

Classification of pseudoparticle solutions in gauge theories

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We discuss the classification of pseudoparticle solutions obtained using all inequivalent embeddings of SU(2) in a semisimple algebra G . We also discuss some simple consequences of this classification on the interaction of the pseudoparticles of Belavin *et al.*

I. INTRODUCTION

It has been suggested by Belavin *et al.*¹ and more recently by Wilczek² that one may construct pseudoparticle³ solutions of the gauge field equations in the group G by embedding the group SU(2) in G . Owing to the fact that there exist inequivalent embeddings of SU(2) in SU(N) ($N > 2$), Wilczek was able to draw some conclusions about the 4-particle interactions from pseudoparticle solutions in SU(3), and to construct solutions of winding number $q > 1$ in SU(N).

An embedding of SU(2) in SU(N), for example, is completely defined by a set of N numbers known as the defining vector of the embedding, and can be graphically represented by a diagram known as the Dynkin characteristic.⁴ Half the square of the length of the defining vector is called the index of the embedding. In most cases, the knowledge of the index is sufficient to specify completely the embedding. However, it may happen that inequivalent embeddings admit the same index; in this case reference should be made to the corresponding Dynkin characteristics.

We show that the charge, or Pontryagin index, is a multiple of the index of a particular embedding of SU(2) in G . We illustrate our classification by performing explicitly the classification of the embeddings for some simple algebras G [SU(N), for $N = 3, 4, 5, 6, 7, 8$ and the exceptional algebra E_7].

The gauge-transformation properties of the solutions are completely determined by the Dynkin characteristic; this allows us to assign relevant quantum numbers for the various solutions obtained in this manner. We do this in the cases $G = \text{SU}(3)$ and $\text{SU}(4)$ as examples. Moreover, a mathematical property of the embeddings is used to discuss the interaction of pseudoparticles.

This paper is organized as follows. In Sec. II, we recall the pseudoparticle solution of Belavin *et al.* for $G = \text{SU}(2)$, $q = 1$, and their extension to $G = \text{SU}(N) \supset \text{SU}(2)$, $q > 1$, by Wilczek. Section III deals with the mathematical apparatus needed for our classification. This classification of the pseudoparticle solutions is achieved in Sec. IV.

Finally, Sec. V is devoted to a discussion of some simple properties of these solutions.

II. PSEUDOPARTICLE SOLUTIONS

We consider the gauge group SU(2) in the four-dimensional Euclidean space, and define the gauge fields by

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + i\epsilon_{ijk} A_\mu^j A_\nu^k, \quad i = 1, 2, 3 \text{ and } \mu, \nu = 1, 2, 3, 4. \quad (2.1)$$

If we use matrix notation and write

$$A_\mu = A_\mu^i \frac{\sigma_i}{2}, \quad (2.2)$$

where σ_i are the usual Pauli matrices, then

$$F_{\mu\nu} = F_{\mu\nu}^i \frac{\sigma_i}{2}, \quad (2.3)$$

and the effective interaction is

$$S = \frac{1}{2g^2} \text{Tr} \int F_{\mu\nu} F^{\mu\nu} d^4x. \quad (2.4)$$

It has been shown recently by Belavin *et al.*¹ that there exist nontrivial solutions to the classical equations of motion, characterized by a topological quantum number, known as the Pontryagin index, and given by

$$q = \frac{1}{16\pi^2} \text{Tr} \int F_{\mu\nu} \tilde{F}^{\mu\nu} d^4x, \quad (2.5)$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ is the dual of $F_{\mu\nu}$. This index q may be shown to take always integer values.

An important inequality holds for the effective action. For, since

$$\text{Tr} \int (F_{\mu\nu} - \tilde{F}_{\mu\nu})^2 d^4x \geq 0, \quad (2.6)$$

then

$$S \geq \frac{8\pi^2}{g^2} |q|. \quad (2.7)$$

It is clear that if there exists a field such that for $q \neq 0$, $F_{\mu\nu} = \tilde{F}_{\mu\nu}$, then the inequality is satur-

ated and this $F_{\mu\nu}$ is a nontrivial solution to the classical field equations.

Belavin *et al.* construct such a solution for $q=1$. This is given by

$$A_\mu = \frac{x^2}{x^2 + \lambda^2} g^{-1} \partial_\mu g, \quad g = \frac{x_4 + i\vec{x} \cdot \vec{\sigma}}{|x|}, \quad (2.8)$$

where λ is an arbitrary parameter, or, using the Jackiw-Rebbi gauge transformation,⁵

$$A_\mu = -\frac{\lambda^2}{x^2 + \lambda^2} (\partial_\mu g) g^{-1}. \quad (2.9)$$

If we write this solution in the form $A_\mu = A_\mu^i(x)(\sigma_i/2)$ leading to the field strength, $F_{\mu\nu} = F_{\mu\nu}^i(x)(\sigma_i/2)$, then if T_i , $i=1, 2, 3$, form any representation of $SU(2)$, $[T_i, T_j] = 2i\epsilon_{ijk} T_k$, the potential $A_\mu = A_\mu^i(x)(T_i/2)$ will give the field strength, $F_{\mu\nu}^i(x)(T_i/2)$, which is self-dual and hence saturates the inequality for some q . Thus, if the gauge group is G , all such pseudoparticle solutions in G may be obtained in this manner by embedding $SU(2)$ into G in all inequivalent ways.^{1,2} All such solutions are thus characterized by all inequivalent embeddings of $SU(2)$ in G .

III. EMBEDDING OF THE $SU(2)$ ALGEBRA IN A (SEMI) SIMPLE ALGEBRA G

The mathematical problem of embedding a simple subalgebra \tilde{G} in a simple algebra G has been discussed by Dynkin.⁴

Consider a semisimple complex algebra G and K its Cartan subalgebra. Let \tilde{G} be a subalgebra of G , with \tilde{K} its Cartan subalgebra. An embedding of \tilde{G} into G is completely defined by a mapping from \tilde{K} into K

$$f(\tilde{H}_i) = \sum_{k=1}^n f_{ik} H_k, \quad i=1, \dots, l, \quad (3.1)$$

with \tilde{H}_i and H_i the elements of \tilde{K} and K , respectively, where l is the rank of \tilde{G} and n the rank of G .

The relation

$$(f(\tilde{X}), f(\tilde{Y})) = j_f (\tilde{X}, \tilde{Y}), \quad \tilde{X}, \tilde{Y} \in \tilde{G}, \quad (3.2)$$

in which $(\tilde{X}, \tilde{Y}) = \text{Tr ad}\tilde{X} \cdot \text{ad}\tilde{Y}$ is the Killing form relative to \tilde{X} and \tilde{Y} , determines a scalar factor j_f independent of \tilde{X}, \tilde{Y} , and is called the index of the embedding.

The set of numbers f_{ik} form the matrix of the embedding. The embedding of the shift operators $\tilde{E}_{\tilde{\alpha}}$ of the subalgebra \tilde{G} in G is given as

$$f(\tilde{E}_{\tilde{\alpha}}) = \sum_{\alpha \in \Gamma_{\tilde{\alpha}}} C_{\tilde{\alpha}\alpha} E_\alpha, \quad (3.3)$$

with the $C_{\tilde{\alpha}\alpha}$ complex numbers determined through f_{ik} by the embedding of the roots $\tilde{\alpha}$ into the sub-

set $\Gamma_{\tilde{\alpha}}$ of the root system Γ of the algebra G , defined by

$$\Gamma_{\tilde{\alpha}} = \{ \alpha \in \Gamma \mid f^*(\alpha) = \tilde{\alpha} \}, \quad (3.4)$$

where the map f^* is defined through the relation

$$(H, f(\tilde{H})) = (f^*(H), \tilde{H}), \quad (3.5)$$

for any two elements H and $f(\tilde{H})$ in K . The index j_f in the embedding f may also be defined by

$$j_f = \sum_{\alpha \in \Gamma_{\tilde{\alpha}}} |C_{\tilde{\alpha}\alpha}|^2, \quad (3.6)$$

as long as G and \tilde{G} are different from C_n .⁶ The index j_f is a non-negative integer. Embeddings of the same algebra \tilde{G} in an algebra G which are related through an inner automorphism of G are equivalent embeddings and the subalgebras of G which correspond to these embeddings are conjugate subalgebras of G . (Two algebras are called conjugate if they are related by an inner automorphism.) Two equivalent embeddings have the same matrix (f_{ik}) and hence the same index j_f .

In most cases the indices j_f corresponding to inequivalent embeddings are different and hence may be used to label the embeddings. However, the indices of inequivalent embeddings may coincide, in which case one must refer to the defining matrix (f_{ik}) .

If \tilde{G} is isomorphic to $SU(2)$ then the matrix (f_{ik}) becomes a vector (f_k) , $k=1, \dots, n$, called the defining vector of the embedding. The index is in this case given by

$$j_f = \frac{1}{2} \sum_{k=1}^n f_k^2, \quad (3.7)$$

which is half the square of the length of the defining vector, in any m -dimensional Euclidean space with $m \geq n$.

It is well known that one can choose among the set of (positive) roots of an algebra G a subset of n roots $\alpha^{(1)}, \dots, \alpha^{(n)}$ such that any root is a linear combination of $\alpha^{(i)}$ with either all non-negative or all nonpositive integers. This subset of n roots is called a system of simple roots for G . It holds that

$$\frac{(\alpha^{(i)}, \alpha^{(j)})^2}{(\alpha^{(i)}, \alpha^{(i)})(\alpha^{(j)}, \alpha^{(j)})} = \frac{r}{4}, \quad r=0, 1, 2, 3; i \neq j, \quad (3.8)$$

and

$$(\alpha^{(i)}, \alpha^{(j)}) \leq 0.$$

The longest root is normalized through the usual scalar product

$$(\alpha, \alpha) = 2, \quad (3.9)$$

ments⁹ by $m_J = 2J, 2(J - 1), \dots, -2J; m_{J'},$
 $= 2J', \dots, 3J', \dots$, where $(2J + 1) + (2J' + 1) + \dots$
 $= N$, then the index of the embedding becomes

$$j_f = \frac{1}{2} \sum_{J, J', \dots} m_J^2 \tag{3.14}$$

or

$$j_f = \frac{1}{2} \text{Tr} T_3^2. \tag{3.15}$$

Note that Eq. (3.15) may be directly obtained from Eqs. (3.1), (3.2), taking proper care of normalization factors.

It is obvious to generalize this property for any semisimple algebra. Note that if the gauge group is of the form $G^{(1)} \times G^{(2)} \times \dots \times G^{(k)}$ with $G^{(1)}, G^{(2)}, \dots, G^{(k)}$ simple, then the corresponding defining vectors are sets of k vectors $f^{(1)}, \dots, f^{(k)}$, with $f^{(i)}$ characterizing the embedding of $SU(2)$ in $G^{(i)}$. In this case the index is the sum of the indices $j_{f^{(i)}}$.

IV. CLASSIFICATION OF PSEUDOPARTICLE SOLUTIONS

Let $F^i_{\mu\nu} T_i$ represent solutions obtained by embedding $SU(2)$ into G . The Pontryagin index is given by

$$\begin{aligned} q &= \frac{1}{16\pi^2} \text{Tr} \int F_{\mu\nu} F^{\mu\nu} d^4x \\ &= \frac{1}{16\pi^2} \text{Tr} (T_3)^2 \int F^i_{\mu\nu} F_i^{\mu\nu} d^4x \\ &= j_f \frac{1}{8\pi^2} \int F^i_{\mu\nu} F_i^{\mu\nu} d^4x, \end{aligned} \tag{4.1}$$

where j_f is the index of the embedding.

For the solution (3.8) of Belavin *et al.*, $q = j_f = 1$. In other words, the integral $(1/8\pi^2) \int F^i_{\mu\nu} F_i^{\mu\nu} d^4x = 1$. A new set of solutions¹⁰ where this integral takes all integral values has been obtained by Witten; i.e., $q = j_f w$, where $j_f = 1$, with w any integer.

For any embedding of $SU(2)$ in G , we get a generalization of the Belavin *et al.* solutions with

$$q = j_f w, \tag{4.2}$$

where j_f is the index of the embedding.

It must be clear from our discussion of Sec. III that the index j_f is not enough to classify the embedding, and hence the solutions. One must refer to the Dynkin characteristic of the embeddings. In the following, we discuss the solutions for $G = SU(N)$, $N = 3, 4, \dots, 8$, and for E_7 , one of the exceptional groups, which are also of physical interest.

Let us mention at this point that the classification of the embeddings of $SU(2)$ into simple algebras of rank ≤ 6 are given in Ref. 11, and into

the exceptional algebras in Ref. 4.

Case $G = SU(N)$. The inequivalent embeddings of $SU(2)$ into $SU(N)$, $N = 3, 4, 5, 6, 7, 8$, are given in Table I.

In Table II we show the characteristics for the cases $SU(3)$ and $SU(4)$. Let us recall that the Dynkin characteristic is simply a graphical description of the defining vector. The second column shows the minimal including regular subalgebra. One calls G' a regular subalgebra of G if the set $\Gamma_{\alpha'}$ contains one and only one root α of G for every root $\alpha' \in \Gamma'$ in G' [see Eqs. (3.4), (3.5)]. The knowledge of these subalgebras will be useful in the discussion of Sec. V to visualize the positions of the different $SU(2)$ in the matrix adjoint representation of G .

We notice that when N is odd, the $\frac{1}{2}(N+1)$ th coordinate of the defining vector is 0, and the other coordinates are symmetric with respect to it. When N is even, the coordinates are symmetric with respect to the $\frac{1}{2}N$ th comma.

Every embedding specifies the transformation properties of the pseudoparticle solution under G . For example, in the case of $SU(3)$, the solutions with $j_f = 1$ and 4 transform like λ_3 ; in the case of $SU(4)$ they transform like $\lambda_3 - (1/\sqrt{3})\lambda_8 + (\frac{2}{3})^{1/2}\lambda_{15}$ for $j_f = 2$, and $\lambda_3 + \sqrt{3}\lambda_8 + \sqrt{6}\lambda_{15}$ for $j_f = 10$.

Note that in the case of $SU(8)$, one obtains two solutions with the same index but different defining vector. The degeneracy of j_f , already seen by Wilczek,² emphasizes the need of the full defining vector (or characteristic) to classify solutions.

Case $G = E_7$. As another particular case of different characteristics corresponding to the same index, we mention in Table III a few of the embeddings of $SU(2)$ into E_7 . A complete list is given in Ref. 4.

For all other simple groups B_ν, C_ν, D_ν and G_2, F_4, E_6, E_8 , complete listings of all $SU(2)$ embeddings are given in Refs. 4 and 11.

V. SUMS OF PSEUDOPARTICLE SOLUTIONS

We consider now the problem of adding two or more solutions to get a third. It is clear that when two or more solutions are simply added, the sum does not form in general a solution, except finally when their positions coincide. This later possibility arises when the sum of two (or more) embeddings of $SU(2)$ in G is itself an embedding.

Consider the sum of solutions

$$A_\mu = A^i_\mu (x - t_a)^{(a)} T_i + A^i_\mu (x - t_b)^{(b)} T_i + \dots$$

From the general form of the action, one clearly sees that

$$S > \frac{8\pi^2}{g^2} (|q_{(a)}| + |q_{(b)}| + \dots).$$

TABLE I. Embeddings of $SU(2)$ into $SU(N)$, $3 \leq N \leq 8$.

Algebra	Minimal including regular subalgebra	Index j_f	Defining vector
SU(3)	A_1	1	(1, 0, -1)
	A_2	4	(2, 0, -2)
SU(4)	A_1	1	(1, 0, 0, -1)
	$2A_1$	2	(1, 1, -1, -1)
	A_2	4	(2, 0, 0, -2)
	A_3	10	(3, 1, -1, -3)
SU(5)	A_1	1	(1, 0, 0, 0, -1)
	$2A_1$	2	(1, 1, 0, -1, -1)
	A_2	4	(2, 0, 0, 0, -2)
	$A_2 + A_1$	5	(2, 1, 0, -1, -2)
	A_3	10	(3, 1, 0, -1, -3)
	A_4	20	(4, 2, 0, -2, -4)
SU(6)	A_1	1	(1, 0, 0, 0, 0, -1)
	$2A_1$	2	(1, 1, 0, 0, -1, -1)
	$3A_1$	3	(1, 1, 1, -1, -1, -1)
	A_2	4	(2, 0, 0, 0, 0, -2)
	$A_2 + A_1$	5	(2, 1, 0, 0, -1, -2)
	$2A_2$	8	(2, 2, 0, 0, -2, -2)
	A_3	10	(3, 1, 0, 0, -1, -3)
	$A_3 + A_1$	11	(3, 1, 1, -1, -1, -3)
	A_4	20	(4, 2, 0, 0, -2, -4)
	A_5	35	(5, 3, 1, -1, -3, -5)
SU(7)	A_1	1	(1, 0, 0, 0, 0, 0, -1)
	$2A_1$	2	(1, 1, 0, 0, 0, -1, -1)
	$3A_1$	3	(1, 1, 1, 0, -1, -1, -1)
	A_2	4	(2, 0, 0, 0, 0, 0, -2)
	$A_2 + A_1$	5	(2, 1, 0, 0, 0, -1, -2)
	$A_2 + 2A_1$	6	(2, 1, 1, 0, -1, -1, -2)
	$2A_2$	8	(2, 2, 0, 0, 0, -2, -2)
	A_3	10	(3, 1, 0, 0, 0, -1, -3)
	$A_3 + A_1$	11	(3, 1, 1, 0, -1, -1, -3)
	$A_3 + A_2$	14	(3, 2, 1, 0, -1, -2, -3)
	A_4	20	(4, 2, 0, 0, 0, -2, -4)
	$A_4 + A_1$	21	(4, 2, 1, 0, -1, -2, -4)
	A_5	35	(5, 3, 1, 0, -1, -3, -5)
	A_6	56	(6, 4, 2, 0, -2, -4, -6)
SU(8)	A_1	1	(1, 0, 0, 0, 0, 0, 0, -1)
	$2A_1$	2	(1, 1, 0, 0, 0, 0, -1, -1)
	$3A_1$	3	(1, 1, 1, 0, 0, -1, -1, -1)
	$4A_1$	4'	(1, 1, 1, 1, -1, -1, -1, -1)
	A_2	4''	(2, 0, 0, 0, 0, 0, 0, -2)
	$A_2 + A_1$	5	(2, 1, 0, 0, 0, 0, -1, -2)
	$A_2 + 2A_1$	6	(2, 1, 1, 0, 0, -1, -1, -2)
	$2A_2$	8	(2, 2, 0, 0, 0, 0, -2, -2)
	$2A_2 + A_1$	9	(2, 2, 1, 0, 0, -1, -2, -2)
	A_3	10	(3, 1, 0, 0, 0, 0, -1, -3)
	$A_3 + A_1$	11	(3, 1, 1, 0, 0, -1, -1, -3)
	$A_3 + 2A_1$	12	(3, 1, 1, 1, -1, -1, -1, -3)
	$A_3 + A_2$	14	(3, 2, 1, 0, 0, -1, -2, -3)
	$2A_3$	20'	(3, 3, 1, 1, -1, -1, -3, -3)
	A_4	20''	(4, 2, 0, 0, 0, 0, -2, -4)
	$A_4 + A_1$	21	(4, 2, 1, 0, 0, -1, -2, -4)
	$A_3 + A_2$	24	(4, 2, 2, 0, 0, -2, -2, -4)
	A_5	35	(5, 3, 1, 0, 0, -1, -3, -5)
	$A_5 + A_1$	36	(5, 3, 1, 1, -1, -1, -3, -5)
	A_6	56	(6, 4, 2, 0, 0, -2, -4, -6)
	A_7	86	(7, 5, 3, 1, -1, -3, -5, -7)

TABLE II. Dynkin characteristics for $SU(2) \subset SU(3)$ and $SU(4)$.

	Index	Dynkin characteristic
SU(3)	SU(2) case: 1	$\begin{array}{c} \\ \circ - \circ \\ \end{array}$
	SO(3) case: 4	$\begin{array}{c} 2 \\ \circ - \circ \\ 2 \end{array}$
SU(4)	1	$\begin{array}{c} & 0 & \\ \circ - \circ - \circ \\ \end{array}$
	2	$\begin{array}{c} 0 & 2 & 0 \\ \circ - \circ - \circ \\ \end{array}$
	4	$\begin{array}{c} 2 & 0 & 2 \\ \circ - \circ - \circ \\ \end{array}$
	10	$\begin{array}{c} 2 & 2 & 2 \\ \circ - \circ - \circ \\ \end{array}$

If A_μ were a solution the equality sign would hold. When we let all $t_{(a)}, t_{(b)}, \dots$ go to zero, the form A_μ becomes a solution only if the sum

$${}^{(a)}T_i + {}^{(b)}T_i + \dots = T_i$$

TABLE III. Embeddings of $SU(2)$ into E_7 for $j_f \leq 5$.

Index	Minimal including regular subalgebra	Characteristic
1	A_1	$\begin{array}{c} \circ - \circ - \circ - \circ - \circ - \circ - \circ \\ \\ \circ \end{array}$
2	$2A_1$	$\begin{array}{c} \circ - 1 - \circ - \circ - \circ - \circ - \circ \\ \\ \circ \end{array}$
3'	$[3A_1]'$	$\begin{array}{c} \circ - \circ - \circ - \circ - 1 - \circ - \circ \\ \\ \circ \end{array}$
3''	$[3A_1]''$	$\begin{array}{c} 2 - \circ - \circ - \circ - \circ - \circ - \circ \\ \\ \circ \end{array}$
4'	$[4A_1]'; A_2$	$\begin{array}{c} \circ - \circ - \circ - \circ - \circ - \circ - 2 \\ \\ \circ \end{array}$
4''	$[4A_1]''$	$\begin{array}{c} 1 - \circ - \circ - \circ - \circ - \circ - \circ \\ \\ \circ \end{array}$
5	$5A_1; A_2 + A_1$	$\begin{array}{c} \circ - 1 - \circ - \circ - \circ - \circ - \circ - 1 \\ \\ \circ \end{array}$

with $[T_i, T_j] = 2i\epsilon_{ijk} T_k$. This clearly would be the case if the sum of the embeddings (a), (b), ... is itself an embedding of $SU(2)$ in G . Then the action becomes

$$S = \frac{8\pi^2}{g^2} (|q_{(a)}| + |q_{(b)}| + \dots).$$

Using theorem D II of Sec. III, we can see that this would happen in the following cases.

(i) $SU(4)$. The embedding with $j_f = 2$ may be obtained from the sum of two embeddings of $j_f = 1$. This is because one can add two equivalent embeddings of the form $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}$, where T are 2×2 matrices corresponding to $j_f = 1$, to form the inequivalent embedding $\begin{pmatrix} T & 0 \\ 0 & T' \end{pmatrix}$ with $j_f = 2$ in $SU(4)$.

(ii) $SU(5)$. For the same reason as in the case of $SU(4)$ above, $j_f = 2$ solutions may be obtained from the sum of two $j_f = 1$ solutions. Moreover, the $j_f = 5$ solution can be obtained as the sum of the $j_f = 1$ and $j_f = 4$ solutions. This is because for the $j_f = 5$ solutions the corresponding matrices are of the form $\begin{pmatrix} T & 0 \\ 0 & T' \end{pmatrix}$ with T 2×2 matrices and T' 3×3 matrices, whereas for $j_f = 4$ these are of the form $\begin{pmatrix} 0 & 0 \\ 0 & T' \end{pmatrix}$ and for $j_f = 1$ of the form $\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$.

(iii) $SU(6)$. In this case, just as in $SU(5)$, the $j_f = 2$ may be obtained as the sum of two $j_f = 1$ and $j_f = 5$ as the sum of $j_f = 4$ and $j_f = 1$. Moreover, the $j_f = 3$ solution may be considered as the sum of three $j_f = 1$ solutions, or of one $j_f = 1$ and one $j_f = 2$ solutions, for now the matrices for $j_f = 3$ are of the form

$$\begin{pmatrix} T & & \\ & T & \\ & & T \end{pmatrix}.$$

Similarly the $j_f = 8$ may be the sum of two $j_f = 4$, and $j_f = 11$ the sum of $j_f = 10$ and $j_f = 1$.

(iv) $SU(7)$. The list grows and in addition to the case of $SU(6)$ we have $j_f = 6$ the sum of $j_f = 4$ and $j_f = 2$, or $j_f = 5$ and $j_f = 1$, or $j_f = 4$ and $j_f = 1$ and $j_f = 1$; $j_f = 14$ the sum of $j_f = 10$ and $j_f = 4$; and $j_f = 21$ the sum of $j_f = 20$ and $j_f = 1$.

(v) $SU(8)$. In $SU(8)$, the list goes further. However, we note that $j_f = 4'$ (following notation of Table I) may be obtained as the sum of four $j_f = 1$, or two $j_f = 2$, or $j_f = 1$ and $j_f = 3$, or two $j_f = 1$ and one $j_f = 2$, while for the inequivalent embedding $j_f = 4''$ theorem D II does not apply. Nevertheless, as shown by Wilczek² in $SU(3)$ and hence in all $SU(N)$, $N > 3$, one can consider $j_f = 4''$ as the sum of four $j_f = 1$.

A similar discussion can be done in the case of all other semisimple groups, see for example Table III for the case of E_7 .

Another set of cases where solutions can be

seen as the sum of two or more may be obtained using the sum constructed by Wilczek in the case of SU(3). For example, one can see that the $j_f=8$ solution in SU(6) can be obtained as the sum of four $j_f=2$ solutions. We notice, however, that the defining vectors corresponding to $j_f=1$ and $j_f=4$ are parallel; so are the defining vectors for $j_f=2$ and $j_f=8$ in SU(6). We conjecture that in any semisimple algebra G , when two embeddings f and f' are such that their defining vectors are parallel, i.e., $f' = \alpha f$, $\alpha > 1$, then it is possible to sum α^2 embeddings f and obtain an embedding f' .

We remark here that the solutions¹⁰ in SU(2) with $q > 1$ appear to be a sum of separated solutions of $j_f=1$, which is itself a solution because of the

special field configuration it has in space. The individual pseudoparticles in this case are in a sense free as there is no interaction energy. These solutions, however, are of the general form $A_\mu^i(x)(\sigma_i/2)$, and hence similar solutions may be obtained in G by replacing σ_i by T_i , the various embeddings of SU(2) in G , and the interaction energy would still be zero.

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⁶In the case of the algebras C_n , the longest root is normalized such that $(\alpha, \alpha) = 4$. When $\hat{G} = \text{SU}(2)$, the index j_f is $= 2 \sum_{\alpha \in \Gamma_{\hat{G}}} |C_{\alpha'} \alpha|^2 = \sum_k f_k f_k$

⁷Ref. 4, Theorem 8.2.

⁸Ref. 4, Theorem 2.3.

⁹Note that the commutation relations defined in (3.9) imply matrices which are twice the usual one.

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