PHYSICAL REVIEW D VOLUME 16, NUMBER 2 15 JULY 1977

Canonical quantization of fields with higher-derivative couplings*

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(Received 28 March 1977)

It is shown with the use of a model of scalar fields how canonical quantization can be carried out when the free part of the Lagrangian density involves only the first derivatives, while the coupling terms involve higher derivatives. Cancellation of noncovariant terms in the scattering operator is found to occur for such couplings provided that $\delta(0)$ terms are eliminated with the use of dimensional regularization.

I. INTRODUCTION

Canonical formalism for fields involving higher derivatives in the free part of the Lagrangian density has been extensively investigated.[1] We shall here consider a related but different problem. We shall show with the use of a model of scalar fields how the canonical quantization can be carried out when the free part of the Lagrangian density involves only the first derivatives, while the coupling terms involve higher derivatives. It will then be demonstrated that cancellation of noncovariant terms in the scattering operator, which is well known for first-derivative couplings,[2] also occurs for higher-derivative couplings provided that $\delta(0)$ terms are eliminated with the use of dimensional regularization.[3]

Although our quantization procedure for higher-derivative couplings is mathematically more complicated than that for first-derivative couplings, it does not present any difficulty in principle. Apart from providing an interesting generalization of the canonical formalism, this investigation is useful because higher-derivative couplings are encountered in some physical problems.[4]

II. CANONICAL FORMALISM

Let us consider a Lagrangian density of the form

$$L=L(\phi_r,\partial_\mu\phi_r,\partial_\mu\partial_\nu\phi_r,\partial_\mu\partial_\nu\partial_\lambda\phi_r,\ldots),\tag{2.1}$$

where $r=1,2,\ldots,n$. It is convenient to define partial derivatives as

$$\frac{\partial(\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_m}\phi_r)}{\partial(\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_m}\phi_s)}=\frac{\delta_{rs}}{m!}\sum_p\delta_{\alpha_1\mu_1}\delta_{\alpha_2\mu_2}\cdots\delta_{\alpha_m\mu_m},\tag{2.2}$$

where the summation extends over the $m!$ permutations of the indices $\mu_1,\mu_2,\ldots,\mu_m$. This definition of partial derivatives ensures that

$$\frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi_r)}=\frac{\partial L}{\partial(\partial_\nu\partial_\mu\phi_r)},\ \text{etc.},\tag{2.3}$$

while the variation in L is expressible as

$$\delta L=\sum_r\left[\frac{\partial L}{\partial\phi_r}\delta\phi_r+\frac{\partial L}{\partial(\partial_\mu\phi_r)}\delta(\partial_\mu\phi_r)+\frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi_r)}\delta(\partial_\mu\partial_\nu\phi_r)+\cdots\right].\tag{2.4}$$

The resulting field equations are

$$\frac{\partial L}{\partial\phi_r}-\partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi_r)}+\partial_\mu\partial_\nu\frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi_r)}-\partial_\mu\partial_\nu\partial_\lambda\frac{\partial L}{\partial(\partial_\mu\partial_\nu\partial_\lambda\phi_r)}+\cdots=0,\tag{2.5}$$

and the canonical (unsymmetrized) energy-momentum tensor is given by

$$S_{\mu\nu}=L\delta_{\mu\nu}-\sum_r\left[\frac{\partial L}{\partial(\partial_\nu\phi_r)}\partial_\mu\phi_r-\left(\partial_\lambda\frac{\partial L}{\partial(\partial_\lambda\partial_\nu\phi_r)}\partial_\mu\phi_r-\frac{\partial L}{\partial(\partial_\lambda\partial_\nu\phi_r)}\partial_\lambda\partial_\mu\phi_r\right)\right.$$
$$\left.+\left(\partial_\lambda\partial_\rho\frac{\partial L}{\partial(\partial_\lambda\partial_\rho\partial_\nu\phi_r)}\partial_\mu\phi_r-\partial_\lambda\frac{\partial L}{\partial(\partial_\lambda\partial_\rho\partial_\nu\phi_r)}\partial_\rho\partial_\mu\phi_r+\frac{\partial L}{\partial(\partial_\lambda\partial_\rho\partial_\nu\phi_r)}\partial_\lambda\partial_\rho\partial_\mu\phi_r\right)-\cdots\right]\tag{2.6}$$

with

$$\partial_\nu S_{\mu\nu}=0.\tag{2.7}$$

When the free part of the Lagrangian density is

$$L_0=-\tfrac{1}{2}\sum_r[(\partial_\mu\phi_r)^2+m_r^2\phi_r^2],\tag{2.8}$$

it is possible to remove second and higher time derivatives from (2.1) by carrying out either covariant or noncovariant field transformations, so that the transformed Lagrangian density involves only higher space derivatives.[5] In the absence of higher time-derivatives in L, the energy density takes the form

$$
\begin{aligned}
H &= -S_{44} \\
&= -L + \sum_r \left[\frac{\partial L}{\partial(\partial_4\phi_r)}\partial_4\phi_r - \left(\partial_i \frac{\partial L}{\partial(\partial_i\partial_4\phi_r)}\partial_4\phi_r - \frac{\partial L}{\partial(\partial_i\partial_4\phi_r)}\partial_i\partial_4\phi_r\right)\right. \\
&\qquad \left. + \left(\partial_i\partial_j \frac{\partial L}{\partial(\partial_i\partial_j\partial_4\phi_r)}\partial_4\phi_r - \partial_i \frac{\partial L}{\partial(\partial_i\partial_j\partial_4\phi_r)}\partial_j\partial_4\phi_r + \frac{\partial L}{\partial(\partial_i\partial_j\partial_4\phi_r)}\partial_i\partial_j\partial_4\phi_r\right) - \cdots\right],
\end{aligned}
$$

or, since three-divergences can be added to H without altering the total energy,

$$
H = -L + \sum_r \left(\frac{\partial L}{\partial(\partial_4\phi_r)} - 2\partial_i \frac{\partial L}{\partial(\partial_i\partial_4\phi_r)} + 3\partial_i\partial_j \frac{\partial L}{\partial(\partial_i\partial_j\partial_4\phi_r)} - \cdots\right)\partial_4\phi_r . \tag{2.9}
$$

Further, by defining π_r, the canonical conjugate of ϕ_r, as

$$
i\pi_r = \frac{\partial L}{\partial(\partial_4\phi_r)} - 2\partial_i \frac{\partial L}{\partial(\partial_i\partial_4\phi_r)} + 3\partial_i\partial_j \frac{\partial L}{\partial(\partial_i\partial_j\partial_4\phi_r)} - \cdots , \tag{2.10}
$$

the energy density can be expressed in the usual form

$$
H = \sum_r i\pi_r\partial_4\phi_r - L . \tag{2.11}
$$

As a specific example, we shall take a system of three interacting scalar fields with the Lagrangian density

$$
L = L_0 + L_{\text{int}} \tag{2.12}
$$

with

$$
L_0 = -\tfrac{1}{2}(\partial_\mu\phi_1)^2 - \tfrac{1}{2}m_1^2\phi_1^2 - \tfrac{1}{2}(\partial_\mu\phi_2)^2 - \tfrac{1}{2}m_2^2\phi_2^2 - \tfrac{1}{2}(\partial_\mu\phi_3)^2 - \tfrac{1}{2}m_3^2\phi_3^2 , \tag{2.13}
$$

$$
L_{\text{int}} = g\partial_\mu\phi_1\partial_\nu\phi_2\partial_\mu\partial_\nu\phi_3 . \tag{2.14}
$$

The second time derivative in the coupling term can be removed by subjecting (2.12) to the transformation

$$
\phi_3 \to \phi_3 - g\partial_4\phi_1\partial_4\phi_2 , \tag{2.15}
$$

and remembering that the Lagrangian density can be simplified by dropping terms of the form $\partial_i\Lambda_i$ or $\partial_4\Lambda_4$. This procedure generates second-order coupling terms involving second time derivatives, which can again be removed by an appropriate field transformation. Thus, it is possible to remove second and higher time derivatives from the coupling terms up to any desired order by successive applications of field transformations. Accordingly, (2.12) is transformed into

$$
\begin{aligned}
L = {} & L_0 + g\phi_{ij}\partial_i\partial_j\phi_3 + 2g\phi_{i4}\partial_i\partial_4\phi_3 - g\phi_{44}\Delta_3\phi_3 - g^2\partial_i\partial_j\phi_{ij}\phi_{44} - 2g^2\partial_i\phi_{i4}(\Delta_1\phi_1\partial_4\phi_2 + \partial_4\phi_1\Delta_2\phi_2) \\
& -\tfrac{1}{2}g^2(\partial_i\phi_{44})^2 + \tfrac{1}{2}g^2(\Delta_1\phi_1\partial_4\phi_2 + \partial_4\phi_1\Delta_2\phi_2)^2 - \tfrac{1}{2}g^2m_3^2\phi_{44}^2 + g^3(2\partial_i\phi_{i4} - \Delta_1\phi_1\partial_4\phi_2 - \partial_4\phi_1\Delta_2\phi_2) \\
& \times\{\partial_4\phi_1[\partial_i(\partial_\mu\phi_1\partial_i\partial_\mu\phi_3) + \partial_i\partial_4\phi_1\partial_i\partial_4\phi_3 - \partial_\mu\phi_1\Delta_3\partial_\mu\phi_3 + \Delta_1\phi_1\Delta_3\phi_3] \\
& \quad + \partial_4\phi_2[\partial_i(\partial_\mu\phi_2\partial_i\partial_\mu\phi_3) + \partial_i\partial_4\phi_2\partial_i\partial_4\phi_3 - \partial_\mu\phi_2\Delta_3\partial_\mu\phi_3 + \Delta_2\phi_2\Delta_3\phi_3]\} \\
& + O(g^4),
\end{aligned} \tag{2.16}
$$

where

$$
\phi_{\mu\nu} = \tfrac{1}{2}(\partial_\mu\phi_1\partial_\nu\phi_2 + \partial_\nu\phi_1\partial_\mu\phi_2) \tag{2.17}
$$

and

$$
\Delta_1 = \partial_i^2 - m_1^2, \quad \Delta_2 = \partial_i^2 - m_2^2, \quad \Delta_3 = \partial_i^2 - m_3^2. \tag{2.18}
$$

From (2.16), we can derive the canonical conjugates of ϕ_1, ϕ_2, and ϕ_3 as defined by (2.10), and express the energy density (2.11) in terms of the canonical variables and their space derivatives. Then, passing over to the interaction picture in the usual manner, we obtain for the interaction energy density

$$
H_{\text{int}} = H_{\text{int}}^{(1)} + H_{\text{int}}^{(2)} + H_{\text{int}}^{(3)} + O(g^4), \tag{2.19}
$$

where

$$H^{(1)}_{\text{int}} = -g\partial_\mu\phi_1\partial_\nu\phi_2\partial_\mu\partial_\nu\phi_3, \tag{2.20}$$

$$H^{(2)}_{\text{int}} = g^2\partial_i\partial_j\phi_{ij}\phi_{44} - 2g^2\partial_i\phi_{i4}\partial_\mu\phi_{\mu4} + \tfrac{1}{2}g^2(\partial_i\phi_{44})^2 - \tfrac{1}{2}g^2(\partial_4\phi_{44})^2 + \tfrac{1}{2}g^2 m_3{}^2\phi_{44}{}^2 - \tfrac{1}{2}g^2\theta_4\theta_4 - \tfrac{1}{2}g^2\tilde{\theta}_4\tilde{\theta}_4, \tag{2.21}$$

$$\begin{aligned} H^{(3)}_{\text{int}} = {}& -g^3(2\partial_i\phi_{i4} + \partial_4\phi_{44})(\partial_4\phi_1\partial_\mu\theta_\mu + \partial_4\phi_2\partial_\mu\tilde{\theta}_\mu + \partial_4{}^2\phi_1\theta_4 + \partial_4{}^2\phi_2\tilde{\theta}_4) \\ & + g^3(\partial_i\partial_j\phi_{ij} + m_3{}^2\phi_{44})(\partial_4\phi_1\theta_4 + \partial_4\phi_2\tilde{\theta}_4) + 2g^3\partial_i\partial_j\phi_{j4}(\partial_i\phi_1\theta_4 + \partial_i\phi_2\tilde{\theta}_4) \\ & + g^3\partial_i\phi_{44}\partial_i(\partial_4\phi_1\theta_4 + \partial_4\phi_2\tilde{\theta}_4) - g^3\partial_4\phi_{44}\partial_i(\partial_i\phi_1\theta_4 + \partial_i\phi_2\tilde{\theta}_4) - g^3\theta_4\tilde{\theta}_4\partial_4{}^2\phi_3, \end{aligned} \tag{2.22}$$

with

$$\begin{aligned} \phi_{\mu\nu} &= \tfrac{1}{2}(\partial_\mu\phi_1\partial_\nu\phi_2 + \partial_\mu\phi_2\partial_\nu\phi_1), \\ \theta_\nu &= \partial_\mu\phi_1\partial_\mu\partial_\nu\phi_3, \\ \tilde{\theta}_\nu &= \partial_\mu\phi_2\partial_\mu\partial_\nu\phi_3. \end{aligned} \tag{2.23}$$

III. LORENTZ INVARIANCE OF SCATTERING OPERATOR

For the evaluation of the contributions of the scattering operator with the interaction energy density (2.19), we require the contractions

$$\begin{aligned} \phi(x)^\bullet\phi(x')^\bullet &= -i\Delta_F(x-x';m), \\ \partial_\mu\phi(x)^\bullet\phi(x')^\bullet &= -i\partial_\mu\Delta_F(x-x';m), \\ \partial_\mu\phi(x)^\bullet\partial'_\alpha\phi(x')^\bullet &= -i\partial_\mu\partial'_\alpha\Delta_F(x-x';m) + i\delta_{\mu4}\delta_{\alpha4}\delta(x-x'), \\ \partial_\mu\partial_\nu\phi(x)^\bullet\phi(x')^\bullet &= -i\partial_\mu\partial_\nu\Delta_F(x-x';m) - i\delta_{\mu4}\delta_{\nu4}\delta(x-x'), \\ \partial_\mu\partial_\nu\phi(x)^\bullet\partial'_\alpha\phi(x')^\bullet &= -i\partial_\mu\partial_\nu\partial'_\alpha\Delta_F(x-x';m) + i\delta_{\mu4}\delta_{\nu4}\delta_{\alpha4}\partial_4\delta(x-x') \\ &\quad + i(\delta_{\mu k}\delta_{\nu4}\delta_{\alpha4} + \delta_{\mu4}\delta_{\nu k}\delta_{\alpha4} + \delta_{\mu4}\delta_{\nu4}\delta_{\alpha k})\partial_k\delta(x-x'), \\ \partial_\mu\partial_\nu\phi(x)^\bullet\partial'_\alpha\partial'_\beta\phi(x')^\bullet &= -i\partial_\mu\partial_\nu\partial'_\alpha\partial'_\beta\Delta_F(x-x';m) - i(\delta_{\mu j}\delta_{\nu k}\delta_{\alpha4}\delta_{\beta4} + \delta_{\mu j}\delta_{\nu4}\delta_{\alpha k}\delta_{\beta4} \\ &\quad + \delta_{\mu j}\delta_{\nu4}\delta_{\alpha4}\delta_{\beta k} + \delta_{\mu4}\delta_{\nu j}\delta_{\alpha k}\delta_{\beta4} \\ &\quad + \delta_{\mu4}\delta_{\nu j}\delta_{\alpha4}\delta_{\beta k} + \delta_{\mu4}\delta_{\nu4}\delta_{\alpha j}\delta_{\beta k})\partial_j\partial_k\delta(x-x') \\ &\quad - i(\delta_{\mu k}\delta_{\nu4}\delta_{\alpha4}\delta_{\beta4} + \delta_{\mu4}\delta_{\nu k}\delta_{\alpha4}\delta_{\beta4} + \delta_{\mu4}\delta_{\nu4}\delta_{\alpha k}\delta_{\beta4} + \delta_{\mu4}\delta_{\nu4}\delta_{\alpha4}\delta_{\beta k})\partial_k\partial_4\delta(x-x') \\ &\quad - i\delta_{\mu4}\delta_{\nu4}\delta_{\alpha4}\delta_{\beta4}(\partial_4{}^2 - \partial_i{}^2 + m^2)\delta(x-x'), \end{aligned} \tag{3.1}$$

where ϕ is a scalar field of mass m, and

$$\Delta_F(x;m) = \lim_{\epsilon\to+0}\frac{1}{(2\pi)^4}\int dk\, e^{ik\cdot x}\frac{1}{k^2+m^2-i\epsilon}. \tag{3.2}$$

Let us consider the second-order term in the scattering operator

$$S_2 = \frac{(-i)^2}{2!}\int dx\int dx'\,T[H^{(1)}_{\text{int}}(x)H^{(1)}_{\text{int}}(x')]. \tag{3.3}$$

After carrying out single contractions, the contribution involving the noncovariant parts of the contractions can be expressed, after some integrations by parts, as

$$S'_2 = i\int dx\, H^{(2)}_{\text{int}}(x), \tag{3.4}$$

which is canceled by the contribution of $H^{(2)}_{\text{int}}$ to the scattering operator.

We next consider the third-order term

$$S_3 = \frac{(-i)^3}{3!}\int dx\int dx'\int dx''\,T[H^{(1)}_{\text{int}}(x)H^{(1)}_{\text{int}}(x')H^{(1)}_{\text{int}}(x'')]. \tag{3.5}$$

Carrying out double contractions, we find that the contribution involving the noncovariant parts of the contractions is expressible, after some integrations by parts, as

$$S'_3 = -ig^3\int dx\int dx'[F(x,x') + \tilde{F}(x,x')] - ig^3\int dx[G(x) + \tilde{G}(x)] + i\int dx\, H^{(3)}_{\text{int}}(x), \tag{3.6}$$

where

$$F(x,x') = [\partial_\mu \partial_\nu \phi_{\mu\nu}(x)\partial_4\phi_2(x)\partial_4 + \partial_\mu\partial_\nu\phi_{44}(x)\partial_\mu\phi_2(x)\partial_\nu + 2\partial_i\partial_j\phi_{i4}(x)\partial_4\phi_2(x)\partial_j + 2\partial_i\partial_j\phi_{i4}(x)\partial_j\phi_2(x)\partial_4 - (\partial^2 - m_3{}^2)\phi_{44}(x)\partial_4\phi_2(x)\partial_4 - \theta_4(x)\partial_\mu\partial_4\phi_3(x)\partial_\mu]\Delta_F(x-x';m_1)\partial'_\alpha\theta_\alpha(x') + \theta_4(x)\partial_\mu\phi_1(x)\partial_\mu\partial_4\Delta_F(x-x';m_3)\partial'_\nu\partial'_\lambda\phi_{\nu\lambda}(x') \quad (3.7)$$

and

$$G = 2(\partial_\mu\partial_\nu\phi_{\mu\nu} - \partial^2\phi_{44} + m_3{}^2\phi_{44})\partial_4\phi_1\theta_4 + 2\partial_\mu\partial_4\phi_{44}\partial_\mu\phi_1\theta_4 + 4\partial_i\partial_j\phi_{j4}\partial_i\phi_1\theta_4 - (2\partial_i\phi_{i4} + \partial_4\phi_{44})\partial_4\phi_1\partial_k\theta_k - \theta_4\tilde{\theta}_4\partial_4{}^2\phi_3\ , \quad (3.8)$$

while $\tilde{F}$ and $\tilde{G}$ can be obtained from F and G by interchanging ϕ_1, m_1, and θ_μ with ϕ_2, m_2, and $\tilde{\theta}_\mu$, respectively. The first two integrals in (3.6) are canceled by the contribution arising from single contractions in

$$(-i)^2 \int dx \int dx' T[H^{(1)}_{\rm int}(x) H^{(2)}_{\rm int}(x')],$$

while the third integral in (3.6) is directly canceled by the contribution of $H^{(3)}_{\rm int}$ to the scattering operator.

In the above treatment of the cancellation of noncovariant terms, we have considered only the tree diagrams. However, cancellation of noncovariant terms for tree diagrams ensures their cancellation also for closed loops except for residual $\delta(0)$ terms, which can be eliminated with the use of dimensional regularization.[3]

We have demonstrated by explicit calculations of the contributions of the scattering operator up to third order that we can take the effective interaction energy density for the system as

$$H_{\rm eff} = -g\partial_\mu\phi_1\partial_\nu\phi_2\partial_\mu\partial_\nu\phi_3, \quad (3.9)$$

and drop noncovariant terms in the contractions of the field operators.

*Work supported in part by the U. S. Energy Research and Development Administration under Contract No. EY-76-S-02-2302.

[1]See, for instance, B. Podolsky and P. Schwed, Rev. Mod. Phys. 20, 40 (1948). Lorentz invariance of the scattering operator in such field theories has recently been demonstrated by C. Bernard and A. Duncan, Phys. Rev. D 11, 848 (1975).

[2]P. T. Matthews, Phys. Rev. 76, 684 (1949).

[3]D. M. Capper and G. Leibbrandt, Lett. Nuovo Cimento 6, 117 (1973); J. Math. Phys. 15, 86 (1974); L. Tataru, Phys. Rev. D 12, 3351 (1975).

[4]It is interesting to note that higher-derivative couplings are unavoidable in the treatment of higher-spin fields with the use of ghost fields.

[5]This result holds whenever the free part of the Lagrangian density is of the conventional form, because the effect of field transformations is equivalent to simplifying coupling terms of a given order with the use of free-field equations and adding higher-order terms.