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Pseudoparticle contributions to the energy spectrum of a one-dimensional system*

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We show that classical solutions of the Euclidean action can be used to calculate the shift in energy levels due to tunneling through a potential barrier. In particular, we use the path integral to compute the kernel of the double-well anharmonic oscillator for a large, but finite, Euclidean time interval by expanding about pseudoparticle solutions (i.e., the kink). This allows us to determine the ground-state energy plus that of the first excited state (the splitting is due to barrier penetration). We find that not only the classical solution must be expanded about, but also nearly stationary trajectories corresponding to kink plus kink-antikink pairs. The quasitranslational invariance must also be dealt with carefully. We compare with the WKB result and find our result more accurate, because it avoids the errors introduced by the linear (Airy functions) connecting formulas.

I. INTRODUCTION

The classical (Minkowski) action has minima corresponding to periodic oscillations about a potential minimum. These classical solutions can be used in connection with Dashen-Hasslacher-Neveu[1] scheme to find information about the spectrum of the Hamiltonian. However, in cases where the potential has more than one minimum, it was previously noted by one of us[2] that such classical solutions yield no information about the shift in energy levels due to tunneling through the potential barrier. Since then, solutions which minimize the classical Euclidean action of several theories have been discovered,[3] and it has been noted that such solutions correspond to tunneling between vacuum states.[4] (These solutions are now generically known as pseudoparticles.) In this paper we demonstrate how such solutions can be used to learn about the spectrum of the Hamiltonian of the system.

For simplicity we consider a theory with one scalar field in one time and zero space dimensions. The Lagrangian is

$$L=\tfrac{1}{2}(\partial_t\varphi)^2-\frac{\lambda}{4}\left(\varphi^2-\frac{\mu^2}{\lambda}\right)^2 \tag{1.1}$$

with μ^2, $\lambda>0$. This is just the anharmonic oscillator with a double-well potential (Fig. 1). In a recent paper proposing several models which possess pseudoparticle solutions, Patrascioiu[5] pointed out that for this anharmonic oscillator the kink is a pseudoparticle. For weak coupling ($\lambda\hbar/\mu^3\ll 1$), we use the kink solution in applying the steepest-descent approximation to the path integral to compute the Euclidean kernel $\langle\varphi_1,t_1|\varphi_2,t_2\rangle$, and from it learn about the energy spectrum. We compare these results with that of an ordinary WKB approach, and to our pleasure find that the pseudoparticle approximation to the Euclidean path integral gives the more accurate result. This is true because the path-integral approach avoids the errors introduced into the WKB approximation by the introduction of connecting formulas. (It is amusing to note that despite the care and effort needed for the path-integral analysis, we probably spent more time coming to grips with some of the subtleties entailed in actually having to execute a WKB program.)

To obtain the correct kernel for a large Euclidean time interval, it turns out that one has to be very careful to expand about not only the classical solution which minimizes the action, but also about trajectories for which the action is almost stationary. Thus not only the one-kink contribution (Fig. 2) must be taken into account, but also that of one kink plus any number of kink-antikink pairs (Fig. 4). These correspond to those processes in which the particle tunnels back and forth between the two wells before finally arriving at (φ_2,t_2).

All translations of the kinks must be reckoned with. For the case of an infinite time interval the action is translationally invariant. In expanding about the kink, this invariance implies a zero-frequency-mode contribution to the propagator. It is known that this problem can be circumvented by the introduction of collective coordinates.[6]

Since we calculate the kernel for a large but

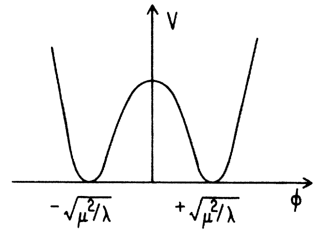

FIG. 1. Anharmonic potential with two wells.

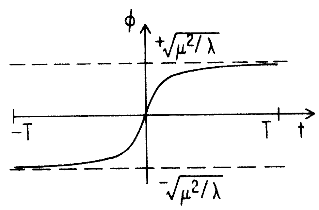

FIG. 2. The kink in a box.

finite time interval there is a quasitranslational invariance. This quasi-invariance results in an eigenmode contribution to the propagator whose eigenvalue decreases exponentially with time; thus the perturbation expansion is no longer useful. Again the problem is surmounted by introducing a collective coordinate. The measure of integration is obtained via the Faddeev-Popov technique[7]; however, the application of this technique to our problem requires great care since we are dealing with only a quasisymmetry.

The inclusion of multikink trajectories requires a refinement beyond the considerations of quasitranslation invariance: One must deal carefully with the problem of joining kink-antikink pairs. A careful analysis leads to the inclusion of an additional determinant.

Our presentation commences with a short discussion of the desired form of the kernel for a large Euclidean time interval. We assign separate sections to the one-kink and multikink contributions. For completeness we briefly review and compare with the WKB result.

II. PROJECTION OF THE GROUND STATE

First let us consider the general case of a field theory in one dimension (one time, zero space dimensions, i.e., quantum mechanics). The only assumption will be that the Hamiltonian is bounded from below and has at least one discrete eigenvalue. Then we know that for a large-enough Euclidean time interval the transition amplitude between a state S_1 at time $-T$ and S_2 at time T is

$$\langle S_2, T|S_1, -T\rangle \simeq \langle S_2|0\rangle\langle 0|S_1\rangle e^{-2E_g T/\hbar}. \tag{2.1}$$

Thus the ground-state eigenvalue, E_g, is projected out. If the lowest eigenstates correspond to a slightly split degeneracy, then one can choose to keep track of all the associated eigenvalues. If the spectrum is completely continuous, then the sum over intermediate states would be an integral, and the result would become

$$\langle S_2, T|S_1, -T\rangle \simeq \frac{1}{(2\pi i\hbar T)^{1/2}}\langle S_2|0\rangle\langle 0|S_1\rangle e^{-2E_g T/\hbar}. \tag{2.2}$$

We now apply these results to the scalar field theory given by Eq. (1.1). Let $|L_n\rangle$ and $|R_n\rangle$ denote the energy eigenstates of the left and right wells when one ignores the presence of the other well. Then the potential barrier splits the L-R degeneracy such that to lowest order in $\hbar$ the two lowest eigenstates are

$$|0\rangle = \frac{1}{\sqrt{2}}(|L_0\rangle + |R_0\rangle), \tag{2.3a}$$

$$|1\rangle = \frac{1}{\sqrt{2}}(|L_0\rangle - |R_0\rangle), \tag{2.3b}$$

with corresponding eigenvalues $E_0 - \Delta E/2$ and $E_0 + \Delta E/2$. Consider the amplitude $\langle\varphi_2, T|\varphi_1, -T\rangle$, where φ_1 and φ_2 are near the bottom of the right and left wells, respectively. Retaining only the contributions of $|0\rangle$ and $|1\rangle$ to this amplitude and dropping the small overlap terms $\langle\varphi_1|L_0\rangle$ and $\langle\varphi|R_0\rangle$, our amplitude for large Euclidean T is

$$\langle\varphi_2, T|\varphi_1, -T\rangle = \langle\varphi_2|L_0\rangle\langle R_0|\varphi_1\rangle e^{-2E_0 T/\hbar} \times \sinh\frac{\Delta E}{\hbar}T. \tag{2.4}$$

Thus if one can compute $\langle\varphi_2, T|\varphi_1, -T\rangle$, one determines E_0 and ΔE. In the next two sections this amplitude will be computed via the pseudoparticle contributions to the path integral. The results will be compared in Sec. V to the WKB approximation.

III. ONE-KINK CONTRIBUTION TO THE KERNEL

In this section we compute the one-kink contribution to the path integral. For pedagogic reasons we find it convenient to first present the naive treatment, which ignores the fact that translations of the kink are quasisymmetries of the action. The correct treatment of translations is given in Sec. III B.

A. Naive treatment

We will compute the Euclidean kernel

$$\langle\varphi_2, T|\varphi_1, -T\rangle \text{ for } -\varphi_1 = \varphi_2 = \left(\frac{\mu^2}{\lambda}\right)^{1/2}\tanh\frac{\mu T}{\sqrt{2}}. \tag{3.1}$$

It is given by the following path integral:

$$\langle\varphi_2, T|\varphi_1, -T\rangle = \int \mathfrak{D}\varphi(t)\exp\left\{\frac{1}{\hbar}\int_{-T}^{T} dt\left[-\frac{\dot{\varphi}^2}{2} - \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2\right]\right\}, \tag{3.2}$$

with the boundary conditions $\varphi(-T) = \varphi_1$ and $\varphi(T) = \varphi_2$. The classical solution obeys the equation

$$\ddot{\varphi} + \mu^2\varphi - \lambda\varphi^3 = 0. \tag{3.3}$$

The boundary conditions were chosen so as to select as the correct solution the kink:

$$\varphi_k(t)=\left(\frac{\mu^2}{\lambda}\right)^{1/2}\tanh\frac{\mu t}{\sqrt{2}}\,. \tag{3.4}$$

The corresponding classical action is

$$\begin{aligned}S_k&=\int_{-T}^{T}dt\mathcal{L}[\varphi_k]\\&=-\sqrt{2}\,\frac{\mu^3}{\lambda}\left(-\frac{1}{3}\tanh^3\frac{\mu T}{\sqrt{2}}+\tanh\frac{\mu T}{\sqrt{2}}\right).\end{aligned}$$

For large T

$$S_k\simeq-\frac{2\sqrt{2}}{3}\,\frac{\mu^3}{\lambda}\,. \tag{3.5}$$

Expanding about the kink, we introduce new variables of integration:

$$y(t)=\varphi(t)-\varphi_k(t)\,. \tag{3.6}$$

The kernel becomes

$$\langle\varphi_2,T|\varphi_1,-T\rangle=\exp\left(-\frac{2\sqrt{2}}{3}\,\frac{\mu^3}{\hbar\lambda}\right)$$

$$\times\int\mathfrak{D}y(t)\exp\left\{-\frac{1}{\hbar}\int_{-T}^{T}dt\left[\tfrac{1}{2}\dot{y}^2-\frac{\mu^2}{2}\,y^2+\frac{3\mu^2}{2}\,y^2\tanh^2\frac{\mu t}{\sqrt{2}}+(\mu^2\lambda)^{1/2}\tanh\left(\frac{\mu t}{\sqrt{2}}\right)y^3+\frac{\lambda}{4}y^4\right]\right\}. \tag{3.7}$$

For weak coupling ($\lambda\hbar\ll\mu^3$), we drop the cubic and quartic terms and approximate the kernel by

$$\langle\varphi_2,T|\varphi_1,-T\rangle=\exp\left(-\frac{2\sqrt{2}}{3}\,\frac{\mu^3}{\hbar\lambda}\right)\int\mathfrak{D}y(t)\exp\left[-\frac{1}{\hbar}\int_{-T}^{T}dt\left(\tfrac{1}{2}\dot{y}^2-\frac{\mu^2}{2}y^2+\frac{3\mu^2}{2}y^2\tanh^2\frac{\mu t}{\sqrt{2}}\right)\right], \tag{3.8}$$

with the boundary conditions $y(-T)=y(T)=0$. The cubic and quartic terms can be taken into account perturbatively, but in the sequel we shall ignore them.

The Gaussian functional integral in (3.8) can be computed in two ways. We do the calculation both ways, not as an exercise in computational prowess, but because it enables us to determine the Jacobian involved in transforming to the normal modes.

(*i*) *Change of variables*. We introduce the mapping

$$z(t)=y(t)-\int_{-T}^{t}\frac{\dot{N}(\tau)}{N(\tau)}\,y(\tau)d\tau\,, \tag{3.9}$$

where $N(t)$ is defined by the equation

$$\ddot{N}=\left(-\mu^2+3\mu^2\tanh^2\frac{\mu t}{\sqrt{2}}\right)N\,. \tag{3.10}$$

We denote the functional integral in (3.8) by I and obtain

$$I=\int\mathfrak{D}z(t)d\alpha\left|\frac{\mathfrak{D}y}{\mathfrak{D}z}\right|\exp\left\{\frac{1}{\hbar}\left[\int_{-T}^{T}dt\left(-\frac{\dot{z}^2}{2}\right)+i\alpha\left(z(T)+N(T)\int_{-T}^{T}dt\,\frac{\dot{N}(t)}{N^2(t)}\,z(t)\right)\right]\right\}. \tag{3.11}$$

Here α is a Lagrange multiplier which inserts the constraint on $z(t)$ induced by $y(T)=0$. The integrations were carried out by Dashen *et al.*[1] and give

$$I=\left(\frac{1}{2\pi\hbar}\right)^{1/2}\left(\frac{1}{N(T)N(-T)}\right)^{1/2}\left(\int_{-T}^{T}\frac{dt}{N^2(t)}\right)^{-1/2}. \tag{3.12}$$

A solution of (3.10) is

$$N(t)=\dot{\varphi}_k(t)=\left(\frac{\mu^2}{\lambda}\right)^{1/2}\frac{\mu}{\sqrt{2}}\,\frac{1}{\cosh^2(\mu t/\sqrt{2})}, \tag{3.13}$$

so that finally we have

$$I=\left(\frac{\mu\sqrt{2}}{2\pi\hbar}\right)^{1/2}. \tag{3.14}$$

(ii) *Expansion in normal modes*. Performing an integration by parts, we can write I as

$$I=\int\mathfrak{D}y(t)\exp\left[-\frac{1}{\hbar}\int_{-T}^{T}dty\left(-\tfrac{1}{2}\partial_t^{\,2}-\frac{\mu^2}{2}+\frac{3\mu^2}{2}\tanh^2\frac{\mu t}{\sqrt{2}}\right)y\right]. \tag{3.15}$$

The matrix

$$M\equiv-\tfrac{1}{2}\partial_t^{\,2}-\frac{\mu^2}{2}+\frac{3\mu^2}{2}\tanh^2\frac{\mu t}{\sqrt{2}} \tag{3.16}$$

is Hermitian, and it can be diagonalized by solving for its eigenvalues $E_n^{\,2}$ and eigenfunctions Ψ_n,

$$M\Psi_n=E_n^{\,2}\Psi_n, \tag{3.17}$$

subject to the boundary conditions $\Psi_n(-T)=\Psi_n(T)=0$. We can expand y in terms of normal modes

$$y(t)=\sum c_n\Psi_n(t), \tag{3.18}$$

and change variables of integration from $y(t)$ to $\{c_n\}$. Since the transformation is linear [Eq. (3.18)], the Jacobian $|\mathfrak{D}y/\mathfrak{D}c_n|$ can be factored out. The integrations over c_n's are simple Gaussians, and we obtain

$$I=\left|\frac{\mathfrak{D}y}{\mathfrak{D}c_n}\right|\prod\left(\frac{\pi\mu^2}{E_n^2}\right)^{1/2}. \tag{3.19}$$

In deriving (3.19), we have normalized the Ψ_n's such that

$$\int_{-T}^{T}dt\,\Psi_n(t)\Psi_n(t)=\hbar/\mu^2. \tag{3.20}$$

The product in (3.19) extends over all eigenvalues of M. If T were infinite, the spectrum of M would contain two bound states of energies $\bar{E}_0^2=0$ and $\bar{E}_1^2=\frac{3}{4}\mu^2$ and a continuum of states for $\bar{E}^2>\mu^2$.[8] We show in Appendix A (using boundary perturbations) that imposing the boundary condition $\Psi_n(-T)=\Psi_n(T)=0$ produces a spectrum of bound states with energies

$$E_0^2=24\mu^2e^{-2\sqrt{2}\mu T}, \tag{3.21a}$$

$$E_1^2=\tfrac{3}{4}\mu^2+O(e^{-cT}), \quad \text{with } c \text{ some constant} \tag{3.21b}$$

$$E_{n+1}^2=\mu^2+n^2\pi^2/2T^2,\ n=1,2,\ldots. \tag{3.21c}$$

B. Nonperturbative treatment of kink translations

As stated in the Introduction, the presence of the quasizero eigenvalue (3.21a) destroys any hope of computing the deviations from Gaussian behavior of (3.7) in perturbation theory. Moreover, we notice from (3.8) and (3.14) that the naive treatment does not reproduce the desired result presented in Sec. II [Eq. (2.4)]. To account for the quasisymmetry represented by kink translations in a nonperturbative way, we use the Faddeev-Popov technique[7] and define

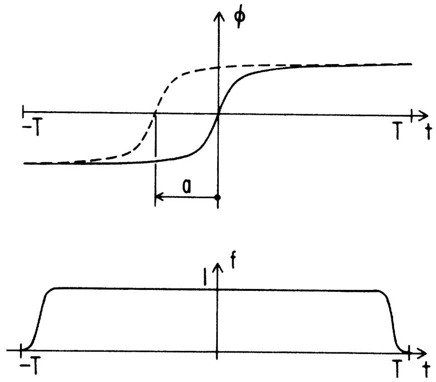

FIG. 3. (a) The kink (solid line) and its translation by a. (b) $f(t)$, which is used with translations in order to fix the end points.

$$\Delta[F(\varphi(t))]\int_{-T+c}^{T-c}da\,\delta\left(\int_{-T}^{T}dt\,\Psi_0(t)\varphi(t+af(t))\right)=1, \tag{3.22}$$

where $f(t)$ (which is chosen so as to ensure that under translations the end points remain fixed at φ_1 and φ_2) is shown in Fig. 3(b), c will be specified shortly, and $\Psi_0(t)$ is the eigenfunction which corresponds to E_0^2 [in fact, $\Psi_0(t)\propto\dot{\varphi}_k(t)$]. It follows that for $\varphi(t)$ such that

$$\int_{-T}^{T}dt\,\Psi_0(t)\varphi(t)=0, \tag{3.23a}$$

we have

$$\Delta[F(\varphi(t+Af(t)))]=\Delta[F(\varphi(t))] \quad \text{for any } |A|<T-c, \tag{3.23b}$$

and in fact

$$\Delta[F(\varphi(t))]\approx\int_{-T}^{T}dt\,\Psi_0(t)\dot{\varphi}(t). \tag{3.24}$$

Next we multiply Eqs. (3.2) and (3.22) and interchange the order of integration to obtain

$$\begin{aligned}\langle\varphi_2,T|\varphi_1,-T\rangle&=\int_{-T+c}^{T-c}da\int\mathfrak{D}\varphi(t)\exp\left(\frac{1}{\hbar}S[\varphi(t)]\right)\Delta[F(\varphi(t))]\delta\left(\int_{-T}^{T}dt\,\Psi_0(t)\varphi(t+af(t))\right)\\&=\int_{-T+c}^{T-c}da\int\mathfrak{D}\varphi(t)\exp\left(\frac{1}{\hbar}S[\varphi(t-af(t))]\right)\Delta[F(\varphi(t))]\delta\left(\int_{-T}^{T}dt\,\Psi_0(t)\varphi(t)\right).\end{aligned} \tag{3.25}$$

In the second step we have used (3.23b). To do the $\varphi(t)$ integration, we again write

$$\varphi(t)=\varphi_k(t)+y(t) \tag{3.26}$$

and obtain

$$\exp\left(\frac{1}{\hbar}\int_{-T+a}^{T+a} dt\,\mathcal{L}_{\rm cl}\right)\int \mathfrak{D}y(t)\exp\left[-\frac{1}{\hbar}\int_{-T+a}^{T+a} dt\left(\tfrac{1}{2}\dot{y}^2-\frac{\mu^2}{2}y^2+\frac{3\mu^2}{2}\tanh^2\frac{\mu t}{\sqrt{2}}y^2+(\mu^2\lambda)^{1/2}\tanh\frac{\mu t}{\sqrt{2}}y^3+\frac{\lambda}{4}y^4\right)\right]$$
$$\times\left(\int_{-T}^{T} dt\,\Psi_0(\dot{\phi}_k+\dot{y})\right)\delta\left(\int_{-T}^{T} dt\,\Psi_0(\varphi_k+y)\right). \tag{3.27}$$

The $y(t)$ integration can be performed by first computing

$$\bar{I}\equiv\int \mathfrak{D}y(t)\exp\left[-\frac{1}{\hbar}\int_{-T+a}^{T+a} dt\left(\tfrac{1}{2}\dot{y}^2-\frac{\mu^2}{2}y^2+\frac{3\mu^2}{2}\tanh^2\frac{\mu t}{\sqrt{2}}y^2\right)\right]\left(\int_{-T}^{T} dt\,\Psi_0\dot{\phi}_k\right)\delta\left(\int_{-T}^{T} dt\,\Psi_0 y\right), \tag{3.28}$$

and then taking care of the remaining terms perturbatively. We notice that if in (3.22) we chose $c=O(1/\mu)$ (i.e., the "width" of the kink), then neither the propagator nor the vertices produced by (3.27) will depend on a (corrections are of order $e^{-\sqrt{2}\mu T}$). Also we have

$$\int_{-T+a}^{T+a} dt\,\mathcal{L}_{\rm cl}\simeq\int_{-T}^{T} dt\,\mathcal{L}_{\rm cl}. \tag{3.29}$$

The Faddeev-Popov determinant,

$$\int_{-T}^{T} dt\,\Psi_0\mathring{\varphi}_k=\left(\frac{2\sqrt{2}\hbar\mu}{3\lambda}\right)^{1/2}, \tag{3.30}$$

can be factored out in (3.28). The remaining integral can be done by normal-mode decomposition. The δ function eliminates the quasizero eigenvalue E_0^2, and we obtain

$$\bar{I}=\left(\frac{2\sqrt{2}\hbar\mu}{3\lambda}\right)^{1/2}\left|\frac{\mathfrak{D}y}{\mathfrak{D}c_n}\right|\prod_{E_n\neq E_0}\left(\frac{\pi\mu^2}{E_n^2}\right)^{1/2}\left(\frac{\mu^2}{\hbar}\right). \tag{3.31}$$

Combining (3.14) and (3.19), we can rewrite this equation as

$$\bar{I}=\left(\frac{2\sqrt{2}\hbar\mu}{3\lambda}\right)^{1/2}\left(\frac{\mu\sqrt{2}}{2\pi\hbar}\right)^{1/2}\left(\frac{E_0^2}{\pi\mu^2}\right)^{1/2}\left(\frac{\mu^2}{\hbar}\right), \tag{3.32}$$

or, substituting for E_0^2 [(3.21a)],

$$\bar{I}=\left(\frac{\mu\sqrt{2}}{\pi\hbar}\right)^{1/2}\frac{\mu}{\sqrt{2}}\left(\frac{16\sqrt{2}\mu^3}{\pi\lambda\hbar}\right)^{1/2}e^{-\sqrt{2}\mu T}. \tag{3.33}$$

Returning finally to the kernel (3.25), we obtain in the Gaussian approximation

$$\langle\varphi_2,T|\varphi_1,-T\rangle=\left(\frac{\mu\sqrt{2}}{\pi\hbar}\right)^{1/2}e^{-\sqrt{2}\mu T}(\sqrt{2}\mu T)\times\left(\frac{16\sqrt{2}\mu^3}{\pi\lambda\hbar}\right)^{1/2}\exp\left(-\frac{2\sqrt{2}\mu^3}{3\lambda\hbar}\right). \tag{3.34}$$

In obtaining this formula, we have neglected $O(1/\mu)$ compared to T.

In (3.34) we have written the terms in a form reminiscent of the desired form. The product of the last three terms is the first term in the expansion of $\sinh[(\Delta E)T/\hbar]$, as will be seen in the next section, where we include the effect of all antikink-kink pairs. We notice that a perturbation expansion would now be fruitful since the previously bothersome quasizero-frequency mode is eliminated from the propagator by the δ function.

IV. CONTRIBUTIONS FROM KINK PLUS ANTIKINK-KINK PAIRS

We begin our discussion again with the path-integral formula for the kernel [Eq. (3.2)]. We notice first that the boundary condition requires summation over all paths which begin at φ_1 (<0) and end at φ_2 (>0). Such paths fall into classes according to the number of times they cross the line $\varphi(t)=0$. This number must be odd. For simplicity we will discuss the case of three zeros, the generalization being obvious.

The path shown in Fig. 4(a) is not a classical trajectory such that $\varphi(-T)=\varphi_1$ and $\varphi(T)=\varphi_2$, but it is nearly stationary. It has the property that for large T, under local translations labeled by

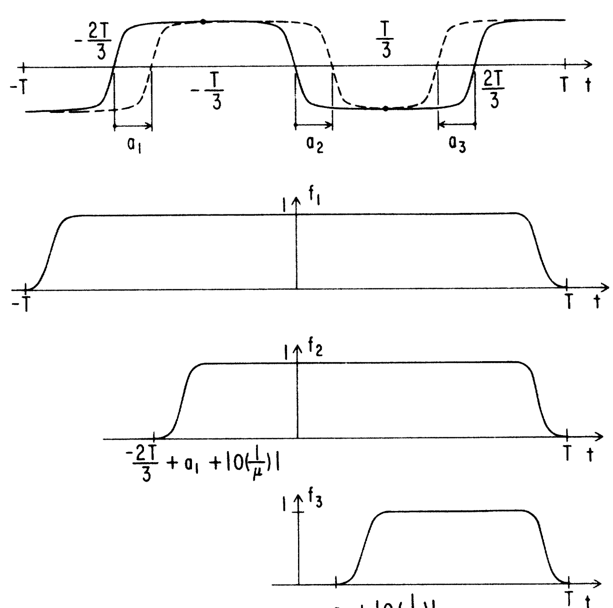

FIG. 4. (a) Three kinks (solid line) and a translation by the a_i's. (b) $f_i(t)$'s, used in performing local translations [Eq. (4.4)].

a_1, a_2, and a_3 [Fig. 4(a)], the action changes only by terms of order e^{-cT} (some c). Thus we have a quasisymmetry in our problem, much as in the case of one kink translations. Hence, we will need three δ functions to fix the zeros at the locations of our choice, say $-\frac{2}{3}T$, 0, and $\frac{2}{3}T$. Having done that, we notice that we can use the completeness of the kernel to write

$$\langle\varphi_2, T|\varphi_1, -T\rangle = \int_{-\infty}^{\infty} d\tilde{\varphi}_1 \int_{-\infty}^{\infty} d\tilde{\varphi}_2 \langle\varphi_2, T|\tilde{\varphi}_2, \tfrac{1}{3}T\rangle \times\langle\tilde{\varphi}_2, \tfrac{1}{3}T|\tilde{\varphi}_1, -\tfrac{1}{3}T\rangle \times\langle\tilde{\varphi}_1, -\tfrac{1}{3}T|\varphi_1, -T\rangle. \quad (4.1)$$

The steepest-descent method applied to compute the intermediate-time kernels in (4.1) and to do the $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ integrations will select as the optimum paths (nearly stationary) the kink plus antikink-kink pair, so that $-\tilde{\varphi}_2 = \tilde{\varphi}_1 = \mu^2/\lambda$. The Gaussian integrations over $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ will introduce two determinants of the form

$$\Delta = \left(\frac{\pi}{\frac{1}{2\hbar}\frac{\partial^2 S[\varphi_k]}{\partial\varphi_{1k}{}^2}}\right)^{1/2} \quad (4.2)$$

We compute $\partial^2 S/\partial\varphi_{1k}{}^2$ in Appendix B and we find

$$\Delta = \left(\frac{\pi\hbar}{\sqrt{2}\,\mu}\right)^{1/2}. \quad (4.3)$$

As mentioned in the Introduction, the kink plus antikink-kink pair describes tunneling back and forth between the two wells; in Sec. V we will associate the determinant (4.3) with the ground-state wave function.

Having outlined the procedure for including contributions from kink plus antikink-kink pairs, we will now give the details of the computation. We begin by defining the Faddeev-Popov determinant associated with the three δ functions:

$$\Delta^{(3)}[F(\varphi(t))]\int_{-T/3}^{5T/3} da_1 \int_{-2T/3+a_1}^{T} da_2 \int_{-2T/3+a_2}^{T/3} da_3\, \delta\left(\int_{-T}^{T} dt\,\Psi_0(t+\tfrac{2}{3}T)\varphi(t+\tfrac{2}{3}T - a_1 f_1(t))\right)\delta\left(\int_{-T}^{T} dt\,\Psi_0(t)\varphi(t-a_2 f_2(t))\right) \times \delta\left(\int_{-T}^{T} dt\,\Psi_0(t-\tfrac{2}{3}T)\varphi(t-\tfrac{2}{3}T - a_3 f_3(t))\right) = 1. \quad (4.4)$$

Here a_1, a_2, and a_3 label local translations, and f_1, f_2, and f_3 are shown in Fig. 4(b). The translations are made so as to preserve the order of the kinks. We notice the invariance of $\Delta^{(3)}[F(\varphi(t))]$ under independent, but appropriately chosen local time translations. We also note that

$$\Delta^{(3)}[F(\varphi(t))] \simeq \{\Delta[F(\varphi(t))]\}^3, \quad (4.5)$$

and that

$$\int_{-T/3}^{5T/3} da_1 \int_{-2T/3+a_1}^{T} da_2 \int_{-2T/3+a_2}^{T/3} da_3 = \frac{(2T)^3}{3!}. \quad (4.6)$$

In (4.4) and (4.6) we have neglected the width of the kink, $O(1/\mu)$, compared to T [see Eq. (3.22)].

We have at hand all the pieces of the calculation; to assemble them we follow the principle outlined in (4.1). The manipulations are familiar from the one-kink case. The contribution to the kernel of kink plus antikink-kink is

$$\langle\varphi_2, T|\varphi_1 - T\rangle = \left(\frac{\mu\sqrt{2}}{\pi\hbar}\right)^{1/2} e^{-\sqrt{2}\mu T}\frac{(\sqrt{2}\,\mu T)^3}{3!} \times\left(\frac{16\sqrt{2}\,\mu^3}{\pi\lambda\hbar}\right)^{3/2}\left[\exp\left(-\frac{2\sqrt{2}\,\mu^3}{3\lambda\hbar}\right)\right]^3. \quad (4.7)$$

[For the reader's benefit: In evaluating (4.1), we multiplied three quasizero frequencies $\sqrt{24}\mu\exp(-\sqrt{2}\,\mu\frac{1}{3}T)$, hence the factor $\exp(-\sqrt{2}\,\mu T)$ in (4.7).]

The generalization to the case of kink plus any number of antikink-kink pairs is obvious. The sum over all contributions is trivial and we obtain the final expression for the kernel:

$$\langle\varphi_2, T|\varphi_1, -T\rangle = \left(\frac{\mu\sqrt{2}}{\pi\hbar}\right)^{1/2} e^{-\sqrt{2}\mu T} \times\sinh\left[\sqrt{2}\,\mu T\left(\frac{16\sqrt{2}\,\mu^3}{\pi\lambda\hbar}\right)^{1/2} \times\exp\left(-\frac{2\sqrt{2}\,\mu^3}{3\lambda\hbar}\right)\right]. \quad (4.8)$$

Comparing this with the desired form [Eq. (2.4)], we can identify

$$\langle\varphi = (\mu^2/\lambda)^{1/2}|R_0\rangle = \left(\frac{\mu\sqrt{2}}{\pi\hbar}\right)^{1/4}, \quad (4.9)$$

$$E_0 = \frac{\hbar\sqrt{2}\,\mu}{2}, \quad (4.10)$$

$$\Delta E = \sqrt{2}\,\mu\hbar\left(\frac{16\sqrt{2}\,\mu^3}{\pi\lambda\hbar}\right)^{1/2}\exp\left(-\frac{2\sqrt{2}\,\mu^3}{3\lambda\hbar}\right). \quad (4.11)$$

V. COMPARISON WITH THE WKB APPROXIMATION

[*Note*. In this section we shall mean by the "standard" WKB result that one which is based on the linear (Airy functions) connecting formulas. These connecting formulas are derived for the highly excited states, but have been shown to give a very good approximation for even the lowest state.[9] If one uses the quadratic (harmonic-oscillator functions) connecting formulas, one will reproduce our result exactly. For those to whom these remarks are obvious, we suggest skipping this section.]

Equation (2.4) can also be evaluated using standard WKB results. One can show that[10]

$$\Delta E = 2\hbar^2 \left[\langle \varphi | R_0 \rangle \frac{\partial}{\partial \varphi} \langle \varphi | R_0 \rangle \right]_{\varphi=0} . \tag{5.1}$$

The WKB wave function inside the barrier is

$$\langle \varphi | R_0 \rangle \Big|_{\varphi=0} = \left(\frac{\sqrt{2}\,\mu}{2\pi |p(0)|} \right)^{1/2} \exp\left(-\frac{1}{\hbar} \int_0^a |p| \, d\varphi \right), \tag{5.2}$$

and its derivative is

$$\frac{\partial}{\partial \varphi} \langle \varphi | R_0 \rangle \Big|_{\varphi=0} = \frac{|p(0)|}{\hbar} \langle \varphi | R_0 \rangle \Big|_{\varphi=0} , \tag{5.3}$$

where $p(\varphi)$ is the classical momentum function and a is the classical turning point at the barrier. Thus

$$\Delta E = \frac{\sqrt{2}\,\mu\hbar}{\pi} \exp\left(-\frac{1}{\hbar} \int_{-a}^{a} |p| \, d\varphi \right), \tag{5.4}$$

where

$$\int_{-a}^{a} |p| \, d\varphi = \int_{-a}^{a} d\varphi \left\{ 2 \left[\frac{\lambda}{4} \left(\varphi^2 - \frac{\mu^2}{\lambda} \right)^2 - \frac{\mu}{\sqrt{2}} \hbar \right] \right\}^{1/2} \tag{5.5}$$

and

$$a = \left[\frac{\mu^2}{\lambda} - \left(\frac{2\sqrt{2}\,\mu\hbar}{\lambda} \right)^{1/2} \right]^{1/2} .$$

This is an elliptic integral. If one expands to lowest nonvanishing order in $(\lambda\hbar/\mu^3)^{1/2}$ the result is

$$\Delta E = \frac{4\mu^2}{\pi} \left(\frac{2\sqrt{2}\,e}{\hbar\lambda} \right)^{1/2} \exp\left(-\frac{2\sqrt{2}}{3} \frac{\mu^3}{\lambda\hbar} \right). \tag{5.6}$$

Comparing with the path-integral result [Eq. 4.11],

$$\frac{\Delta E_{(\mathrm{WKB})}}{\Delta E_{(\mathrm{path\ integral})}} = \left(\frac{e}{\pi} \right)^{1/2} . \tag{5.7}$$

Now note that Eqs. (5.1) and (5.3) imply that ΔE is proportional to the square of the ground-state wave function evaluated at the center of the barrier. Furry[9] has studied the WKB approximation with connecting formulas for the harmonic oscillator. If one takes the square of the ratio of his approximate WKB wave function for the ground state to the exact harmonic-oscillator wave function evaluated deep inside the classically inaccessible region, one gets precisely the value of Eq. (5.7). Thus the path integral does better than the WKB approximation because it avoids the errors introduced by the connecting formulas.

We know from Eq. (2.4) that the kernel is proportional to the value of the square of the wave function evaluated at the bottom of the potential well. The value of the WKB wave function so evaluated is

$$\langle \varphi = (\mu^2/\lambda)^{1/2} | R_0 \rangle_{\mathrm{WKB}} = \left(\frac{4\sqrt{2}\,\mu}{\pi^2 \lambda} \right)^{1/4} . \tag{5.8}$$

The square of the ratio of this WKB wave function to that derived from the functional integral [see Eq. (4.9)] is

$$\frac{\langle \varphi = (\mu^2/\lambda)^{1/2} | R_0 \rangle^2_{\mathrm{WKB}}}{\langle \varphi = (\mu^2/\lambda)^{1/2} | R_0 \rangle^2_{\mathrm{path\ integral}}} = \left(\frac{4}{\pi} \right)^{1/2} \tag{5.9}$$

Again this is exactly what one gets by comparing the corresponding WKB versus exact harmonic-oscillator results.[9] As we by now expect, the path-integral result again avoids the damage done by the connecting formulas.

Note. While completing this work we received a report by Polyakov[11] in which he computes the correlation function $\langle x(0)\, x(\tau) \rangle$ for the anharmonic oscillator. Thus the path integral is evaluated over an infinite time interval. This approach allows one to calculate the energy splitting, but not the ground-state energy. Computing the kernel determines both. Our analysis differs not only in spirit, but also execution. For example, we fail to see in that paper the determinant needed in joining kink-antikink pairs [see Eq. (4.2)].

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APPENDIX A

Imposing the boundary condition $\Psi_n(-T) = \Psi_n(T) = 0$ is equivalent to putting the infinite-domain system into a finite box. Thus the continuum of states is replaced by the discrete infinity of bound states given in (3.21c). The energy of the bound states $\tilde{E}_0^{\,2} = 0$ and $\tilde{E}_1^{\,2} = \frac{3}{4}\mu^2$ is increased owing to the infinite walls at $\pm T$. Since the bound-state wave functions in the infinite domain die off exponentially at large $|t|$, the shifts in the eigenvalues are of order e^{-cT}, where c can be computed. For our purposes the only quantity of interest is

E_0^2, which we will presently determine.

We do this by treating the boundary condition as a small perturbation. We denote by $\tilde{\Psi}_0(t)$ the zero-energy wave function is an infinite domain and by $\Psi_0(t)$ the corresponding wave function in the box. They satisfy the equations ($y \equiv \mu t/\sqrt{2}$)

$$\left(\partial_y^2 + \frac{2}{\cosh^2 y} - 4\tanh^2 y\right)\Psi_0 = -\frac{E_0^2}{\mu^2}\Psi_0\,, \qquad \text{(A1)}$$

$$\left(\partial_y^2 + \frac{2}{\cosh^2 y} - 4\tanh^2 y\right)\tilde{\Psi}_0 = 0\,. \qquad \text{(A2)}$$

Equation (A2) can easily be integrated to give

$$\tilde{\Psi}_0(y) = \frac{1}{\cosh^2 y}\,. \qquad \text{(A3)}$$

Multiplying (A1) by $\tilde{\Psi}_0$ and (A2) by Ψ_0, integrating from $-\mu T/\sqrt{2}$ to $\mu T/\sqrt{2}$, and subtracting we obtain

$$(\tilde{\Psi}_0\partial_y\Psi_0 - \Psi_0\partial_y\tilde{\Psi}_0)\Big|_{-\mu T/\sqrt{2}}^{\mu T/\sqrt{2}} = -\frac{E_0^2}{\mu^2}\int_{-\mu T/\sqrt{2}}^{\mu T/\sqrt{2}} dy\,\tilde{\Psi}_0\Psi_0\,. \qquad \text{(A4)}$$

To evaluate E_0^2 we need $\partial_y\Psi_0$ evaluated at $\pm\mu T/\sqrt{2}$. We can find it using the WKB method. For $y \simeq \mu T/\sqrt{2}$ we have

$$\tilde{\Psi}_0(y) \simeq \frac{c}{\sqrt{p}}\exp\left(-\int_a^y |p(y)|dy\right) \qquad \text{(A5)}$$

$$\Psi_0(y) \simeq \frac{c}{\sqrt{p'}}\left[\exp\left(-\int_a^y |p'(y)|dy\right) - \exp\left(-2\int_a^{\mu T/\sqrt{2}} |p'(y)|dy\right) \times \exp\left(\int_a^y |p'(y)|dy\right)\right], \qquad \text{(A6)}$$

where a is the classical turning point and p the classical momentum. For large y we observe that $p(y) \simeq$ constant, and so for $y = \mu T/\sqrt{2}$ we have

$$\partial_y\tilde{\Psi}_0 = -c|p|\tilde{\Psi}_0\,, \qquad \text{(A7)}$$

$$\partial_y\Psi_0 = -2c|p|\tilde{\Psi}_0 = 2\partial_y\tilde{\Psi}_0\,. \qquad \text{(A8)}$$

Since to leading order

$$\int_{-\mu T/\sqrt{2}}^{\mu T/\sqrt{2}} dy\,\tilde{\Psi}_0\Psi_0 = \int_{-\infty}^{\infty} dy\,\tilde{\Psi}_0\tilde{\Psi}_0\,, \qquad \text{(A9)}$$

we obtain from (A4)

$$E_0^2 = 24\mu^2 e^{-2\sqrt{2}\mu T}\,. \qquad \text{(A10)}$$

APPENDIX B

We know that[12]

$$S(T,\varphi_1,\varphi_2) = W(E,\varphi_1,\varphi_2) - ET\,, \qquad \text{(B1)}$$

and that

$$W(E,\varphi_1,\varphi_2) = \int_{\varphi_1}^{\varphi_2}\left\{2\left[E + \frac{\lambda}{4}\left(\varphi^2 - \frac{\mu^2}{\lambda}\right)^2\right]\right\}^{1/2} d\varphi\,. \qquad \text{(B2)}$$

From (B1) we have

$$\frac{\partial S}{\partial\varphi_1} = \frac{\partial W}{\partial\varphi_1}\,, \qquad \text{(B3)}$$

which leads to

$$\frac{\partial^2 S}{\partial\varphi_1^2} = \frac{\partial^2 W}{\partial\varphi_1^2} + \frac{\partial^2 W}{\partial E\partial\varphi_1}\frac{\partial E}{\partial\varphi_1}\,. \qquad \text{(B4)}$$

Using (B1) again and then (B3), we obtain

$$\frac{\partial E}{\partial\varphi_1} = -\frac{\partial^2 S}{\partial\varphi_1\partial T} = -\frac{\partial}{\partial T}\frac{\partial S}{\partial\varphi_1} = -\frac{\partial}{\partial T}\frac{\partial W}{\partial\varphi_1} = -\frac{\partial^2 W}{\partial E\partial\varphi_1}\frac{\partial E}{\partial T} = -\frac{\partial^2 W}{\partial E\partial\varphi_1}\frac{1}{(\partial T/\partial E)} = -\frac{\partial^2 W}{\partial E\partial\varphi_1}\frac{1}{(\partial^2 W/\partial E^2)} \qquad \text{(B5)}$$

Therefore,

$$\frac{\partial^2 S}{\partial\varphi_1^2} = \frac{\partial^2 W}{\partial\varphi_1^2} - \frac{\partial^2 W}{\partial E\partial\varphi_1}\frac{\partial^2 W}{\partial E\partial\varphi_1}\frac{1}{(\partial^2 W/\partial E^2)}\,. \qquad \text{(B6)}$$

For the kink, ($B2$) and (B6) yield $\partial^2 S/\partial\varphi_1^2 = 2\sqrt{2}\mu$.

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[1] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974).

[2] A. Patrascioiu, Phys. Rev. D 12, 523 (1975).

[3] A. M. Polyakov, Phys. Lett. 59B, 82 (1975); A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. 59B, 85 (1975); A. Patrascioiu, Phys. Rev. D 15, 3051 (1977).

[4] C. Callan, R. Dashen, and D. Gross, Phys. Lett. 63B, 334 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976).

[5] See A. Patrascioiu, Ref. 3.

[6] There is extensive literature on this subject. See for instance J. L. Gervais and B. Sakita, Phys. Rev. D 11, 2943 (1975); E. Tomboulis, *ibid.* 12, 1678 (1975).

[7] L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967).

[8] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. II, Eq. (12.3.22).

[9] W. H. Furry, Phys. Rev. 71, 360 (1946).

[10] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1958), p. 176.

[11] A. M. Polyakov, Nordita Report No. 76/73 (unpublished).

[12] See any classical mechanics text, e.g., H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1950).