)\frac{\delta}{\delta\chi^0_\alpha(x)}W_0$$
$$= \frac{2i}{W_R}\chi^R_\alpha\frac{\delta}{\delta\chi^R_\alpha}W_R \equiv iN_\omega\,, \qquad (2.18)$$

where N_ω is the counting operator[15] for ghosts whose truncated on-shell Green's functions vanish at $q\neq 0$ and at $q=0$ equals the number of external ghosts and antighosts. In particular, therefore, its Green's functions with external gauge bosons and fermions only vanish identically.

(iv) Finally, we consider the term in $\theta^\mu_\mu(x)$:

$$-2\partial^\rho[\xi^{-1}A^\alpha_\rho\partial^\nu A^\alpha_\nu + \bar{\omega}_\alpha(\partial_\rho\omega_\alpha - g_0 c_{\alpha\beta\gamma}A^\gamma_\rho\omega_\beta)]\,. \quad (2.19)$$

Regarding this term we shall show in the Appendix the following:

(a) At $q\neq 0$, its truncated on-shell Green's functions (with physical wave functions attached) vanish identically.

(b) At $q=0$, these Green's functions vanish. This is, of course, obvious once one notes that the expression in the square brackets in (2.19) cannot have a $1/q^2$ singularity because it does not have the quantum numbers of a gauge field.

(c) For the sake of completeness, we shall show that the only other renormalization part in the Green's functions of (2.19), viz., the two-ghost proper vertex with an insertion of the expression (2.19), is finite.

To summarize, we have shown that the terms in $\theta^\mu_\mu(x)$ [see Eq. (2.15)], apart from the term proportional to $(n-4)$, are finite and for *physical matrix elements* of θ^μ_μ,

$$\theta^\mu_\mu(x) - (n-4)[\mathcal{L}_{\rm eff} + \xi_0^{-1}\partial_\rho(A^\alpha_\rho\partial^\nu A^\alpha_\nu)] = m_R N(\bar{\psi}\psi) + iN_\psi\,. \quad (2.20)$$

It thus remains to consider the last term in Eq. (2.15). This gives rise to the anomaly and will be treated in the next two sections.

III. THE TRACE ANOMALY

We will now compute the zero-momentum insertion of the anomalous part of the energy-momentum trace in the limit $n\to 4$, expressed in terms of renormalized operators. The derivation is completed by noting (cf. Sec. IV) that the form obtained must hold for arbitrary momentum, by the general theory[9] of renormalization of gauge-invariant operators.

We begin by noting that, by virtue of the mass-independent renormalization scheme we have chosen, the Z's of Eq. (2.11) all have the functional form

$$Z = Z(g_0(n)\mu^{(n-4)/2}a^{1/2},\ \xi_0(n), n)\,. \quad (3.1)$$

Note the appearance of the loop-counting parameter a accompanying powers of g_0^2.

At zero momentum, the anomalous part of the energy-momentum trace is just [cf. Eq. (2.15)]

$$(\theta^\mu_\mu)^\sim_{\rm anom} = -(n-4)\mathcal{L}^\sim_{\rm eff}\,. \quad (3.2)$$

We have adopted the convention $\theta^\sim \equiv \int d^n x\,\theta(x)$ for an arbitrary local operator at zero momentum. Now we note that zero-momentum insertions of $\mathcal{L}_{\rm eff}$ can be obtained simply by differentiating the bare functional W_0 with respect to the loop parameter a:

$$\left(\frac{1}{W_0}\frac{\partial W_0}{\partial a}\right)_{\mathcal{J}^0,\lambda_0} = -\frac{i}{a^2}\langle\mathcal{L}^\sim_{\rm eff} + (J^0\cdot A)^\sim + \cdots\rangle_{\mathcal{J}^0}$$
$$= -\frac{i}{a^2}\langle\mathcal{L}^\sim_{\rm eff}\rangle_{\mathcal{J}^0} - \frac{1}{a}\int d^n x\,\vec{\mathcal{J}}^0(x)\cdot\frac{\delta W_0}{\delta\vec{\mathcal{J}}^0(x)}\frac{1}{W_0}$$
$$= -\frac{i}{a^2}\langle\mathcal{L}^\sim_{\rm eff}\rangle_{\mathcal{J}^0} - \frac{1}{a}\int d^n x\,\vec{\mathcal{J}}^R(x)\cdot\frac{\delta W_R}{\delta\vec{\mathcal{J}}^R(x)}\frac{1}{W_R}\,. \qquad (3.3)$$

The source terms are evidently finite at $n=4$, so up to terms vanishing at $n=4$

$$-\frac{i}{a^2}(n-4)\langle\mathcal{L}^\sim_{\rm eff}\rangle = (n-4)\left(\frac{\partial W_R}{\partial a}\right)_{\mathcal{J}^0,\lambda_0}\frac{1}{W_R}$$
$$= (n-4)\left[\left(\frac{\partial W_R}{\partial g_R}\frac{\partial g_R}{\partial a}\right)_{\lambda_0} + \left(\frac{\partial W_R}{\partial m_R}\frac{\partial m_R}{\partial a}\right)_{\lambda_0} + \left(\frac{\partial W_R}{\partial \xi_R}\frac{\partial \xi_R}{\partial a}\right)_{\lambda_0}\right.$$
$$\left. + \left(\frac{\partial W_R}{\partial J^R_i}\frac{\partial J^R_i}{\partial a}\right)_{J^0,\lambda_0} + \text{other source terms}\right]\frac{1}{W_R}\,. \qquad (3.4)$$

[The sum over i in (3.4) includes a spatial integration.] From (2.11) and (3.1) it follows that, setting $a=1$,

$$\lim_{n\to 4}(n-4)\left(\frac{\partial g_R}{\partial a}\right)_{\lambda_0}=\beta(g_R)\equiv\lim_{n\to 4}\mu\left(\frac{\partial}{\partial\mu}g_R\right)_{\lambda_0}. \quad (3.5)$$

The other a derivatives in (3.4) reduce similarly to the corresponding anomalous dimensions, yielding ($a=1$), in an obvious notation,

$$-i(n-4)\mathcal{L}^-_{\text{eff}}=\tfrac{1}{2}(\gamma_3 N_A+\gamma_2 N_\psi+\tilde\gamma N_\omega)+\beta\frac{\partial}{\partial g_R}-\gamma_m m_R\frac{\partial}{\partial m_R}-\gamma_3\xi_R\frac{\partial}{\partial\xi_R}. \quad (3.6)$$

The quantities N_A, N_ψ, and N_ω are just the total number of external gauge, fermion, and ghost lines, and arise from the counting operators $J_i^R(\partial/\partial J_i^R)$, etc., in (3.4).

The final stage of the computation involves reexpressing N_A, N_ψ, N_ω and the derivatives with respect to the renormalized parameters in terms of zero-momentum insertions of renormalized normal products.[15] To this end, we write the effective normal-product Lagrangian in terms of dimensionally subtracted normal products:

$$\begin{aligned}
&\mathcal{L}_{\text{eff,NP}}=\mathcal{L}_A+\mathcal{L}_F+\mathcal{L}_{gf}+\mathcal{L}_G\big|_{\zeta_A=\zeta_\psi=\zeta_\omega=1},\\
&\mathcal{L}_A\equiv-\tfrac{1}{4}\zeta_A N[F_{\alpha\mu\nu}F_\alpha^{\mu\nu}],\\
&\mathcal{L}_F\equiv\zeta_\psi N[\bar\psi(i(\not\partial-ig_R\zeta_A^{1/2}\not A_\alpha t_\alpha)-m_R)\psi],\\
&\mathcal{L}_{gf}\equiv-\frac{\zeta_A}{2\xi_R}N[(\partial_\mu A_\alpha^\mu)^2],\\
&\mathcal{L}_G\equiv\zeta_\omega N[\bar\omega_\alpha\overleftarrow{\partial}^\mu(\overrightarrow{\partial}_\mu\delta_{\alpha\beta}-\zeta_A^{1/2}g_R c_{\alpha\beta\gamma}A_{\gamma\mu})\omega_\beta],
\end{aligned} \quad (3.7)$$

with

$$F_{\alpha\mu\nu}\equiv\partial_\mu A_{\alpha\nu}-\partial_\nu A_{\alpha\mu}+g_R\zeta_A^{1/2}c_{\alpha\beta\gamma}A_{\beta\mu}A_{\gamma\nu}. \quad (3.8)$$

Now recall that differentiation of a renormalized Green's function with respect to a parameter λ appearing in (3.7) amounts to insertion of $i(\partial\mathcal{L}_{\text{eff}}/\partial\lambda)$. Furthermore, the dependence of renormalized Green's functions on $\zeta_A, \zeta_\psi, \zeta_\omega$ is just

$$G_R(p;\lambda^R,\zeta_A,\zeta_\psi,\zeta_\omega)=\zeta_A^{-N_A/2}\zeta_\psi^{-N_\psi/2}\zeta_\omega^{-N_\omega/2}\times G_R(p;\lambda^R,1,1,1),$$

so that, for example, N_A can be replaced by the differential operator $-2\partial/\partial\zeta_A$. These remarks lead directly to the following correspondences:

$$N_A\to-i\left\{N\left[A_{\alpha\mu}\frac{\delta\mathcal{L}_A}{\delta A_{\alpha\mu}}\right]^-+2\mathcal{L}^-_{gf}-g_R c_{\alpha\beta\gamma}N[\bar\omega_\alpha\overleftarrow{\partial}^\mu A_{\gamma\mu}\omega_\beta]^-+g_R N[\bar\psi\not A_\alpha t_\alpha\psi]^-\right\}, \quad (3.9)$$

$$N_\psi\to-2i\mathcal{L}^-_F, \quad (3.10)$$

$$N_\omega\to-2i\mathcal{L}^-_G, \quad (3.11)$$

$$g_R\frac{\partial}{\partial g_R}\to iN\left[A_{\alpha\mu}\frac{\delta\mathcal{L}_A}{\delta A_{\alpha\mu}}\right]^--2i\mathcal{L}^-_A-ig_R c_{\alpha\beta\gamma}N[\bar\omega_\alpha\overleftarrow{\partial}^\mu A_{\gamma\mu}\omega_\beta]^-+ig_R N[\bar\psi\not A_\alpha t_\alpha\psi]^-, \quad (3.12)$$

$$\xi_R\frac{\partial}{\partial\xi_R}\to-i\mathcal{L}^-_{gf}, \quad (3.13)$$

$$m_R\frac{\partial}{\partial m_R}\to-iN[\bar\psi m_R\psi]^-, \quad (3.14)$$

In deriving (3.9) and (3.12), it is convenient to note the following scaling properties of $\mathcal{L}_A$:

$$\left.\frac{\partial\mathcal{L}_A}{\partial\zeta_A}\right|_{\zeta_A=1}=\mathcal{L}_A+\tfrac{1}{2}g_R\frac{\partial\mathcal{L}_A}{\partial g_R}, \quad (3.15a)$$

$$\left.\frac{\partial\mathcal{L}^-_A}{\partial\zeta_A}\right|_{\zeta_A=1}=\tfrac{1}{2}\int d^4x\,A_{\alpha\mu}(x)\frac{\delta\mathcal{L}^-_A}{\delta A_{\alpha\mu}(x)}. \quad (3.15b)$$

Substituting (3.9)–(3.14) in (3.6) gives

$$-(n-4)\mathcal{L}^-_{\text{eff}}=-\frac{2\beta}{g_R}\mathcal{L}^-_A+\gamma_m N(\bar\psi m_R\psi)^-+\left(\frac{\beta}{g_R}-\frac{\gamma_3}{2}\right)\left\{N\left[A_{\alpha\mu}\frac{\delta\mathcal{L}_A}{\delta A_{\alpha\mu}}\right]^--g_R c_{\alpha\beta\gamma}N(\bar\omega_\alpha\overleftarrow{\partial}^\mu A_{\gamma\mu}\omega_\beta)^-+g_R N(\bar\psi\not A_\alpha t_\alpha\psi)^-\right\}-\gamma_2\mathcal{L}^-_F-\tilde\gamma\mathcal{L}^-_G. \quad (3.16)$$

Note the appearance of non-gauge-invariant operators and ghost terms in the anomaly. In fact, it is known[9] that the following set of bare operators are the *only* ones to mix under renormalization with the gauge-invariant operator $O_1^b\equiv-\frac{1}{4}F_{\alpha\mu\nu}F_\alpha^{\mu\nu}$:

$$\begin{aligned}
&O_2^b\equiv A_{\alpha\mu}\frac{\delta\mathcal{L}^-_A}{\delta A_{\alpha\mu}}-\bar\omega_\alpha\overleftarrow{\partial}^\mu\partial_\mu\omega_\alpha+g_0\bar\psi\not A_\alpha t_\alpha\psi,\\
&O_3^b\equiv-(\Box\bar\omega_\alpha)\omega_\alpha-g_0 c_{\alpha\beta\gamma}\bar\omega_\alpha\overleftarrow{\partial}^\mu A_{\gamma\mu}\omega_\beta,\\
&O_4^b\equiv\bar\psi(i\not D-m_0)\psi,\\
&O_5^b\equiv\bar\psi m_0\psi.
\end{aligned} \quad (3.17)$$

Furthermore, this set is closed under renormalization to all orders. Defining corresponding renormalized normal-produce operators, we find that (3.16) becomes simply

$$(\theta^\mu_\mu)^-_{\rm anom} = -\frac{2\beta}{g_R}N[O_1]^- + \left(\frac{\beta}{g_R} - \frac{\gamma_3}{2}\right)N[O_2+O_3]^- -\tilde{\gamma}N[O_3]^- - \gamma_2 N[O_4]^- + \gamma_m N[O_5]^- . \tag{3.18}$$

If (as will be demonstrated in Sec. IV) θ^μ_μ at arbitrary momenta is a linear combination of just these operators, we may remove the restriction to zero momentum in (3.18):

$$(\theta^\mu_\mu)_{\rm anom} = -\frac{2\beta}{g_R}N[O_1] + \left(\frac{\beta}{g_R} - \frac{\gamma_3}{2}\right)N[O_2] + \left(\frac{\beta}{g_R} - \frac{\gamma_3}{2} - \tilde{\gamma}\right)N[O_3] - \gamma_2 N[O_4] + \gamma_m N[O_5] . \tag{3.19}$$

In particular, on the mass shell (*with* external wave functions) and at nonzero momentum, $N[O_3]$ and $N[O_4]$ vanish by the ghost and fermion equations of motion. It can also be shown[16] that $N[O_2]$ vanishes on-shell at nonzero momentum. Under these circumstances, therefore,

$$(\theta^\mu_\mu)_{\rm anom}\big|_{\substack{\text{on-shell}\\ q\neq 0}} = -\frac{2\beta}{g_R}N[O_1] + \gamma_m N[O_5] = +\frac{\beta}{2g_R}N[F_{\alpha\mu\nu}F^{\mu\nu}_\alpha] + \gamma_m N[\bar\psi m_R \psi] . \tag{3.20}$$

Putting this result together with Eq. (2.20) we obtain the following simple result for the on-mass-shell full trace[17] at nonzero momentum:

$$\theta^\mu_\mu\big|_{\substack{\text{on-shell}\\ q\neq 0}} = (1+\gamma_m)N[\bar\psi m_R\psi] + \frac{\beta}{2g_R}N[F_{\alpha\mu\nu}F^{\mu\nu}_\alpha] . \tag{3.21}$$

Finally, we note that the recently obtained result[8] for the anomalous trace of the Belinfante tensor in QED is obtainable directly from (3.19) by setting $\beta = \frac{1}{2}g_R\gamma_3$, $\tilde\gamma = 0$. Also, it is amusing to note that, by (3.21), the eigenvalue condition $\beta(g_R)=0$ is a necessary and sufficient condition for the restoration of softness of the trace in on-mass-shell matrix elements, though not [because of the appearance of the dimension-four operator $N[O_2]$) in the general Green's functions.

IV. EXTENSION TO ARBITRARY MOMENTUM

Here we shall derive the form of the anomaly in $\tilde\theta^\mu_\mu(q)$ valid at arbitrary momentum q. This form will determine the set of linearly independent operators that can be present in the anomaly: Comparison with the known value of the anomaly at $q=0$ as given by Eq. (3.18) will determine the coefficients of all the operators and thus establish Eq. (3.19). The method is a minor modification of that employed in Ref. 8.

The terms in $\theta^\mu_\mu(x)$ proportional to $(n-4)$ are

$$-(n-4)\{\mathcal{L}_{\rm eff}(x) + \xi_0^{-1}\partial^\mu[A^\alpha_\mu(x)\partial^\nu A^\alpha_\nu(x)]\} \equiv -(n-4)\Theta(x) . \tag{4.1}$$

$\Theta(x)$ has a simple transformation property under the Becchi-Rouet-Stora (BRS) superfield transformations[18]:

$$\begin{aligned}
\delta A^\alpha_\mu(x) &= [\partial_\mu\omega_\alpha(x) + g_0 c_{\alpha\beta\gamma}A^\beta_\mu(x)\omega_\gamma(x)]\delta\lambda \\
&\equiv D^{\alpha\beta}_\mu\omega_\beta(x)\delta\lambda ,\\
\delta\omega_\alpha(x) &= \tfrac{1}{2}g_0 c_{\alpha\beta\gamma}\,\omega_\beta\omega_\gamma(x)\delta\lambda ,\\
\delta\bar\omega_\alpha(x) &= \frac{1}{\xi_0}\partial_\nu A^\nu_\alpha(x)\delta\lambda ,\\
\delta\psi(x) &= ig_0 t_\alpha\psi(x)\omega_\alpha(x)\delta\lambda ,\\
\delta\bar\psi(x) &= -ig_0\bar\psi(x)t_\alpha\omega_\alpha(x)\delta\lambda ,
\end{aligned} \tag{4.2}$$

where $\delta\lambda$ is an x-independent anticommuting c number. These transformations satisfy

$$0 = \delta(D^{\alpha\beta}_\mu\omega_\beta) = \delta(-\tfrac{1}{2}c_{\alpha\beta\gamma}\omega_\beta\omega_\gamma) = \delta(t_\alpha\psi\omega_\alpha) = \delta(\bar\psi t_\alpha\omega_\alpha) . \tag{4.3}$$

The gauge-invariant part of $\mathcal{L}_{\rm eff}$ is clearly invariant under the BRS transformations, as they are just a special gauge transformation on $A, \psi, \bar\psi$. Furthermore, one finds

$$\delta\left(-\frac{1}{2\xi_0}(\partial^\nu A^\alpha_\nu)^2 - \bar\omega_\alpha\bar\partial^\mu D^{\alpha\beta}_\mu\omega_\beta\right) = 0 , \tag{4.4}$$

$$\delta\left(\frac{1}{\xi_0}A^\alpha_\mu\partial^\nu A^\alpha_\nu + \bar\omega_\alpha D^{\alpha\beta}_\mu\omega_\beta\right) = \frac{1}{\xi_0}A^\alpha_\mu\partial^\nu(D^{\alpha\beta}_\nu\omega_\beta)\delta\lambda , \tag{4.5}$$

which lead immediately to

$$\delta\Theta = \frac{1}{\xi_0}\partial^\mu(A^\alpha_\mu\partial^\nu D^{\alpha\beta}_\nu\omega_\beta)\delta\lambda . \tag{4.6}$$

Consider now the generating functional

$$W'[\mathcal{J}^0, K^0, H^0, \bar H^0, L^0, M] = \int (dA\, d\omega\, d\bar\omega\, d\psi\, d\bar\psi)\exp\left\{i\left[\bar S + \int d^n x\, M(x) + \text{source terms for fields}\right]\right\}, \tag{4.7}$$

where the source terms for fields are identical to those in the exponent in Eq. (2.1), and $\bar{S}$ contains sources for the BRS invariants of Eq. (4.3),

$$\bar{S} \equiv S_{eff} + \int d^n x [K^{0\mu}_\alpha D^{\alpha\beta}_\mu \omega_\beta(x) + ig_0(\bar{H}^0 t_\alpha \psi - \bar{\psi} t_\alpha H^0)\omega_\alpha(x) + \tfrac{1}{2} L^0_\alpha g_0 c_{\alpha\beta\gamma}\omega_\beta\omega_\gamma(x)] , \tag{4.8}$$

and is fully invariant under the BRS transformations.

Now consider the effect of BRS transformations on the integration variables of W'. The Jacobian of the transformation is unity. Furthermore, the value of W' should not change under the change of integration variables. This leads to

$$0 = \int (dA\, d\omega\, d\bar{\omega}\, d\psi\, d\bar{\psi}) \int d^n x \Big\{ J^0_{\alpha\mu}(x) \frac{\delta}{\delta K^0_{\alpha\mu}(x)} + \bar{\chi}^0_\alpha(x) \frac{\delta}{\delta L^0_\alpha(x)} - \frac{i}{\xi_0}[\partial_\nu A^\nu_\alpha(x)]\chi^0_\alpha(x)$$
$$+ \bar{\eta}^0(x) \frac{\delta}{\delta \bar{H}^0(x)} - \frac{\delta}{\delta H^0(x)}\eta^0(x) - \frac{i}{\xi_0} M(x)\partial^\mu[A^\alpha_\mu(x)\chi^0_\alpha(x)] \Big\} \exp[i(\bar{S} + \cdots)] . \tag{4.9}$$

The antighost equation of motion has been used to replace $\partial^\nu D^{\alpha\beta}_\nu \omega_\beta(x)$ by $-\chi^0_\alpha(x)$ in the last term in curly brackets. Had we had a gauge-invariant operator multiplying $M(x)$ instead of $\Theta(x)$, this term would have been absent. Thus the Ward-Takahashi (WT) identity satisfied by W' is identical to that satisfied by a generating functional for insertions of a gauge-invariant operator, *except* for this term. Now we can, as usual, define the generating functional for proper vertices Γ' corresponding to W', by making a Legendre transformation $Z' \equiv -i \ln W'$:

$$\Gamma'(a^0, \Omega^0, \bar{\Omega}^0, \Psi^0, \bar{\Psi}^0, K^0, \text{etc.}) \equiv Z' - \int d^n x (J_{\alpha\mu} a^{0\mu}_\alpha + \bar{\eta}^0 \Psi^0 + \bar{\Psi}^0 \eta^0 + \bar{\chi}^0_\alpha \Omega^0_\alpha + \bar{\Omega}^0_\alpha \chi^0_\alpha) , \tag{4.10}$$

with $a^0_\alpha(x) \equiv \delta Z'/\delta J^\mu_\alpha(x)$, etc. For any source K not entering the Legendre transformation, we have $\delta Z'/\delta K|_{J^0,\dots} = \delta\Gamma'/\delta K|_{a^0,\dots}$. Thus Eq. (4.9) becomes

$$\int d^n x \Big\{ \frac{\delta\Gamma'}{\delta a^{0\mu}_\alpha(x)} \frac{\delta\Gamma'}{\delta K^0_{\alpha\mu}(x)} - \frac{\delta\Gamma'}{\delta \Omega^0_\alpha(x)} \frac{\delta\Gamma'}{\delta L^0_\alpha(x)} - \frac{1}{\xi_0}[\partial_\nu a^{0\nu}_\alpha(x)] \frac{\partial\Gamma'}{\delta\bar{\Omega}^0_\alpha(x)}$$
$$- \frac{\delta\Gamma'}{\delta\Psi^0(x)} \frac{\delta\Gamma'}{\delta\bar{H}^0(x)} - \frac{\delta\Gamma'}{\delta H^0(x)} \frac{\delta\Gamma'}{\delta\bar{\Psi}^0(x)} \Big\} = \frac{1}{\xi_0} \int d^n x\, \partial^\mu \Big[a^0_{\alpha\mu}(x) \frac{\delta\Gamma'}{\delta\bar{\Omega}^0_\alpha(x)} \Big] M(x) . \tag{4.11}$$

Differentiating with respect to $M(y)$ and setting $M = 0$,

$$\int d^n x \Big[\frac{\delta\Gamma'}{\delta a^{0\mu}_\alpha(x)} \frac{\delta}{\delta K^0_{\alpha\mu}(x)} + \frac{\delta\Gamma'}{\delta K^0_{\alpha\mu}(x)} \frac{\delta}{\delta a^{0\mu}_\alpha(x)} + \cdots - \frac{1}{\xi_0}\partial_\nu a^{0\nu}_\alpha(x) \frac{\delta}{\delta\bar{\Omega}^0_\alpha(x)} \Big] \frac{\delta\Gamma'}{\delta M(y)} \Big|_{M=0} = \frac{1}{\xi_0}\partial^\mu \Big[a^0_{\alpha\mu}(y) \frac{\delta\Gamma'}{\delta\bar{\Omega}^0_\alpha(y)} \Big] . \tag{4.12}$$

The dotted terms $(\cdots)$ involve "conjugate" pairs $(\Omega^0_\alpha, L^0_\alpha)$, $(\Psi^0, \bar{H}^0)$, $(\bar{\Psi}^0, H^0)$. The sources $K^0, L^0, H^0, \bar{H}^0$ are all multiplicatively renormalized such that when each term on the left-hand side is expressed in terms of renormalized quantities, a factor of $(Z_3\tilde{Z})^{-1/2}$ can be extracted uniformly.
For example,

$$\frac{\delta\Gamma'}{\delta a^0_{\alpha\mu}} \frac{\delta\Gamma'}{\delta K^0_{\alpha\mu}} = \frac{\delta\Gamma'}{\delta a^R_{\alpha\mu}} \frac{\delta\Gamma'}{\delta K^R_{\alpha\mu}} (Z_3\tilde{Z})^{-1/2} . \tag{4.13}$$

But the right-hand side of Eq. (4.12) is

$$\frac{1}{\xi_0}\partial^\mu \Big(a^0_{\alpha\mu} \frac{\delta\Gamma'}{\delta\bar{\Omega}^0_\alpha} \Big) = \frac{1}{\xi_R} Z_3^{-1} \partial^\mu \Big(Z_3^{1/2} a^R_{\alpha\mu} \tilde{Z}^{-1/2} \frac{\delta\Gamma'}{\delta\bar{\Omega}^R_\alpha} \Big)$$
$$= (Z_3\tilde{Z})^{-1/2} \frac{1}{\xi_R}\partial^\mu \Big(a^R_{\alpha\mu} \frac{\delta\Gamma'}{\delta\bar{\Omega}^R_\alpha} \Big) . \tag{4.14}$$

Thus, expressed in terms of renormalized quantities, Eq. (4.12) has the form

$$\int d^n x \Big[\frac{\delta\Gamma'}{\delta a^{R\mu}_\alpha(x)} \frac{\delta}{\delta K^R_{\alpha\mu}(x)} + \cdots \Big] \frac{\delta\Gamma'}{\delta M(y)} \Big|_{M=0} = \text{finite functional.} \tag{4.15}$$

But this WT identity is identical in form to that for a gauge-invariant operator except that the right-hand side is finite instead of zero. Thus the one-loop divergence in

$$\frac{\delta\Gamma[a^R, \Omega^R, \bar{\Omega}^R]}{\delta M(x)} \Big|_{K^R = L^R = H^R = \bar{H}^R = N = 0}$$

can be expressed in terms of the field operators that can mix under renormalization with a dimension-four gauge-invariant operator that is a Lorentz scalar and has even charge conjugation. But they are precisely the five operators listed in Eq. (3.17).[9] In other words,

$$\left\{\frac{\delta\Gamma[a^R,\Omega^R,\bar{\Omega}^R]}{\delta M(x)}\bigg|_{K^R=L^R=H^R=\bar{H}^R=M=0}\right\}_1^{\mathrm{div}} = \sum b_i^{(1)}(n,g,\xi)O_i[a^R,\Omega^R,\bar{\Omega}^R;g]\,, \quad (4.16)$$

where $b_i(n,g,\xi)$ may be divergent as $n\to 4$. Therefore, we can write

$$\{\langle\Theta(x)\rangle\} = \text{finite terms} + \left\{\sum_i b_i^{(1)}\langle O_i[A,\omega,\bar{\omega},g_0,\xi_0]\rangle\right\}. \quad (4.17)$$

The above arguments can be extended to higher orders in the same way as done for gauge-invariant operators in Sec. III of Ref. 9. Basically, the argument consists in showing that the above set of operators that appear as renormalization counterterms for $\Theta(x)$ themselves satisfy the homogeneous WT identity [i.e., Eq. (4.15) with the right-hand side set equal to zero.] With such an argument one can conclude that the overall divergence in $[\delta\Gamma/\delta M(x)]$ in the N-loop ($N>1$) approximation has the same form as that given by Eq. (4.16). In other words, we can write

$$\left\langle\Theta(x) - \sum_{i=1}^{5}\sum_{l=1}^{N-1} b_i^{(l)}a^l O_i[A,\omega,\bar{\omega},g_0]\right\rangle = \left\langle\sum_{i=1}^{5} b_i^{(N)}a^N O_i[A,\omega,\bar{\omega},g_0]\right\rangle + \text{finite terms},$$

i.e.,

$$\langle\Theta(x)\rangle_{gR} = \left\langle\sum_{i=1}^{5} b_i O_i[A,\omega,\bar{\omega},g_0]\right\rangle_{gR} + \text{finite terms} \quad (4.18)$$

valid up to an N-loop approximation. Here the b_i are polynomials in $1/(n-4)$. Now using the fact that this set of operators is closed under renormalization we can write

$$\langle O_i\rangle = \sum_j Z_{ij} N[O_j[A,\omega,\bar{\omega},g]]\,, \quad (4.19)$$

where Z_{ij} has an expansion in a and $1/(n-4)$. Substituting Eq. (4.19) into Eq. (4.18), we can rewrite Eq. (4.18) as

$$\langle\Theta(x)\rangle_g = \sum_{i=1}^{5} b_i'\langle N[O_i[A,\omega,\bar{\omega},g]]\rangle + \text{finite terms}. \quad (4.20)$$

We are interested in the limit $(n\to 4)$ of

$$-(n-4)\langle\Theta(x)\rangle_g = -\sum_{i=1}^{5}(n-4)b_i'\langle N[O_i(x)]\rangle_g - (n-4)\times(\text{finite terms}). \quad (4.21)$$

Note that the above equation is valid at each x and hence for $\int\Theta(x)d^nx$, i.e., for the zero-momentum insertion case worked out in the previous section. Comparison of Eq. (4.21) in the limit $(n\to 4)$ with Eq. (3.18) tells us that the quantities $\lim_{n\to 4}(n-4)b_i'$ are finite and given by the coefficients of the respective operators in Eq. (3.18). We thus establish Eq. (3.19) for an arbitrary momentum q (i.e., at each x). It should be noted that the extension from the $q=0$ to the $q\neq 0$ case has been possible because the Green's functions of $N[O_i]$ for the five operators in Eq. (4.21) remain linearly independent at $q=0$.

V. ANOMALIES IN CURRENT MATRIX ELEMENTS

In this section, we shall discuss the dilatation anomalies which appear in matrix elements of current operators, in order to clarify the content of the "partially conserved dilatation current" (PCDC) hypothesis.[5] Specifically, we shall consider the electromagnetic current in a theory with fermions interacting via non-Abelian gauge gluons. The electromagnetic interaction will be treated to lowest nontrivial order, and the strong gauge interaction to all orders throughout. We shall find that the dilatation anomaly (to be defined below) in the two-current matrix element of $\theta_\lambda^{\lambda-}$, although renormalized infinitely in any *finite* order of the strong interactions, is completely *unrenormalized* provided we sum to all orders in g_R (the renormalized strong-coupling constant) before passing to the limit of physical dimensions, $n=4$.

Consider the matrix element of θ_λ^λ with two renormalized photon fields: Using gauge invariance, we have

$$p_1^2p_2^2\langle 0|T\tilde{A}_\mu^R(p_1)A_\nu^R(0)\tilde{\theta}_\lambda^\lambda(q)|0\rangle \equiv (p_1\cdot p_2 g_{\mu\nu} - p_{1\nu}p_{2\mu})V_1(p_1,p_2) + (p_{1\mu}p_{1\rho} - g_{\mu\rho}p_1^2)(p_2^\rho p_{2\nu} - g_\nu^\rho p_2^2)V_2(p_1,p_2), \quad (5.1)$$

where $q \equiv -(p_1+p_2)$. We now employ the methods of Sec. III to relate the zero-momentum insertion of $\theta_\lambda^{\lambda-}$ into the renormalized photon propagator itself:

$$\langle 0|T\tilde{A}_\mu^R(p)A_\nu^R(0)\theta_\lambda^{\lambda-}|0\rangle = -i\left[\beta_e\frac{\partial}{\partial e_R}+\beta_g\frac{\partial}{\partial g_R}-(1+\gamma_m)m_R\frac{\partial}{\partial m_R}-\gamma_3\xi_R\frac{\partial}{\partial \xi_R}+\gamma_3\right]\langle 0|T\tilde{A}_\mu^R(p)A_\nu^R(0)|0\rangle$$

$$= -i\left(p^\mu\frac{\partial}{\partial p^\mu}+2\right)\langle 0|T\tilde{A}_\mu^R(p)A_\nu^R(0)|0\rangle\,, \tag{5.2}$$

where we have used the renormalization-group equation for the renormalized photon propagator. Writing the latter (to order e_R^2) as

$$D_{\mu\nu}^R(p)=\frac{g_{\mu\nu}-p_\mu p_\nu/p^2}{p^2[1+e_R^2\Pi_R(p,\mu,m_R,g_R,n)]}+\xi_R\frac{p_\mu p_\nu}{(p^2)^2}\,, \tag{5.3}$$

we have

$$\langle 0|T\tilde{A}_\mu^R(p)A_\nu^R(0)\theta_\lambda^{\lambda-}|0\rangle=\frac{1}{(p^2)^2}(p_\mu p_\nu-g_{\mu\nu}p^2)\times e_R^2p_\rho\frac{\partial}{\partial p_\rho}\Pi_R\,. \tag{5.4}$$

Together with Eq. (5.1) (at $q=0$), Eq. (5.4) implies

$$V_1(p,-p)-p^2V_2(p,-p)=e_R^2p_\mu\frac{\partial}{\partial p_\mu}\Pi_R(p,\mu,m_R,g_R,n)\,. \tag{5.5}$$

Now suppose we attempt to saturate $\bar{\theta}_\lambda^\lambda$ in the mass-shell matrix element $\langle p_1\epsilon_1,p_2\epsilon_2|\bar{\theta}_\lambda^\lambda(q)|0\rangle$ with a single ϵ ("dilaton") pole contribution. Only $V_1(p_1,p_2)$ is relevant to this matrix element, whereas the relation (5.5) involves the combination $V_1-p^2V_2$. However, at $p^2=0$, the right-hand side of (5.5) vanishes (for nonvanishing bare quark masses), implying that $V_1(p,-p)|_{p^2=0}$ is zero. In other words, the use of the *total* energy-momentum tensor $\bar{\theta}_\lambda^\lambda(q)$ to extrapolate the ϵ field off-shell is incompatible with the assumption of maximal smoothness for the ϵ-2 photon coupling (i.e., an effective $\epsilon F_{\mu\nu}F^{\mu\nu}$ interaction). This is because the electromagnetic and hadronic contributions in $\bar{\theta}_\lambda^\lambda(q)$ cancel exactly in this matrix element at $q=0$.

The assumption of PCDC is simply that the *hadronic* part of θ_λ^λ is saturated with an ϵ pole. In order to implement this assumption, one must define a *finite* "hadronic part" of θ_λ^λ. We shall now show that to finite orders in strong perturbation theory, it is not possible to unambiguously separate finite hadronic and electromagnetic contributions in θ_λ^λ (although one can certainly *define* a finite hadronic piece, this definition amounts to a specific choice of finite part of the counterterms). However, if one sums to all orders of g_R *before* passing to four dimensions, the hadronic part of the bare trace remains finite and in fact has a completely calculable zero-momentum vacuum to two-photon matrix element, given exactly by the lowest-order result.

To proceed, we first relate the matrix element of $\theta_\lambda^{\lambda-}$ with two currents to that with two unrenormalized fields, as follows:

$$\langle 0|T\tilde{A}_\mu(p)A_\nu(0)\theta_\lambda^{\lambda-}|0\rangle\equiv\int d^4x\,d^4y\,e^{ip\cdot x}\langle 0|TA_\mu(x)A_\nu(0)\theta_\lambda^\lambda(y)|0\rangle$$

$$=-e_0^2\Delta_{\rho\mu}(p)\Delta_{\sigma\nu}(p)\langle 0|T\tilde{j}^\rho(p)j^\sigma(0)\theta_\lambda^{\lambda-}|0\rangle$$

$$-\frac{n-4}{2}[p_\rho D_{\mu\sigma}(p)-p_\sigma D_{\mu\rho}(p)][p^\rho D_\nu^\sigma(p)-p^\sigma D_\nu^\rho(p)]+O(e_0^4)\,. \tag{5.6}$$

Here Δ and D are the free and unrenormalized photon propagators respectively [the latter is computed to $O(e_0^2)$, all orders in g_0]:

$$\Delta_{\rho\mu}(p)=\frac{g_{\rho\mu}-p_\rho p_\mu/p^2}{p^2}+\xi_0\frac{p_\rho p_\mu}{(p^2)^2}\,, \tag{5.7}$$

$$D_{\rho\mu}(p)=\frac{g_{\rho\mu}-p_\rho p_\mu/p^2}{p^2[1+e_0^2\Pi_0(p^2;g_0(n),m_0(n),n)]}+\xi_0\frac{p_\rho p_\mu}{(p^2)^2}$$

$$=[1+e_0^2\mu^{n-4}z_3(g_0(n)\mu^{(n-4)/2},n)]D_{\rho\mu}^R(p)+O(e_0^4)\,, \tag{5.8}$$

where

$$z_3(g_0(n)\mu^{(n-4)/2},n)\equiv-\text{pole part of }[\mu^{4-n}\Pi_0(p^2;g_0(n),m_0(n),n)]\,. \tag{5.9}$$

The second term in Eq. (5.6) arises from graphs in which the electromagnetic part of θ^λ_λ (expressed in terms of bare fields), namely $\frac{1}{4}(n-4)F_{\mu\nu}F^{\mu\nu}$, inserts directly onto an external photon leg. The graphical alternatives are summarized in Fig. 1.

The renormalized form of Eq. (5.6) is just

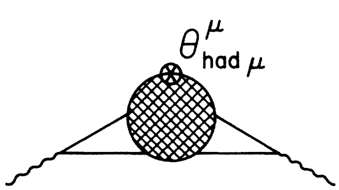
$$\langle 0|T\tilde{A}^R_\mu(p)A^R_\nu(0)\theta^{\lambda\sim}_\lambda|0\rangle = -e_R{}^2\frac{1}{(p^2)^2}\langle 0|T\tilde{j}_\mu(p)j_\nu(0)\theta^{\lambda\sim}_\lambda|0\rangle$$
$$-e_R{}^2(n-4)\frac{1}{(p^2)^2}(g_{\mu\nu}p^2-p_\mu p_\nu)z_3(g_0(n)\mu^{(n-4)/2},n)+O(e_R{}^4)$$
$$+(\text{terms vanishing at } n=4)\,. \tag{5.10}$$

The left-hand side of Eq. (5.10) can be related via Eq. (5.4) to the renormalized photon self-energy, so finally we obtain

$$\langle 0|T\tilde{j}_\mu(p)j_\nu(0)\theta^{\lambda\sim}_\lambda|0\rangle = (g_{\mu\nu}p^2-p_\mu p_\nu)\left[p_\lambda\frac{\partial}{\partial p_\lambda}\Pi_R(p^2)-(n-4)z_3(g_0(n)\mu^{(n-4)/2},n)\right]. \tag{5.11}$$

The last term on the right-hand side of (5.11) corresponds precisely, in zeroth order of g_0, to the "canonical trace anomaly" of Chanowitz and Ellis.[6] We will refer to it as the *dilatation* anomaly, to distinguish it from the anomaly in the trace of the energy-momentum tensor itself, discussed in the preceding sections. Standard renormalization-group arguments[19] show that the dilatation anomaly is finite up to order $g_R{}^2$, but diverges in order $g_R{}^4$ (owing to a double pole in z_3) and all higher orders. This, of course, implies that the two-current matrix element of θ^λ_λ is divergent[20] in perturbation theory: It is, in fact, subtractively renormalizable by precisely the divergence of the anomaly. We conclude that the assumption of PCDC is nonsense *in perturbation theory*.

Nevertheless, it is possible to resurrect the PCDC assumption on a purely nonperturbative basis, as we now demonstrate. Let Z_3 be the complete photon wave-function renormalization constant:



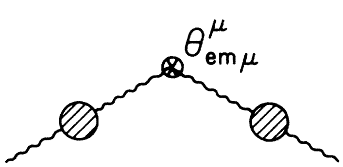


FIG. 1. (a) Insertion of the hadronic part of the bare trace in a two-photon matrix element. (b) Insertion of the electromagnetic part of the bare trace.

$$Z_3(g_0(n)\mu^{(n-4)/2}, e_0(n)\mu^{(n-4)/2},n)$$
$$=1+e_R{}^2z_3(g_R,n)+O(e_R{}^4)\,. \tag{5.12}$$

Thus

$$\gamma_3(e_R,g_R)\equiv\mu\frac{\partial}{\partial\mu}\ln Z_3$$
$$\equiv e_R{}^2\gamma(g_R)+O(e_R{}^4)\,. \tag{5.13}$$

Here $\gamma(g_R)$ is of order unity as $g_R\to 0$.

Note that with minimal subtraction,[12] γ_3, and hence γ, are independent of n. Let $\beta_g(g_R)$ be the strong-interaction β function in the theory with e_0 set equal to zero:

$$\beta_g(g_R)\equiv\mu\frac{\partial}{\partial\mu}g_R\bigg|_{e_0=0;\ g_0,n\ \text{fixed}}\,. \tag{5.14}$$

One easily derives the following inhomogeneous renormalization-group equation for z_3:

$$\beta_g(g_R,\epsilon)\frac{\partial z_3(g_R,n)}{\partial g_R}+\epsilon z_3(g_R,n)=\gamma(g_R)\,, \tag{5.15}$$

where $\epsilon\equiv n-4$. For fixed $n\neq 4$, we may assume $\lim_{g\to 0}g^2z_3(g,n)=0$. We also have[12]

$$\beta_g(g_R,\epsilon)=\tfrac{1}{2}g_R[\epsilon+\tilde{\gamma}(g_R)]\,, \tag{5.16}$$

where $\tilde{\gamma}(g_R)$ is of order $g_R{}^2$ for g_R approaching zero. The solution of (5.15) is

$$z_3(g_R,n)=\int_0^{g_R}dg\frac{\gamma(g)}{\beta_g(g,\epsilon)}$$
$$\times\exp\left[-\epsilon\int_g^{g_R}\frac{dg'}{\beta_g(g',\epsilon)}\right]. \tag{5.17}$$

We shall assume (as suggested by scaling[21]) that the physical renormalized coupling g_R is in the domain of attraction of the origin, so that there are no zeros of $\tilde{\gamma}(g)$ in the interval $(0, g_R)$.

Now recall that the anomaly we are interested in is just $\lim_{\epsilon\to 0}\epsilon z_3(g_R, 4+\epsilon)$. However, a Taylor expansion in g_R will inevitably commit us to an expansion around the point at *infinity* in the ϵ plane:

$$\frac{\epsilon}{\beta_g(g_R,\epsilon)} = \frac{2}{g_R}\left[1+\frac{1}{\epsilon}\tilde{\gamma}(g_R)\right]^{-1} = \frac{2}{g_R}\left(1-\frac{1}{\epsilon}\tilde{\gamma}+\frac{1}{\epsilon^2}\tilde{\gamma}^2-\cdots\right),$$

whereas we are interested in the behavior of $z_3(g_R,\epsilon)$ for *small* ϵ. Inspection of Eq. (5.17) shows only the integration regions $g, g' \to 0$ can be responsible[12] for any singularities of z_3 as $\epsilon\to 0$. Specifically, the region of interest is (given the assumptions on $\gamma, \tilde{\gamma}$ stated above) $g^2\sim\epsilon$, $g'^2\sim g^2\sim\epsilon$. Thus, we can neglect the terms of order g_R^4 and higher in $\tilde{\gamma}$ and those of order g_R^2 and higher in γ in Eq. (5.17), as far as the leading singularity of z_3 as $\epsilon\to 0$ is concerned. Consequently, defining $\gamma_0\equiv\gamma(0)$, $a\equiv\lim_{g\to 0}(1/g^2)\tilde{\gamma}(g)$,

$$z_3(g_R, 4+\epsilon)\underset{\epsilon\to 0}{\sim}\int_0^{g_R} dg\,\frac{2\gamma_0}{g(\epsilon+ag^2)} \times\exp\left[-\epsilon\int_g^{g_R}\frac{2dg'}{g'(\epsilon+ag'^2)}\right] = \frac{\gamma_0}{\epsilon},$$

which yields the *exact* result for the anomaly

$$\lim_{\epsilon\to 0}\epsilon z_3(g_R, 4+\epsilon)=\gamma_0\,. \tag{5.18}$$

This concludes the proof of nonrenormalization of the dilatation anomaly.[22] It should be emphasized that, in contrast to the situation with the triangle anomaly, the result (5.18) is necessarily nonperturbative, and *cannot* be checked in any finite order of perturbation theory.[20] This is a consequence of the highly nonuniform (in ϵ) character of the expansion in powers of g_R.

This result does not, of course, circumvent the usual physical objections to the PCDC hypothesis. The enormous width of the ϵ ensures that the assumption that the zero-momentum trace $\theta_\lambda^{\lambda-}$ couples to an arbitrary state via a single pole term is crudely qualitative at best. Furthermore, in contrast to the case with partial conservation of axial-vector current (PCAC), there is not even a fictional world (for PCAC, that of massless bare quarks and pions) where exact calculations can be performed. If we are not at an eigenvalue of the Gell-Mann–Low function, then even in the limit of zero bare quark masses dilatation invariance is explicitly broken, so there is no Goldstone limit in which spontaneous breakdown of dilatation invariance would force the appearance of a zero-mass particle. Nevertheless, one could adopt the PCDC assumption as a testable qualitative hypothesis (at least in principle) provided that one further difficulty can be resolved. Namely, the magnitude of the coupling of the trace to the ϵ pole is undetermined theoretically, so one is forced to consider,[5,6] in addition to $\epsilon\to 2\gamma$, processes such as $\epsilon\to 2\pi$ which involve the dilatation anomalies of chiral current matrix elements. However, chiral currents are notoriously difficult to handle in a dimensional renormalization scheme, which has been central to our analysis. It remains to be seen whether a canonical dilatation anomaly will also emerge nonperturbatively in such cases.[23]

Note Added. After completion of this paper, we received a report[24] from N. K. Nielsen obtaining similar results.

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APPENDIX

Here we shall deal with the term

$$\partial^\rho[\xi_0^{-1}A_\rho^\alpha\partial^\nu A_\nu^\alpha+\bar{\omega}_\alpha(\partial_\rho\omega_\alpha-g_0c_{\alpha\beta\gamma}A_\rho^\gamma\omega_\beta)] \tag{A1}$$

in Eq. (2.15).

We shall, first, give a simple proof using the discussion of Sec. IV that all the renormalization parts of the above quantity are finite without additional subtractions. Consider the discussion of Sec. IV for $\Theta(x)$. We note from Eqs. (4.4) and (4.5) that the only part of $\Theta(x)$ that is not BRS invariant is precisely the expression in (A1). So the whole discussion for the WT identity for $\Theta(x)$ and its renormalization applies to this piece as well. Hence this implies that the renormalization counterterms for this operator can only be those five operators listed in (3.21). But the expression (A1), and hence its renormalization counterterms themselves, is a four-divergence. But there are no such (linear combinations of) operators in (3.21). Hence the proper vertices of (A1) must be finite once the wave-function and coupling-constant renormalizations are done.

Next consider the Green's functions of (A1) with no external ghosts. They can be expressed via the WT identity of Eq. (2.8) in a simpler form, after a straightforward manipulation. The result is

$$\langle \partial^\rho\{\xi_0^{-1}A^\alpha_\rho(x)\partial^\nu A^\alpha_\nu(x)+\bar\omega_\alpha(x)[\partial_\rho\omega_\alpha(x)-g_0c_{\alpha\beta\gamma}A^\gamma_\rho(x)\omega_\beta(x)]\}\rangle$$

$$=\partial^\rho[\bar\omega_\alpha(x)A^\alpha_\rho(x)]\int d^ny\{J^0_{\beta\mu}(y)[\partial_\mu\omega_\beta(y)-ig_0c_{\alpha\beta\gamma}A^\gamma_\mu(y)\omega_\alpha(y)]+ig_0[\bar\eta(y)t^\alpha\psi(y)-\bar\psi(y)t^\alpha\eta(y)]\omega_\alpha(y)\}. \quad \text{(A2)}$$

With the help of Eq. (A2) it is easy to see that the truncated Green's functions of the left-hand side (with physical wave functions attached) vanish for $q\neq 0$, owing to the lack of a single-particle pole in one of the external lines.

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[20]An opposite result that there are no two-loop corrections to the anomaly, is found by H. Sato, Univ. of Minnesota, report (unpublished). However, his Eq. (3.19) appears to be in error: Since Sato's renormalized coupling constant g has dimension $2-\frac{1}{2}n$, this equation needs an extra term $(2-\frac{1}{2}n)g\partial\Pi_{\mu\nu}/\partial g$ in order to agree with the equation of dimensional analysis.

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[22]A corresponding result can be derived in QED, so at first sight the dilatation anomaly would also appear to be unrenormalized there. However, in QED, $\beta=\frac{1}{2}(n-4)g_R+g_R^3/(12\pi^2)+O(g_R^5)$ and has an infrared fixed point at $g_R^2\approx 6\pi^2(4-n)$. Initially we consider the theory with $n<4$ and g_R below this IR fixed point, so its UV behavior is governed by the origin. The physical theory at $n=4$ has $g_R^2>0$ and so lies *above* the IR fixed point. The UV behavior is no longer governed by the origin. Since the construction of the theory crucially involves the UV renormalization, it is not clear that continuation of the theory above the IR fixed point has any relevance to physics. See also D. Gross, Erice lectures, 1976 (unpublished).

[23]Note that Chanowitz and Ellis (Ref. 6) argue that the calculation of $\epsilon\to 2\pi$ is not affected by the dilation anomaly, at least if it is canonical.

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