Massless excitations in pseudoparticle fields*

Lowell S. Brown, Robert D. Carlitz,† and Choonkyu Lee
Department of Physics, University of Washington, Seattle, Washington 98195
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A simple relation is derived which specifies the number of massless excitations of a Dirac field of any isospin in the field of an arbitrary configuration of pseudoparticles. The small fluctuations of the pseudoparticle field are related to those of a Dirac field with unit isospin, proving that a pseudoparticle field with winding number $\pm n$ has $8n$ independent modes of small oscillations. Thus, a general configuration of n pseudoparticles or antipseudoparticles should be characterized by $8n$ parameters.

I. INTRODUCTION

A fascinating aspect of non-Abelian gauge theories involves the existence of topologically stable, classical field solutions in Euclidean space-time.[1] These "pseudoparticle" solutions have a finite action and thus describe significant paths for quantum-mechanical tunneling in Minkowski space-time. This tunneling process has profound implications on the structure of the vacuum state and on the nature of symmetries in gauge theories.[2] The tunneling process may provide a mechanism for the confinement of quarks.[3]

The pseudoparticle solutions are classical solutions to the Euclidean, non-Abelian field equations for the gauge group SU(2). Solutions with n psuedoparticles (or n antipseudoparticles) are characterized by a topological winding number (Pontryagin index) n (or $-n$).[1] Explicit forms have been found corresponding to configurations with arbitrary space-time positions and arbitrary sizes.[4,5] The most general form which has been found[5] entails a total of $5n+4$ parameters. An obvious question concerns the number of parameters needed to describe the most general configuration of n pseudoparticles. A principal result of the present paper is the demonstration that this number is $8n$ (or $8n-3$ if the 3 parameters which characterize a global gauge rotation are not counted). That there are $8n$ independent parameters should not be unexpected. A single-pseudoparticle solution involves 8 parameters: 4 parameters specify its position, 1 parameter specifies its size, and 3 parameters specify its global gauge orientation. A configuration of n pseudoparticles which are widely separated from each other (relative to their sizes) can be approximately described by a "dilute gas" of n noninteracting pseudoparticles (with the field strength being simply the sum of the field strengths of n single pseudoparticles). Since each of the single pseudoparticles is described by 8 parameters, the n-pseudoparticle solution in the dilute-gas approximation is described by $8n$ parameters. Our result shows that this enumeration of parameters is not altered when the pseudoparticles become more closely spaced and an exact solution is necessary.

We shall not attempt here to construct explicit pseudoparticle solutions with a maximal number of parameters. Instead, we shall consider small, massless fluctuations of the vector potential about a pseudoparticle field presumed to be known. The response of a general pseudoparticle field to the variation of one of its continuous parameters yields such a small-fluctuation potential. Hence, the number of independent massless excitations must be at least as great as the number of parameters that characterize a general pseudoparticle field. On the other hand, given the vector potential of a massless excitation, a new pseudoparticle field can be constructed which differs from the old field by an infinitesimal amount α. The full non-Abelian field equations then determine the further corrections of order α^2, $\alpha^3, \ldots,$ and one can formally construct a complete solution for a finite parameter α. Thus, the number of independent parameters in the general pseudoparticle solution should be equal to the number of independent modes of massless excitations which it supports. We shall prove that there are precisely $8n$ modes of independent massless excitations in an n-pseudoparticle field.[6] We should note that each mode of massless excitation (or each continuous parameter in the pseudoparticle field) requires the introduction of a collective coordinate.[7] Our work thus enumerates the number of collective coordinates required in any pseudoparticle calculation.

In the next section, we examine the vector field equations for the small fluctuations about a pseudoparticle field which is self-dual (or anti-self-dual). We find that this vector field must yield a field strength tensor which is itself self-dual (or anti-self-dual). Moreover, we find that the vector field equations are equivalent to the Dirac equation for a spinor with unit isospin and with definite chirality. The problem of counting the number of modes of massless excitations of the vector field is thus reduced to that of counting the number of massless modes of the Dirac field. In the last section, we present a simple derivation of a theorem[8] which relates the number of massless modes of the Dirac field to the winding number of the pseudoparticle field.

II. FIELD EQUATIONS: VECTORS AND SPINORS

We consider an SU(2) gauge theory in Euclidean space-time. It has a field strength tensor[9]

$$F_{\mu\nu a}=\partial_\mu A_{\nu a}-\partial_\nu A_{\mu a}+\epsilon_{abc}A_{\mu b}A_{\nu c} \tag{1}$$

and Lagrange function

$$g^2\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu a}{}^2-\tfrac{1}{2}(\mathcal{G}_{\mu ab}A_{\mu b})^2\;, \tag{2}$$

where the second term is the one which fixes the gauge. (We have omitted irrelevant "ghost" field contributions.) The operator $\mathcal{G}_{\mu ab}$ may depend upon a classical solution to the field equations (the "background field gauge"[10]) but such a field dependence is not to be varied in deriving the field equations. Hence these equations are of the form

$$D_{\mu ab}F_{\mu\nu b}+\bar{\mathcal{G}}_{\nu ab}(\mathcal{G}_{\mu bc}A_{\mu c})=0\;, \tag{3}$$

where

$$D_{\mu ab}=\partial_\mu\delta_{ab}+\epsilon_{acb}A_{\mu c} \tag{4}$$

is the gauge-covariant derivative, and the precise form of $\bar{\mathcal{G}}_{\mu ab}$ follows from the variation of the Lagrangian (2) with respect to $A_{\nu a}$. We shall often suppress isospin indices, and treat them in a matrix notation. Thus we have

$$[D_\mu, D_\nu]=F_{\mu\nu}\;, \tag{5}$$

where the field strength matrix $F_{\mu\nu}$ has the components

$$F_{\mu\nu ab}=F_{\mu\nu c}\,\epsilon_{acb}\;. \tag{6}$$

Accordingly, if we take the gauge-covariant divergence of Eq. (3), making use of the antisymmetry $F_{\mu\nu}=-F_{\nu\mu}$ and the commutator (5), we get

$$D_{\mu ab}\bar{\mathcal{G}}_{\mu bc}(\mathcal{G}_{\nu cd}A_{\nu d})=0\;. \tag{7}$$

In general, the operator $D_{\mu ab}\bar{\mathcal{G}}_{\mu bc}$ is nonsingular and Eq. (7) implies that the vector potential must obey the gauge constraint

$$\mathcal{G}_{\nu ab}A_{\nu b}=0\;. \tag{8}$$

Thus the field equations (3) can be replaced by the constraint (8) and

$$D_{\mu ab}F_{\mu\nu b}=0\;. \tag{9}$$

We shall consider small fluctuations about general pseudoparticle fields obeying Eqs. (8) and (9). These fluctuation fields should be square integrable and thus cannot change the winding number. We shall use a superscript cl to label a pseudoparticle field solution to Eqs. (8) and (9). The covariant derivative $D_{\mu ab}$ [Eq. (4)] will henceforth always involve $A^{\rm cl}$, but for notational simplicity the superscript cl will be deleted from this operator. We shall adopt the background-field gauge specified by

$$\mathcal{G}_{\mu ab}=D_{\mu ab}\;. \tag{10}$$

The general n-pseudoparticle fields which have been found,[4,5] and about which we shall perturb, are either self-dual or anti-self-dual,

$$F^{\rm cl}_{\mu\nu}=\pm {}^*F^{\rm cl}_{\mu\nu}=\pm\tfrac{1}{2}\epsilon_{\mu\nu\lambda\kappa}F_{\lambda\kappa}\;, \tag{11}$$

where $\epsilon_{\mu\nu\lambda\kappa}$ is the completely antisymmetrical symbol with $\epsilon_{1234}=+1$. The action for the small-fluctuation field $\phi_{\mu a}$ is obtained by writing

$$A_{\mu a}=A^{\rm cl}_{\mu a}+\phi_{\mu a} \tag{12}$$

so that

$$F_{\mu\nu a}=F^{\rm cl}_{\mu\nu a}+f_{\mu\nu a}-\phi_{\mu b}\epsilon_{bac}\phi_{\nu c}\;, \tag{13}$$

where

$$f_{\mu\nu a}=D_{\mu ab}\phi_{\nu b}-D_{\nu ab}\,\phi_{\mu b}\;. \tag{14}$$

Inserting these forms into the Lagrange function (2) and identifying the quadratic pieces in ϕ_μ gives the Euclidean action for the small fluctuations:

$$g^2W_2=-\int(d^4_Ex)\,[\tfrac{1}{4}f_{\mu\nu}{}^2-\tfrac{1}{2}\,\phi_\mu F^{\rm cl}_{\mu\nu}\phi_\nu+\tfrac{1}{2}(D_\mu\phi_\mu)^2]\;. \tag{15}$$

Since this action is quadratic in the field ϕ_μ, it vanishes when the field obeys its field equation,

$$g^2W_2=0\;. \tag{16}$$

On the other hand, if we make use of the self-duality property of the pseudoparticle field, Eq. (11), and the commutator (5), we get

$$\int(d^4_Ex)\phi_\mu F^{\rm cl}_{\mu\nu}\phi_\nu=\pm\epsilon_{\mu\nu\lambda\kappa}\int(d^4_Ex)\phi_\mu D_\lambda D_\kappa\phi_\nu =\pm\tfrac{1}{2}\int(d^4_Ex)f_{\mu\nu}\,{}^*f_{\mu\nu}\;, \tag{17}$$

where in the last equality we have performed a partial integration and used the definition (14). (The surface contribution at infinity vanishes because the fluctuation fields ϕ_μ are square integrable.) These results imply that

$$\int(d^4_Ex)[\tfrac{1}{8}(f_{\mu\nu}\mp{}^*f_{\mu\nu})^2+\tfrac{1}{2}(D_\mu\phi_\mu)^2]=0\;. \tag{18}$$

Since a positive sum of squares appears here, we learn that the fluctuation field strength $f_{\mu\nu}$ must have the same duality property as the pseudoparticle field $F^{\rm cl}_{\mu\nu}$,

$$f_{\mu\nu} = \pm {}^*f_{\mu\nu} \ , \tag{19}$$

and that the field ϕ must also obey the gauge constraint obeyed by the pseudoparticle vector potential,

$$D_\mu \phi_\mu = 0 \ . \tag{20}$$

The self-dual (or anti-self-dual) character of the field strength tensor $f_{\mu\nu}$ implies that it transforms according to the irreducible (1,0) [or (0,1)] representation of the Euclidean $O(4) = SU(2) \otimes SU(2)$ group.[11] We can exploit this character if we introduce the antisymmetric symbols

$$\eta^{(\pm)}_{\mu\nu a} = -\eta^{(\pm)}_{\nu\mu a} \tag{21}$$

defined by

$$\eta^{(\pm)}_{kla} = \epsilon_{kla}, \quad \eta^{(\pm)}_{k4a} = \pm\delta_{ka} \ . \tag{22}$$

The symbol $\eta^{(+)}_{\mu\nu a}$ ($\eta^{(-)}_{\mu\nu a}$) is self-dual (anti-self-dual) and the two symbols are projection matrices for antisymmetrical tensors into the irreducible subspaces (1,0) and (0,1) since

$$\eta^{(\pm)}_{\mu\nu a}\eta^{(\pm)}_{\lambda\kappa a} = \delta_{\mu\lambda}\delta_{\nu\kappa} - \delta_{\mu\kappa}\delta_{\nu\lambda} \pm \epsilon_{\mu\nu\lambda\kappa} \ . \tag{23}$$

Thus, we can replace the self-duality condition (19) by

$$\eta^{(\mp)}_{\mu\nu b} f_{\mu\nu a} = 0 \ , \tag{24}$$

which gives three equations (with $b = 1, 2, 3$) for each value of the isospin index a.

The symbols $\eta^{(\pm)}_{\mu\nu a}$ not only play the role of projection matrices of antisymmetrical tensors into irreducible $SU(2) \otimes SU(2)$ subspaces, they also represent the two SU(2) spins in the vector $[(\frac{1}{2}, \frac{1}{2})]$ representation. In order to exhibit this, we define a 2×2 matrix from the vector ϕ_μ,

$$\Phi = \phi_4 + i\sigma_k\phi_k = i\sigma^\dagger_\mu \phi_\mu \ , \tag{25}$$

where σ_k are the usual Hermitian Pauli matrices and $\sigma_4 = i$. Explicitly, Φ has the form

$$\Phi = \begin{pmatrix} \phi_4 + i\phi_3 & \phi_2 + i\phi_1 \\ -\phi_2 + i\phi_1 & \phi_4 - i\phi_3 \end{pmatrix} . \tag{26}$$

Since

$$\mathrm{tr}\,\sigma_\mu \sigma^\dagger_\nu = 2\delta_{\mu\nu} \ , \tag{27}$$

Eq. (25) can be inverted to give

$$\phi_\mu = -\tfrac{1}{2} i\, \mathrm{tr}\,\sigma_\mu \Phi \ . \tag{28}$$

Using

$$\sigma_k\sigma_l = \delta_{kl} + i\epsilon_{klm}\sigma_m \ , \tag{29}$$

we now find that the action of the symbols $\eta^{(\pm)}$ on ϕ is given by

$$-i\eta^{(+)}_{\mu\nu a}\phi_\nu = -\tfrac{1}{2} i\, \mathrm{tr}\,\sigma_\mu\sigma_a\Phi \ , \tag{30a}$$

$$+i\eta^{(-)}_{\mu\nu a}\phi_\nu = -\tfrac{1}{2} i\, \mathrm{tr}\,\sigma_\mu\Phi\sigma_a \ , \tag{30b}$$

which explicitly exhibits their role in representing the SU(2) spins of the irreducible $(\frac{1}{2}, \frac{1}{2})$ representation.

The gauge constraint (20) and the three duality conditions (24) can be united into a single 2×2 matrix equation. In the following development we shall concentrate on the pseudoparticle case ($\eta^{(-)}_{\mu\nu b} f_{\mu\nu a} = 0$) in order to simplify the notation. (As should become evident, the antipseudoparticle case can be treated similarly.) Thus, using Eqs. (30b) and (14), Eqs. (24) and (20) are united into

$$iD_\mu\, \mathrm{tr}\,\sigma_\mu\Phi\sigma_\nu = 0 \tag{31}$$

for $\nu = 1, 2, 3, 4$. Since the σ_ν are a complete set of 2×2 matrices, this requires that

$$iD_\mu\sigma_\mu\Phi = 0 \ . \tag{32}$$

The matrix Φ must obey

$$\sigma_2\Phi^*\sigma_2 = \Phi \tag{33}$$

in order to produce a real field ϕ_μ. The reality constraint (33) is, of course, consistent with Eq. (32) since

$$\sigma_2(i\sigma_\mu)^*\sigma_2 = i\sigma_\mu \ . \tag{34}$$

The colums of the 2×2 matrix Φ are not interchanged by its field equation (32). One of the spinor labels in the $(\frac{1}{2}, \frac{1}{2})$ vector representation is not affected by the field equation. We must be able to construct matrix solutions from spinors χ that obey

$$iD_\mu\sigma_\mu\chi = 0 \ . \tag{35}$$

To make this construction explicit, we note that if

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} \tag{36}$$

is a solution of the spinor field equation (35), then so is

$$-i\sigma_2\chi^* = \begin{pmatrix} -b^* \\ a \end{pmatrix} . \tag{37}$$

Thus, the matrix

$$\Phi^{(1)} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \tag{38}$$

satisfies the field matrix equation (32). The reality constraint (33) uniquely determines the second column of this matrix from its first column as is shown in Eq. (38). From this matrix solution a second, linearly independent solution is obtained by replacing χ by $i\chi$:

$$\Phi^{(2)} = \begin{pmatrix} ia & ib^* \\ ib & -ia^* \end{pmatrix} . \tag{39}$$

We conclude that for each of the linearly independent solutions of the spinor field equation there are precisely two linearly independent solutions of the small-fluctuation, vector field equation.

[Incidentally, $\Phi^{(2)}=\Phi^{(1)}i\sigma_3$, and two other solutions are obtained from $\Phi^{(1)}i\sigma_1$ and $\Phi^{(1)}i\sigma_2$ which obey the reality constraint (33). The solution $\Phi^{(1)}i\sigma_2$ corresponds to a $\Phi^{(1)}$ solution constructed from the spinor $i\sigma_2\chi^*$, and hence these solutions for the vector field correspond to solutions already present in the spinor field equation. Similarly, since $i\sigma_1=-i\sigma_2 i\sigma_3$, the solution $\Phi^{(1)}i\sigma_1$, corresponds to a $\Phi^{(2)}$ solution constructed from $i\sigma_2\chi^*$.]

The spinor wave equation (35) is equivalent to the Dirac equation

$$\gamma_\mu D_\mu \psi=0 \tag{40}$$

for a Dirac field ψ with unit isospin and definite chirality,

$$\gamma_5\psi=-\psi\ . \tag{41}$$

Here in our Euclidean space-time $\gamma_4=i\gamma^0$,

$$\{\gamma_\mu,\gamma_\nu\}=-2\delta_{\mu\nu}\ , \tag{42}$$

and $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$. Since

$$\gamma_k=i\sigma_k\gamma_4\gamma_5 \tag{43}$$

with

$$\sigma_k=\tfrac{1}{2}i\epsilon_{klm}\gamma_l\gamma_m\ , \tag{44}$$

we can write the Dirac equation (40) for a field of the definite chirality (41) as

$$(D_4-i\sigma_k D_k)\gamma_4\psi=0\ , \tag{45}$$

which is indeed identical in form to the spinor wave equation (35). We have written out the development for the case of a pseudoparticle field. For the case of an antipseudoparticle field, the development is entirely similar except that the spinor field is replaced by its adjoint. This replacement yields a Dirac field with the opposite chirality. We shall prove in the following section that the Dirac equation (40) in an n-pseudoparticle field has $4n$ linearly independent solutions. Thus there are $8n$ linearly independent solutions of the small-fluctuation, vector field equation.

III. COUNTING MASSLESS EXCITATIONS

We consider the massless modes of excitation of a Dirac field with arbitrary isospin in an external, classical gauge field. These Dirac fields satisfy

$$\gamma_\mu D_\mu\psi=0\ , \tag{46}$$

where now

$$D_\mu=\partial_\mu-iT_aA_{\mu a}\ , \tag{47}$$

and the T_a form the appropriate representation of the isospin generators. We can establish a theorem concerning the number of these modes by examining the quantity

$$T(m^2)=\mathrm{Tr}\,\frac{m^2}{(\gamma D)^2+m^2}\,\gamma_5\ . \tag{48}$$

Here we employ an operator notation, and the trace is a diagonal sum over all indices, including the space-time coordinates. The quantity $T(m^2)$ is the space-time volume integral of the vacuum expectation value $m\langle\psi^\dagger\gamma_5\psi\rangle$ of a Dirac field of mass m, with the Dirac propagator rationalized and odd powers of γ_μ deleted since they vanish in the trace. The difference $[T(m^2)-T(M^2)]$ is the volume integral of the divergence[12] of a well-defined (Pauli-Villars-regulated) axial-vector current, and hence this difference must vanish. The $M^2\to\infty$ limit of $T(M^2)$ produces the triangle anomaly[13] of the axial-vector current, while the $m^2\to 0$ limit of $T(m^2)$ receives contributions only from the massless modes. Thus, the number of massless modes is related to the triangle anomaly.

Rather than following through the procedure which we just described, it is much simpler to examine the quantity $T(m^2)$ directly. First we note that D_μ obeys the commutation relation

$$[D_\mu,D_\nu]=-iT_aF_{\mu\nu a} \tag{49}$$

and hence by virtue of the anticommutation relation of the γ matrices [Eq. (42)] and the definition

$$\sigma_{\mu\nu}=\tfrac{1}{2}i[\gamma_\mu,\gamma_\nu]\ , \tag{50}$$

we have

$$m^2+(\gamma D)^2=m^2-D^2-\tfrac{1}{2}\sigma_{\mu\nu}T_aF_{\mu\nu a}\ . \tag{51}$$

Equation (48) is not well-defined as it stands, for it involves potentially divergent high-energy contributions. However, we can delete the first two terms of the expansion of $(m^2-D^2-\frac{1}{2}\sigma TF)^{-1}$ in powers of F since they give vanishing contributions in the γ-matrix trace. Hence Eq. (48) can be replaced by the definition

$$T(m^2)=m^2\mathrm{Tr}\,\frac{1}{-D^2+m^2}\,\tfrac{1}{2}\sigma TF\,\frac{1}{-D^2+m^2}\,\tfrac{1}{2}\sigma TF\times\frac{1}{-D^2+m^2-\frac{1}{2}\sigma TF}\,\gamma_5\ , \tag{52}$$

which is well-behaved and which has formal properties that are identical to those which would be inferred from Eq. (48).

The large-m^2 limit of $T(m^2)$ follows quickly from Eq. (52). Only large values of $-D^2$ contribute to this limit and the $-\frac{1}{2}\sigma TF$ term in the last denominator in Eq. (52) can be omitted. Moreover, in this limit $-D^2$ can be replaced by p^2, where the momentum p_μ has the differential-operator realization $-i\partial_\mu$. Thus,

$$\lim_{m^2\to\infty} T(m^2) = \lim_{m^2\to\infty} m^2 \operatorname{Tr} \frac{1}{p^2+m^2}\tfrac{1}{2}\sigma TF \frac{1}{p^2+m^2} \times \tfrac{1}{2}\sigma TF \frac{1}{p^2+m^2}\gamma_5 \,. \tag{53}$$

The commutator of F with $(p^2+m^2)^{-1}$ produces terms of higher order in $(p^2+m^2)^{-1}$ that do not contribute to the limit, and we can evaluate the trace in the coordinate representation to get

$$\lim_{m^2\to\infty} T(m^2) = \int (d_E^4x)\operatorname{tr}(\tfrac{1}{2}\sigma TF)^2\gamma_5 \times \lim_{m^2\to\infty}\left\langle x\left|\frac{m^2}{(p^2+m^2)^3}\right|x\right\rangle, \tag{54}$$

where tr refers to a diagonal sum over spinor and isospin indices. Now

$$\left\langle x\left|\frac{m^2}{(p^2+m^2)^3}\right|x\right\rangle = \int\frac{(d^4p)}{(2\pi)^4}\frac{m^2}{(p^2+m^2)^3} = \frac{1}{2}\frac{1}{(4\pi)^2}\,, \tag{55}$$

while the spin and isospin traces are given by

$$\operatorname{tr}\sigma_{\mu\nu}\sigma_{\lambda\kappa}\gamma_5 = -4\epsilon_{\mu\nu\lambda\kappa}\,, \tag{56}$$

and

$$\operatorname{tr}T_aT_b = \tfrac{1}{3}\delta_{ab}t(t+1)(2t+1)\,, \tag{57}$$

where t is the magnitude of the total isospin represented by the Dirac field. Hence,

$$\lim_{m^2\to\infty} T(m^2) = -\tfrac{1}{3}t(t+1)(2t+1) \times \frac{1}{(4\pi)^2}\int (d_E^4x)F_{\mu\nu a}{}^*F_{\mu\nu a}\,. \tag{58}$$

We remarked previously that $T(m^2)$ is, in fact, independent of the value of m^2. This is established most easily from the formal expression (48) which gives

$$\frac{\partial}{\partial m^2}T(m^2) = \operatorname{Tr}\frac{(\gamma D)^2}{[(\gamma D)^2+m^2]^2}\gamma_5\,. \tag{59}$$

Since γ_5 anticommutes with γD, we can use the cyclic symmetry of the trace to conclude that

$$\frac{\partial}{\partial m^2}T(m^2) = -\frac{\partial}{\partial m^2}T(m^2) = 0\,. \tag{60}$$

Thus, $T(m^2)$ is a constant which can also be evaluated by the small-m^2 limit

$$T = \lim_{m^2\to 0}\operatorname{Tr}\frac{m^2}{(\gamma D)^2+m^2}\gamma_5 = \operatorname{Tr}P_0\gamma_5\,, \tag{61}$$

where P_0 is a projection operator into the subspace of massless modes, the subspace spanned by the solutions to Eq. (46). Hence, if there are n_+ massless modes with positive chirality ($\gamma_5'=+1$) and n_- massless modes with negative chirality ($\gamma_5'=-1$), we have

$$T = n_+ - n_-\,. \tag{62}$$

Comparing this with Eq. (58) yields

$$n_+ - n_- = -\tfrac{1}{3}t(t+1)(2t+1) \times \frac{1}{(4\pi)^2}\int (d_E^4x)F_{\mu\nu a}{}^*F_{\mu\nu a}\,. \tag{63}$$

The integral which appears here is related[1] to the winding number ν,

$$\frac{1}{(4\pi)^2}\int (d_E^4x)F_{\mu\nu a}{}^*F_{\mu\nu a} = 2\nu\,, \tag{64}$$

where $\nu=+n$ for a field with n pseudoparticles, while $\nu=-n$ for a field with n antipseudoparticles. Thus we obtain the theorem[8]

$$n_+ - n_- = -\tfrac{2}{3}t(t+1)(2t+1)\nu\,. \tag{65}$$

The pseudoparticle gauge fields about which we perturb are characterized not only by a nonvanishing winding number, but also by field strengths which are either self-dual (with $\nu>0$) or anti-self-dual (with $\nu<0$). We will now demonstrate that $n_\pm=0$ for $\nu \gtrless 0$. To prove this we first multiply the field equation (46) of the massless mode functions ψ with $-\gamma D$ to get

$$(D^2 + \tfrac{1}{2}\sigma_{\mu\nu}T_aF_{\mu\nu a})\psi = 0\,. \tag{66}$$

Here we may assume that ψ is a field of definite chirality,

$$\gamma_5\psi = \pm\psi\,, \tag{67}$$

since γ_5 anticommutes with γD which has a zero eigenvalue. It is a simple exercise in γ-matrix algebra (similar to the work at the end of the preceding section) to show that for this field of definite chirality

$$\sigma_{\mu\nu}\psi = \eta^{(\mp)}_{\mu\nu a}\sigma_a\psi\,. \tag{68}$$

Recalling that $\eta^{(-)}_{\mu\nu a}F_{\mu\nu b}$ vanishes for a self-dual field and $\eta^{(+)}_{\mu\nu a}F_{\mu\nu b}$ vanishes for an anti-self-dual field, we find that Eq. (66) implies

$$D^2\psi = 0 \tag{69}$$

for a self-dual (or anti-self-dual) pseudoparticle field $F_{\mu\nu a}$ and a Dirac field ψ with positive (or negative) chirality. Multiplying Eq. (69) by $\psi^\dagger$, integrating over all space-time and performing a partial integration gives

$$\int (d_E^4x)\,|D_\mu\psi|^2 = 0\,. \tag{70}$$

(The square integrability of ψ permits this partial integration.) Accordingly, Eq. (69) implies that

$$D_\mu \psi = 0 , \qquad (71)$$

and

$$i[D_\mu, D_\nu]\,\psi = F_{\mu\nu a} T_a \psi = 0 . \qquad (72)$$

The matrices $\eta^{(\pm)}_{\mu\nu b} F_{\mu\nu a} = C_{ba}$ are generally nonsingular for $\nu \gtrless 0$. Hence ψ must vanish, and we conclude that Eq. (69) admits no solution. Therefore $n_+ = 0$ if $\nu = n > 0$ and $n_- = 0$ if $\nu = -n < 0$, and the previous theorem becomes a statement for the nonvanishing mode number,

$$n_\pm = \tfrac{2}{3} t(t+1)(2t+1)n . \qquad (73)$$

Specializing to unit isospin gives

$$n_\pm = 4n , \qquad (74)$$

proving that there are $8n$ independent massless modes of small fluctuations about the pseudoparticle field.

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