Lattice approach to string theory*

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We construct a lattice version of dual string theory, which we hope will be useful in systematically analyzing the properties of that theory. Mass renormalization is discussed, and the familiar tachyon problem is seen to arise so long as counterterms are restricted to be local. The dual-loop expansion is discussed and the origin of the critical dimension elucidated. In the interacting theory, the critical dimension is required for Lorentz invariance, but it is also the only dimension for which the coupling-constant renormalization is finite. The choice of bare coupling constant, $g_0 = 1$, is seen to be particularly attractive. If we make this choice, the string interaction coupling constant is calculable in terms of the fundamental rest tension T_0. The instabilities associated with tachyons are discussed, and we propose a method for discovering whether the interactions stabilize the theory. Application of our ideas to the baryon three-string problem is also mentioned.

I. INTRODUCTION

If dual string theory[1] is to be taken seriously as a theory of strong interactions, the deficiencies of the extant models must be removed. Of course, one of the strongest criticisms of the string model is that no version of the model has exhibited the symmetry structure so evident in strong-interaction phenomena. However, as theories go, the string model is still young, and we expect that more realistic versions can be found. So let us accept the fact that we have not yet found the right string model. The two viable string models, the generalized Veneziano model (GVM) and the Neveu-Schwartz-Ramond (NSR) model, are open to serious criticism on general grounds apart from the inadequacy of the spectrum of these models.

In the absence of interactions both models have tachyons. Interactions are introduced perturbatively in a series of Feynman-type diagrams, and the tachyons render the loop corrections meaningless. This situation is reminiscent of that in a field theory perturbed about a classically unstable solution (e.g. the symmetric solution in a field theory in which the symmetry is spontaneously broken). Bardakci and Halpern[2] have suggested the analogous interpretation of the tachyon problem in string theory: that the dual-loop expansion is a perturbation series about an unstable bare vacuum. They have proposed techniques to search for a suitable vacuum. It is clear that such a search must be nonperturbative.

Another serious criticism of dual models is the absence of a parton picture which could explain scaling in deep-inelastic electroproduction. Various proposals have been made for constructing currents in dual models, but none of them have been completely successful. The most interesting proposals of Schwarz, Corrigan, and Fairlie and of Green and Shapiro[3] explain power-law form factors, but lack an explanation of parton phenomena. It seems possible that the difficulty here is not the way currents are introduced, but rather that in the narrow-resonance approximation the hadronic wave function is dominated by the single string components, and a parton picture is associated with the many string components of the wave function. These will only be dominant for strong coupling. It is therefore important to try to understand how the effective coupling depends on the distance scale of the probe: If the effective coupling at short distances is strong, a parton picture may emerge.

To deal with these problems we are motivated to set up a string formalism which will lend itself to nonperturbative techniques. We shall use a lattice version of Mandelstam's interacting string formalism.[4] We work on a lattice in order to regularize the ultraviolet divergences of the theory. Because there are an infinite number of particles in the string theory, divergences appear in the tree approximation, as well as in higher-order loops. Our lattice prescription regularizes all these divergences at once. All lattice calculations are well defined. We propose to study nonperturbative phenomena on the lattice before taking the continuum limit.

In this paper we set up the general lattice formalism and show how the familiar results of the dual-loop expansion emerge. Interactions are introduced in a natural way, and it seems possible to deal with questions which go beyond perturbation theory. This paper is organized as follows: In Sec. II we review the classical string theory choosing the transverse parametrization of Goddard, Goldstone, Rebbi, and Thorn (GGRT).[5] We then quantize the theory by path integrals defined on a lattice in both σ and τ, and give the general prescription for including interactions. In Sec. III we use our lattice formalism to calculate the propagators for both open and closed strings in the absence of interactions. The divergences arising in the continuum limit are removed by counterterms which, we argue, preserve the causal properties of the interactions. There is no local counterterm which avoids the appearance of tachyons in the spectrum. In the absence of interactions the dimension of space-time is not strictly determined to be the critical one, provided one allows nonlocal counterterms to be added to the Lagrangian. In Sec. IV we study the simplest interactions in the weak-coupling limit. The choice $D=26$ is seen to be required if either (a) the continuum limit is required not to introduce a divergence in the bare coupling constant, or (b) the interactions are required to be Lorentz invariant.

In Sec. V we explain the instabilities in the lattice version of the theory which give rise to tachyons in the continuum limit. We point out that these instabilities may be artifacts of the weak-coupling limit and develop a formalism for going beyond perturbation theory. A relativistic theory of the mass spectrum is presented and an integral equation for the mass eigenvalues is derived. We show that the sum over all planar graphs is equivalent to a string theory in which nearest-neighbor sites are coupled by a modified potential which is attractive but admits continuum energy eigenvalues. If all the sites bind in this potential there will be no tachyons in the "planar" approximation. In Sec. VI we discuss how our formalism may offer a tractable method of calculating the spectrum of the baryon three-string. We also mention various problems for future investigation.

II. GENERAL FORMALISM

A. Review of the classical theory

We can summarize the classical dynamics of the dual string with constant rest tension T_0 by a simple action principle if we choose the transverse parametrization of GGRT[5]:

$$x^+(\sigma,\tau) \equiv \frac{1}{\sqrt{2}}(x^0+x^3)=\tau\,,$$
$$\mathcal{P}^+(\sigma,\tau) \equiv \frac{1}{\sqrt{2}}(\mathcal{P}^0+\mathcal{P}^3)=T_0\,, \tag{1.1}$$

where $\mathcal{P}^\mu(\sigma,\tau)$ is the four-momentum density of the string. σ is thus $1/T_0$ times the quantity of P^+ contained in the portion of string extending from one end ($\sigma=0$) to the point labeled by σ. T_0 must have magnitude 13 long tons to yield a Regge trajectory with slope $\alpha'=1\ (\mathrm{GeV})^{-2}$. With this choice, the minus components are dependent variables:

$$\frac{\partial x^-}{\partial\sigma}\equiv x^{-\prime}(\sigma,\tau)=\dot{\vec{\mathrm{x}}}\cdot\vec{\mathrm{x}}'\,,$$
$$\frac{\partial x^-}{\partial\tau}\equiv \dot{x}^-(\sigma,\tau)=\tfrac{1}{2}(\dot{\vec{\mathrm{x}}}^2+\vec{\mathrm{x}}'^2)\,, \tag{1.2}$$

where the overdot denotes $\partial/\partial\tau$ and the prime denotes $\partial/\partial\sigma$. A vector $\vec{\mathrm{v}}$ represents the $D-2$ components perpendicular to the "0" and "3" axes. The action is, simply,

$$W=T_0\int_{\tau_1}^{\tau_2}dt\int_0^{P^+/T_0}d\tau\,\tfrac{1}{2}(\dot{\vec{\mathrm{x}}}^2-\vec{\mathrm{x}}'^2)\,. \tag{1.3}$$

The classical dynamics is specified by requiring that W be stationary with respect to small variations of $\vec{\mathrm{x}}$ away from the classical trajectory. Notice that we have not incorporated the (trivial) equation of motion for P^+ (i.e., $\dot{P}^+=0$) in this action principle; for the variational principle and the ensuing quantum dynamics P^+ is a (constant) parameter equal to the total P^+ carried by all the strings under consideration.[6]

In interpreting the above action principle, we may choose to describe a closed string for which $\vec{\mathrm{x}}(P^+/T_0,\tau)=\vec{\mathrm{x}}(0,\tau)$, an open string for which $\vec{\mathrm{x}}(0,\tau)$, $\vec{\mathrm{x}}(P^+/T_0,\tau)$ are varied independently, or several open and closed strings for which there are discontinuities at internal points in σ in which case each free end is varied independently. There are even solutions for which the number of strings changes with τ. Thus interactions among strings are present even in the classical theory.

B. The quantum theory without interactions

In the absence of interactions, the string theory may be quantized by straightforward canonical rules as discussed by GGRT.[5] The major obstacle in this approach is the ultraviolet divergence in P^- (the light-cone Hamiltonian) due to zero-point fluctuations. If an arbitrary subtraction is made, one must introduce nonlocal counterterms into P^-. In fact, GGRT found that the subtraction is uniquely determined by requiring consistency with Lorentz invariance. Further, if the quantum-mechanical Lorentz generators were taken to be polynomial analogs of the classical ones, the Lorentz group was not represented except in 26 space-time dimensions. If nonlocal modifications to these generators are allowed, Lorentz invariance can be regained.[7] We wish to follow a quantization scheme which regularizes ultraviolet divergences in a local, causal way, as the cutoff goes to infinity.

We are therefore motivated to introduce a lattice at least in the σ coordinate. Interactions are most easily introduced in the Feynman path-integral formalism, and we accordingly choose this quantization procedure. In order to define path integrals, we shall use imaginary time which we divide into discrete units, i.e., we introduce a lattice in τ and σ. To preserve the symmetry between τ and σ, it is natural to choose equal lattice spacings in σ and τ as the speed of signal propagation in (τ, σ) space is unity. In fact, to each order in perturbation theory in the dual-loop expansion, the detailed cutoff dependence can be absorbed in coupling-constant renormalizations. Also for our theory, the path-integral formalism is well known to be equivalent to standard operator methods, at least in the absence of interactions. For pedagogical simplicity we use our symmetric lattice consistently throughout the remainder of this paper, but we shall be sure to point out those results which are cutoff independent and those which are not as we develop the theory.

We denote the fundamental unit of τ and σ by the letter a. $T\equiv\tau_2-\tau_1$ and P^+/T_0 are then integer multiples of a (see Fig. 1):

$$T\equiv\tau_2-\tau_1=(N+1)a\,,$$
$$P^+=MaT_0\,, \tag{2.1}$$
$$\vec{x}_{ij}\equiv\vec{x}(ia,\tau_1+ja),\quad \vec{x}_{M+1,j}\equiv\vec{x}_{1,j}\,.$$

The quantum dynamics is then specified by defining the transition amplitude for observing a string with transverse coordinates $\{\vec{x}_{i,N+1}\}$ at $\tau=\tau_2$ given that the string has coordinates $\{\vec{x}_{i,0}\}$ at $\tau=\tau_1$. For discrete τ and σ this amplitude is $(\hbar=c=1)$

$$\langle\{\vec{x}_{i,N+1}\},\tau_2\,|\,\{\vec{x}_{i,0}\},\tau_1\rangle=\left[\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^{M(N+1)(D-2)}\prod_{i=1}^{M}\prod_{j=1}^{N}\int d^{D-2}x\,e^{iW}\,, \tag{2.2}$$

where iW is a discrete imaginary-time version of the classical action

$$iW=-T_0\,\frac{1}{2}\left[\sum_{j=0}^{N}\sum_{i=1}^{M}(\vec{x}_{i,j+1}-\vec{x}_{ij})^2+\sum_{j=1}^{N}\sum_{i=1}(\vec{x}_{i+1,j}-\vec{x}_{ij})^2+\tfrac{1}{2}\sum_{i=1}(\vec{x}_{i+1,0}-\vec{x}_{i0})^2+\tfrac{1}{2}\sum_{i=1}(\vec{x}_{i+1,N+1}-\vec{x}_{i,N+1})^2\right]\,, \tag{2.3}$$

where the range of i is $1\le i\le M$ for a closed string and $1\le i\le M-1$ for an open string. The factor $[(T_0/2\pi)^{1/2}]^{M(N+1)(D-2)}$ is the two-dimensional analog of the normalization factor normally introduced in the path integral. The reader may verify that this factor is determined by the requirement

$$\int\prod_{i=1}^{M}d^{D-2}x_i\langle\{\vec{x}_i\},a\,|\,\{\vec{x}_{i,0}\},0\rangle\xrightarrow[M\to\infty]{}1\,.$$

We remark that

$$\langle\{\vec{x}_{i,N+1}\},\tau_2\,|\,\{\vec{x}_{i,0}\},\tau\rangle$$

is independent of the lattice spacing a. The continuum limit is therefore obtained by letting $M,N\to\infty$ in fixed ratios: $(N+1)/M=[(\tau_2-\tau_1)/P^+]T_0$.

C. Quantum theory (with interactions)

Using the continuum path-integral formalism, Mandelstam[4] has developed a consistent theory of interacting strings, which in fact yields the unitarized GVM. However, because of the divergences associated with the continuum, he could only infer the normalization factors indirectly by imposing the closure property of functional integrals. Our lattice formulation allows us to calculate these factors directly.

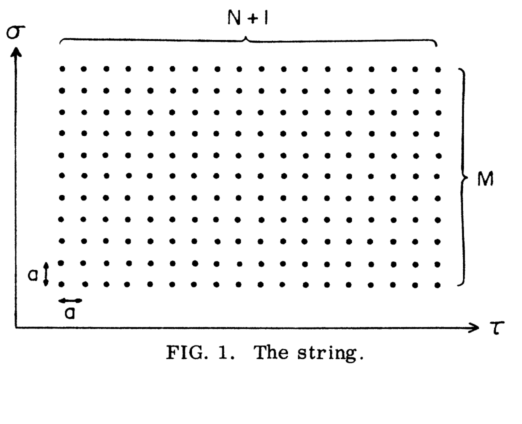


FIG. 1. The string.

Interactions are naturally included on the lattice by summing over paths in which nearest-neighbor couplings in the i variable [we call each term $(\vec{x}_{i+1,j}-\vec{x}_{ij})^2$ a link] are allowed to disappear, appear, or be interchanged between sites from time to time, these changes being restricted only by a topological consideration. In this general picture, the fully interacting system may be viewed as an assembly of M identical bosons (lattice sites) interacting with possible links, each of which can be "on" or "off." A state of the system at any time, j, is specified by giving the transverse position of each site $(\vec{x}_j)$ and the state of each of the $M(M-1)/2$ possible links ("on" or "off"). For the open or closed string we restrict allowed states to those in which *no* site is linked to more than two others. (For a baryonic string, one site is linked to three others.)

We can represent any particular contribution to the sum over histories by a lattice diagram in which lines are drawn between sites when the link is present and are omitted when the link is absent. (A planar topology is shown in Fig. 2.) Once a string has broken into several pieces, the pieces may rejoin in different orders, as for example, the nonplanar topology of Fig. 3. Finally, it is

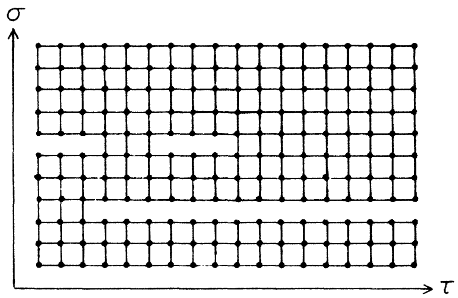


FIG. 2. A simple planar graph.

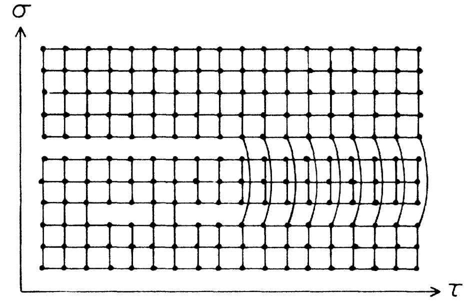


FIG. 3. A simple nonplanar graph.

well known[8] that crossing symmetry requires the interchange interaction depicted in Fig. 4, another kind of nonplanar topology. The complete interacting amplitude is obtained by summing over all allowed topologies. In Sec. IV, we shall discuss interactions in more detail, including the modifications required by renormalization.

III. SINGLE STRING PROPAGATORS; GROUND-STATE ENERGIES AND ZERO-POINT SUBTRACTIONS

The propagators are computed by evaluating the Gaussian integrals in (2.2). Each transverse dimension involves the same calculation and in the following we compute the contribution of a single transverse dimension. At the end of the calculation we shall quote the result including all $D-2$ transverse coordinates. The first step is to obtain the dependence on $\{x_{i,N+1}\}$ and $\{x_{i,0}\}$ by translating x_{ij} by the solution of the classical equations, $(\partial W/\partial x_{ij})(x^c_{ij})=0$, which satisfies $x^c_{i,N+1}=x_{i,N+1}$, $x^c_{i,0}=x_{i,0}$ (see Appendix A). Then

$$\langle\{\vec{x}_{i,N+1}\},\tau_2|\{\vec{x}_{i0}\},\tau_1\rangle=e^{iW(x^{cl}_{ij})}\langle\{\vec{0}\},\tau_2|\{\vec{0}\},\tau_1\rangle\,. \tag{3.1}$$

If we define an $MN\times MN$ matrix $\mathfrak{M}$ by

$$iW_{\vec{0},\vec{0}}=-\tfrac{1}{2}T_0\sum_{ij,i'j'}x_{ij}\mathfrak{M}_{ij;i'j'}x_{i'j'}\,, \tag{3.2}$$

then

$$\langle\{0\},\tau_2|\{0\},\tau_1\rangle\equiv\mathfrak{D}(\tau_2-\tau_1)=(\det{}^{-1/2}\mathfrak{M})\left[\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^M\,. \tag{3.3}$$

$\mathfrak{M}$ has the simple structure

$$\mathfrak{M}=I^M\otimes A^N+B^M\otimes I^N\,, \tag{3.4}$$

where I^M is the identity in the i indices and I^N is that in the j indices,

$$A^N=\begin{bmatrix}2&-1&0&0&\cdot&\cdot&\cdot&&&&0\\-1&2&-1&0&&&&&&&0\\0&-1&2&-1&&&&&&&0\\0&0&-1&2&&&&&&&0\\\cdot&&&&\cdot&&&&&&\cdot\\\cdot&&&&&\cdot&&&&&\cdot\\\cdot&&&&&&\cdot&&&&\cdot\\0&&&&&&&2&-1&0&0\\0&&&&&&&-1&2&-1&0\\0&&&&&&&0&-1&2&-1\\0&&&&\cdot&\cdot&\cdot&0&0&-1&2\end{bmatrix} \tag{3.5}$$

and for the open string,

$$B^M = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdot & \cdot & \cdot & & & & 0 \\ -1 & 2 & -1 & 0 & & & & & & & 0 \\ 0 & -1 & 2 & -1 & & & & & & & 0 \\ 0 & 0 & -1 & 2 & & & & & & & 0 \\ \cdot & & & & \cdot & & & & & & \cdot \\ \cdot & & & & & \cdot & & & & & \cdot \\ \cdot & & & & & & \cdot & & & & \cdot \\ 0 & & & & & & & 2 & -1 & 0 & 0 \\ 0 & & & & & & & -1 & 2 & -1 & 0 \\ 0 & & & & & & & 0 & -1 & 2 & -1 \\ 0 & & & & \cdot & \cdot & \cdot & 0 & 0 & -1 & 1 \end{pmatrix} \tag{3.6}$$

and

$$\det{}^{-1/2}\mathfrak{M} = \prod_{n,m} (\alpha_n + \beta_m)^{-1/2},$$

where

$$\alpha_n = 4\sin^2\frac{n\pi}{2(N+1)}, \quad n = 1, 2, \ldots, N \tag{3.7}$$

and

$$\beta_m = 4\sin^2\frac{m\pi}{2M}, \quad m = 0, 1, \ldots, M-1 \tag{3.8}$$

are the eigenvalues of A^N and B^M, respectively. The product over n can be done because

$$\prod_{n=1}^{N} (\alpha_n - z) = \det(A^N - zI^N) = \frac{\sin(N+1)\kappa}{\sin\kappa}, \tag{3.9}$$

where κ satisfies $z = 4\sin^2(\frac{1}{2}\kappa)$. The reader may easily verify (3.9) by observing that the right-hand side has precisely the same zeros as the left-hand side, and checking the normalization by letting $z \to \infty$. Setting $i\kappa = 2\sinh^{-1}\sin(m\pi/2M)$, we have

$$\begin{aligned} \mathfrak{D}^{\text{open}}(T) &= \left[\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^M \prod_{m=0}^{M-1} \frac{\sinh 2(N+1)\sinh^{-1}\sin(m\pi/2M)}{\sinh 2\sinh^{-1}\sin(m\pi/2M)} \\ &= \left[\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^M M^{1/4}\left(\frac{\sinh 2M\sinh^{-1}1}{\sinh 2\sinh^{-1}1}\right)^{1/4} \frac{1}{(N+1)^{1/2}} \\ &\quad\times \exp\left[-(N+1)\sum_{m=1}^{M-1}\sinh^{-1}\sin\frac{m\pi}{2M}\right] \prod_{m=1}^{M-1}\left\{1 - \exp\left[-4(N+1)\sinh^{-1}\sin\frac{m\pi}{2M}\right]\right\}^{-1/2}. \end{aligned} \tag{3.10}$$

The continuum limit is $M, N \to \infty$ with $(N+1)/M = T_0T/P^+$ fixed. In Appendix A we show that

$$\sum_{m=1}^{M-1}\sinh^{-1}\sin\frac{m\pi}{2M} \underset{M\to\infty}{\sim} \frac{2M}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - \tfrac{1}{2}\sinh^{-1}1 - \frac{\pi}{24M}. \tag{3.11}$$

So

$$(N+1)\sum_{m=1}^{M-1}\sinh^{-1}\sin\frac{m\pi}{2M} \underset{M\to\infty}{\longrightarrow} T\left(\frac{2P^+}{\pi T_0 a^2}G - \frac{1}{2a}\sinh^{-1}1 - \frac{\pi T_0}{24P^+}\right), \tag{3.12}$$

where

$$G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \cong 0.9159656$$

is Catalan's constant. Putting everything together, we have

$$\mathfrak{D}^{\text{open}}(T) \underset{M,N\to\infty}{\sim} \left[\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^M \frac{1}{M^{1/4}}\left(\frac{1}{2\sqrt{2}}\right)^{1/4}\left(\frac{P^+}{TT_0}\right)^{1/2}(\tfrac{1}{2})^{1/4}(1+\sqrt{2})^{M/2}$$
$$\times\exp\left[-T\left(\frac{2MG}{\pi a}-\frac{1}{2a}\sinh^{-1}1-\frac{\pi T_0}{24P^+}\right)\right] \tag{3.13}$$
$$\prod_{m=1}^{\infty}\left[1-\exp\left(-\frac{2\pi T_0 m T}{P^+}\right)\right]^{-1/2}.$$

It is a straightforward matter to repeat the foregoing analysis for the closed string with the result (M odd)

$$\mathfrak{D}^{\text{closed}}(T)=\left[\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^M\left(\frac{M}{N+1}\right)^{1/2}(\sinh M\sinh^{-1}1)^{1/2}\exp\left[-(N+1)\sum_{m=1}^{M-1}\sinh^{-1}\sin\frac{m\pi}{M}\right]$$
$$\times\prod_{m=1}^{M-1}\left\{1-\exp\left[-4(N+1)\sinh^{-1}\sin\frac{m\pi}{M}\right]\right\}^{-1/2} \tag{3.14}$$
$$\underset{M,N\to\infty}{\sim}\left[\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^M\left(\frac{P^+}{2TT_0}\right)^{1/2}(1+\sqrt{2})^{M/2}\exp\left[-T\left(\frac{2MG}{\pi a}-\frac{\pi T_0}{6P^+}\right)\right]\prod_{m=1}^{\infty}\left[1-\exp\left(\frac{-4\pi m TT_0}{P^+}\right)\right]^{-1}. \tag{3.15}$$

For $(D-2)$ transverse dimensions these results are simply raised to the $(D-2)$ power.

To extract information about the ground state, we recall that

$$\mathfrak{D}(T)=\langle\{\vec{0}\},0\,|e^{-P^-T}|\{\vec{0}\},0\rangle,$$
$$\mathfrak{D}(T)=\sum_n\int d^{D-2}p_\perp\exp\left(-\frac{\vec{p}_\perp^{\,2}+m_n^2}{2P^+}T\right)|\langle\{\vec{0}\},0\,|n,\vec{p}_\perp\rangle|^2, \tag{3.16}$$
$$\mathfrak{D}(T)=\left[\left(\frac{2\pi P^+}{T}\right)^{1/2}\right]^{D-2}\sum_n\exp\left(-\frac{m_n^2}{2P^+}T\right)|\langle\{\vec{0}\},0\,|n,\vec{0}\rangle|^2.$$

Comparing with our results we have for the ground state

$$m_G^{2\ \text{open}}=(D-2)\left[\frac{(2P^+)^2}{\pi a^2T_0}G-\frac{P^+}{a}\sinh^{-1}1-\frac{2\pi T_0}{24}\right], \tag{3.17}$$

$$m_G^{2\ \text{closed}}=(D-2)\left[\frac{(2P^+)^2}{\pi a^2T_0}G-\frac{2\pi T_0}{6}\right], \tag{3.18}$$

and we must be able to subtract the noncovariant (and divergent) terms from these expressions. If we had used an asymmetric lattice, with spacing b in σ and a in τ, the coefficients of P^{+2} and P^+ in Eqs. (3.17) and (3.18) would have depended on b/a. The constant term is independent of b/a. Thus, after renormalization, the asymmetric lattice yields the same ground-state masses as the symmetric lattice.

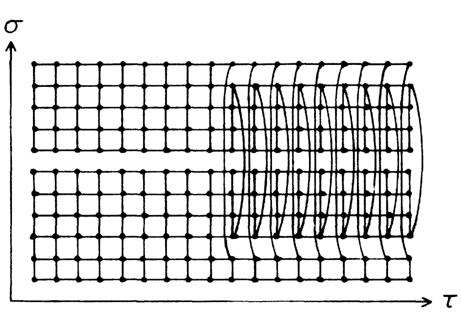


FIG. 4. The interchange interaction on the lattice.

Consider first the most divergent terms, $[(2P^+)^2/\pi a^2T_0]G$. In the functional integral these contributed the expression

$$(e^{-2G/\tau})^{M(N+1)(D-2)}. \tag{3.19}$$

Even for an interacting diagram these contributions to the ground-state energy of each internal line assemble to produce the same overall factor for every diagram. We may therefore subtract these terms without affecting the interactions. Alternatively, we could absorb these factors in the original normalization of the path integral, i.e., replace

$$\left[\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^{M(N+1)(D-2)}$$

by

$$\left[e^{2G/\tau}\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^{M(N+1)(D-2)}$$

in Eq. (2.2). If we do this, these terms will be canceled consistently in every diagram.

There are no further divergent noncovariant factors in $m_G^{2\ \text{closed}}$. The remaining divergent term in $m_G^{2\ \text{open}}$ can only be removed by adding a counterterm to the original action. This term contributes the expression

$$\exp[\tfrac{1}{2}(\sinh^{-1}1)(D-2)(N+1)]$$

to the functional integral. If we interpret an open string as a closed string with a missing link, we see that we shall get a factor

$$\exp[\tfrac{1}{2}(\sinh^{-1}1)(D-2)]$$

for each missing link, and this rule is seen to be consistent for a general multistring configuration. Thus we will remove this term if we add to iW a term $-\frac{1}{2}(\sinh^{-1}1)(D-2)$ for each missing link. Since these counterterms are local (only associated with ends of open strings) we will not destroy the locality of interactions by this procedure; in particular, the argument for crossing symmetry (duality) is not destroyed.

Finally there is no local counterterm which can alter the finite covariant terms, and so these terms have physical significance. After the above subtractions we are left with the results (for noninteracting strings)

$$\frac{m_G^{2\ \mathrm{open}}}{2\pi T_0} = -\frac{D-2}{24},$$

$$\frac{m_G^{2\ \mathrm{closed}}}{2\pi T_0} = -\frac{D-2}{6},$$

so we have recovered the familiar problem with tachyons in dual models.

We could of course continue this analysis for the excited states. But it is evident that the excitation spectrum found in GGRT will emerge.[9] Our aim in this paper is to study the ground state in the hope that interactions will solve the tachyon problem, and thus far we understand how the path-integral formalism leads to tachyons in the absence of interactions. In the following section we shall study interactions in the weak-coupling limit and shall discover how the critical dimension emerges.

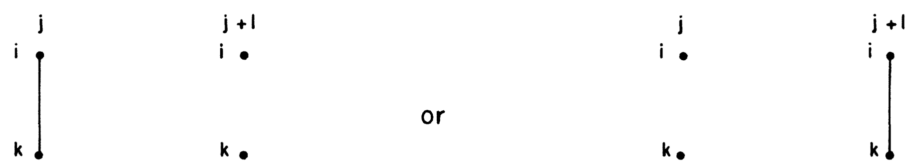
IV. INTERACTIONS AND THE WEAK-COUPLING LIMIT: THE CRITICAL DIMENSION

We first recapitulate the general sum over histories incorporating the zero-point subtractions discussed in the preceding section. We write

$$iW_{\{V_j\}} = -\frac{T_0}{2}\left[\sum_{j=0}^{N}\sum_{i=1}^{M}(\vec{x}_{ij+1}-\vec{x}_{ij})^2+\sum_{j=1}^{N}V_j(\vec{x}_{ij})+\tfrac{1}{2}V_0(\vec{x}_{i0})+\tfrac{1}{2}V_{N+1}(\vec{x}_{i,N+1})\right], \qquad (4.1)$$

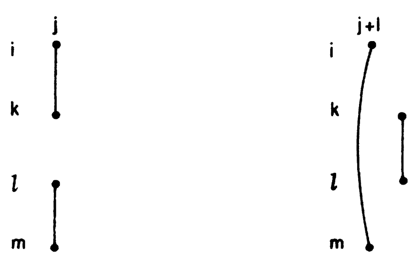
where $V_j(\vec{x}_{ij})$ is a sum of (differences between the $\vec{x}_{ij})^2$, involving up to M links, say $L\leq M$, to which is added the counterterms $(D-2)(M-L)(\sinh^{-1}1)/T_0$. The configuration of links is restricted by the requirement that the configuration in V_{j+1} can be obtained from that in V_j by either

(a) the appearance or disappearance of any number of links,



(4.2)

(b) the interchange of any number of pairs of links,



(4.3)

We remark that this interchange interaction can be simulated by a sequence of link creations and annihilations. However, in the dual-loop expansion such a process is higher order in the coupling. It is an intriguing possibility that for a particular value of the coupling constant the interchange interaction may not be needed, and only interactions of type a may be included. For a particular choice of V_j's the sum over histories is

$$\langle\{\vec{x}_{i,N+1}\},\tau_2|\{\vec{x}_{i0}\},\tau_1\rangle_{\{V_j\}} = \left[e^{2G/\tau}\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^{M(N+1)(D-2)}\prod_{i,j}\int d^{D-2}x_{ij}\,e^{iW\{V_j\}}, \qquad (4.4)$$

and the complete amplitude is the sum over all allowed choices of $\{V_j\}$.

The dual-loop expansion is a power series in a parameter g_0 which is incorporated in the sum over histories in the following fashion. For a given choice of $\{V_j\}$ define the order, $l_{\{V_j\}}$ = the number of link annihilations + the number of link creations + 2 times the number of interchanges. Then multiply this term in the sum over histories by $g_0^{\,l\{V_j\}}$. The dual-loop expansion is obtained by taking the continuum limit order by order in perturbation theory about $g_0=0$. Notice that this expansion is artificial since $g_0=1$ is the natural choice. We believe that the problems of the dual model may be artifacts of this weak-coupling expansion. It is nonetheless of interest to see how the dual-loop expansion emerges in our treatment, and in the remainder of this section we shall discuss the weak-coupling limit. Of particular interest is the way Lorentz covariance emerges in the critical dimension.

We shall treat in detail only the simplest first-order process, namely the transition of the closed-string ground state to the open-string ground state. This will suffice to illustrate the essential features of the weak-coupling expansion. The link structure for this process is illustrated in Fig. 5 (we do not include the factor of g_0 in the definition of the diagram).

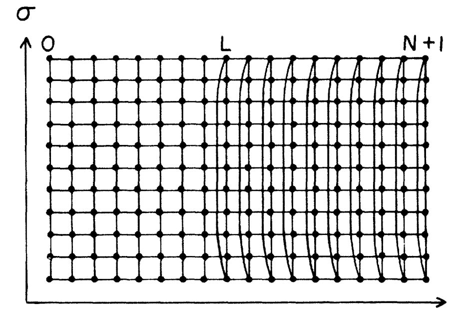


FIG. 5. The lowest-order open to closed two-point function.

The integrations over the variables $j=1,\ldots,L-1$ and $L+1,\ldots,N$ can be immediately performed. The result is just the product of a closed-string propagator and an open-string propagator. Thus calling the vertex function

$$\langle\{x_{i,N+1}\},(N+1-L)a\,|\,V_0\,|\,\{x_{i0}\},\,-La\rangle$$

we have

$$\begin{aligned}\langle\{x_{i,N+1}\},(N+1-L)a\,|\,V_0\,|\,\{x_{i0}\},\,-La\rangle={}&\exp(\tfrac{1}{4}(D-2)\sinh^{-1}1)\\ &\times\int d^{D-2}x_{iL}\langle\{x_{i,N+1}\},(N+1-L)a\,|\,\{x_{iL}\},0\rangle^{\text{closed}}\langle\{x_{iL}\},0\,|\,\{x_{i0}\},\,-La\rangle^{\text{open}}\\ &\times\exp[-\tfrac{1}{4}T_0(\vec{x}_{ML}-\vec{x}_{1L})^2]\,.\end{aligned}\tag{4.5}$$

Since our primary interest is the structure of the ground state, it suffices to work out in detail only the case $\{\vec{x}_{i,N+1}\}=\{\vec{x}_{i0}\}=0$. Also, since each transverse dimension contributes the same factor, in the following we calculate only the contribution of one transverse dimension. With this restriction we have (from Appendix A)

$$\begin{aligned}i(W^{\text{open}}+W^{\text{closed}})=-\frac{T_0}{2}\bigg[&q_0^{\,2}\left(\frac{1}{N+1-L}+\frac{1}{L}\right)\\ &+\sum_{m=1}^{(M-1)/2}(q_m^{c^2}+q_m^{s^2})\sinh\lambda_m^c\coth\lambda_m^c(N+1-L)+\sum_{m=1}^{M-1}q_m^{0^2}\sinh\lambda_m^0\coth\lambda_m^0L\bigg]\end{aligned}\tag{4.6}$$

where

$$\begin{aligned}q_0&=\left(\frac{1}{M}\right)^{1/2}\sum_{i=1}^{M}x_{iL}\,,\\ q_m^0&=\left(\frac{2}{M}\right)^{1/2}\sum_{i=1}^{M}x_{iL}\cos\frac{m\pi}{M}(i-\tfrac{1}{2})\,,\\ q_m^c&=\left(\frac{2}{M}\right)^{1/2}\sum_{i=1}^{M}x_{iL}\cos\frac{2m\pi}{M}(i-\tfrac{1}{2})\,,\\ q_m^s&=\left(\frac{2}{M}\right)^{1/2}\sum_{i=1}^{M}x_{iL}\sin\frac{2m\pi}{M}(i-\tfrac{1}{2})\end{aligned}\tag{4.7}$$

are the normal-mode coordinates for the open and closed strings. Also,

$$\lambda_m^0 = 2\sinh^{-1}\sin\frac{m\pi}{2M},$$
$$\lambda_m^c = 2\sinh^{-1}\sin\frac{m\pi}{M} \tag{4.8}$$

are the discrete time versions of the eigenfrequencies for the open and closed strings, respectively.

We choose the closed-string normal modes q_0, q_m^c, q_m^s as integration variables. We must therefore express the q_m^0 in terms of these. In Appendix B we show that

$$q_m^0 = \begin{cases} q_m^c, & m \text{ even} \\ \frac{2}{M}\sum_{m'=1}^{(M-1)/2} q_{m'}^s U_{mm'}, & m \text{ odd} \end{cases} \tag{4.9}$$

with

$$U_{mm'} = \frac{\sin(m'\pi/M)\cos(m\pi/2M)}{\sin^2(m'\pi/M) - \sin^2(m\pi/2M)}.$$

The Jacobian for the transformation $x_{iL} \to q_m^c, q_m^s, q^0$ is unity so we therefore have

$$\begin{aligned}\mathcal{V}_{2,0}^{0c} &\equiv \langle\{0\}_{\text{closed}}, (N-L+1)a|V_0|\{0\}_{\text{open}}, La\rangle \\ &= \mathfrak{D}^{\text{closed}}((N+1-L)a)\mathfrak{D}^{\text{open}}(La)\exp(\tfrac{1}{4}\sinh^{-1}1) \\ &\times \int dq_0 \int dq_m^c \int dq_m^s \exp\Big(-\frac{T_0}{2}\Big\{q_0^2\Big(\frac{1}{N+1-L}+\frac{1}{L}\Big) + \sum_{m=1}^{(M-1)/2} q_m^{c2}\sinh\lambda_m^0[\coth\lambda_m^c(N+1-L)+\coth\lambda_m^c L] \\ &\qquad + \sum_{m=1}^{(M-1)/2} q_m^{s2}\sinh\lambda_m^c\coth\lambda_m^c(N+1-L) \\ &\qquad + \sum_{m'm''} q_{m'}^s q_{m''}^s\Big[\Big(\frac{2}{M}\Big)^2 \sum_{m=1\text{ odd}}^{M-1} U_{mm'}U_{mm''}\sinh\lambda_m^0\coth\lambda_m^0 L + \frac{4}{M}\sin\frac{m'\pi}{M}\sin\frac{m''\pi}{M}\Big]\Big\}\Big).\end{aligned} \tag{4.10}$$

To extract the ground-state open → ground-state closed matrix element we note that

$$\langle\{\vec{0}\}, T|V_0|\{\vec{0}\}, -T\rangle \underset{T\to\infty}{\simeq} \exp[-T(P_{1G}^- + P_{2G}^-)]\langle G_2|\mathcal{V}_0|G_1\rangle\langle\{\vec{0}\}|G_2\rangle\langle G_1|\{\vec{0}\}\rangle\Big(\frac{2\pi P^+}{2T}\Big)^{1/2}, \tag{4.11}$$

where we have defined

$$\langle G_2, \vec{p}_\perp'|V|G_1, \vec{p}_\perp\rangle \equiv \delta(\vec{p}_\perp - \vec{p}_\perp')\langle G_2, \vec{p}_\perp|\mathcal{V}|G_1, \vec{p}_\perp\rangle$$

and

$$\langle G_2, \vec{0}|\mathcal{V}_0|G_1, \vec{0}\rangle \equiv \langle G_2|\mathcal{V}_0|G_1\rangle. \tag{4.12}$$

Now

$$\mathfrak{D}^{\text{open}}(T)\underset{T\to\infty}{\simeq} |\langle 0|G_1\rangle|^2 e^{-TP_{1G}^-}\Big(\frac{2\pi P^+}{T}\Big)^{1/2},$$
$$\mathfrak{D}^{\text{closed}}(T)\underset{T\to\infty}{\simeq} |\langle 0|G_2\rangle|^2 e^{-TP_{2G}^-}\Big(\frac{2\pi P^+}{T}\Big)^{1/2}. \tag{4.13}$$

So we can set $(N+1)a = 2La \equiv 2T$, and take T large, obtaining

$$\begin{aligned}\mathcal{V}_{2,0}^{0c} \underset{T\to\infty}{\simeq} &\mathfrak{D}^{\text{closed}}(T)\mathfrak{D}^{\text{open}}(T)\int dq_0 \int dq_m^c \int dq_m^s \exp(\tfrac{1}{4}\sinh^{-1}1) \\ &\times \exp\Big\{-\frac{T_0}{2}\Big[q_0^2\Big(\frac{2}{T}\Big) + 2\sum_{m=1}^{(M-1)/2} q_m^{c2}\sinh\lambda_m^c + \sum_{m=1}^{(M-1)/2} q_m^{s2}\sinh\lambda_m^c \\ &\qquad + \sum_{m'm''} q_{m'}^s q_{m''}^s\Big(\frac{2}{M}\Big)^2 \sum_{m=1\text{ odd}}^{M-1} U_{mm'}U_{mm''}\sinh\lambda_m^c \\ &\qquad + \frac{4}{M}\sum_{mm'} q_m^s q_{m'}^s \sin\frac{m\pi}{M}\sin\frac{m'\pi}{M}\Big]\Big\}.\end{aligned} \tag{4.14}$$

The integrals over q^0 and q^c are precisely those one would encounter in the closure relation if the open string were in fact closed. The integrals over q^s_m differ, but if we scale q^s_m by $[(2\sinh\lambda^c_m T_0)/2\pi]\, q^s_m \to \bar{q}_m$, the Jacobian factors together with the results of the q^0 and q^c integrals combine into $\mathfrak{D}^{\text{closed}}(2T)/[\mathfrak{D}^{\text{closed}}(T)]^2$, so

$$\mathcal{V}^{0G}_{2,0} \underset{T\to\infty}{\sim} \frac{\mathfrak{D}^{\text{open}}(T)}{\mathfrak{D}^{\text{closed}}(T)}\, \mathfrak{D}^{\text{closed}}(2T)\exp(\tfrac{1}{4}\sinh^{-1}1)\int\prod_m d\bar{q}_m \exp\Big[-\pi\sum_{m'm''}\bar{q}_{m'}\bar{q}_{m''}(\delta_{m'm''}+\mathcal{A}_{m'm''})\Big] \tag{4.15}$$

with

$$\mathcal{A}_{m'm''} = \tfrac{1}{2}(\mathfrak{M}_{m'm''} - \delta_{m'm''})\,, \tag{4.16}$$

$$\mathfrak{M}_{m'm''} = \frac{1}{(\sinh\lambda^c_{m'}\sinh\lambda^c_{m''})^{1/2}}\left[\sum_{m=1,\,m\ \text{odd}}^{M-1}\sinh\lambda^0_m U_{mm'}U_{mm''}\left(\frac{2}{M}\right)^2 + \frac{4}{M}\sin\frac{m'\pi}{M}\sin\frac{m''\pi}{M}\right]. \tag{4.17}$$

Comparing equations (4.12) and (4.15) we read off

$$\langle G_2, \text{closed}\,|\mathcal{V}_0|\,G_1, \text{open}\rangle = \left|\frac{\langle 0|G_1\rangle}{\langle 0|G_2\rangle}\right|\det{}^{-1/2}(I+\mathcal{A})\exp(\tfrac{1}{4}\sinh^{-1}1)\,. \tag{4.18}$$

Referring to the continuum limit of the open-string and closed-string propagators, Eqs. (3.13) and (3.15), we see that

$$\left|\frac{\langle 0|G_1\rangle}{\langle 0|G_2\rangle}\right|^2 \underset{M\to\infty}{\sim} \frac{1}{(2)^{1/8}M^{1/4}}\,, \tag{4.19}$$

so

$$\langle G_2|\mathcal{V}_0|G_1\rangle \underset{M\to\infty}{\sim} 2^{-1/16}(1+\sqrt{2})^{1/4}\frac{1}{M^{1/8}}\det{}^{-1/2}(I+\mathcal{A})\,. \tag{4.20}$$

In Appendix B we show that

$$\det(I+\mathcal{A}) \underset{M\to\infty}{\sim} \frac{1}{K^2M^{1/8}}\,, \tag{4.21}$$

so that

$$\langle G_2|\mathcal{V}_0|G_1\rangle \sim \left(\frac{1+\sqrt{2}}{(2)^{1/4}}\right)^{1/4}\frac{K}{M^{1/16}}\,, \tag{4.22}$$

where K is a finite numerical constant. The generalization to $D-2$ transverse dimensions is then

$$\langle G_2|\mathcal{V}_0|G_1\rangle \underset{M\to\infty}{\sim} \left(\frac{1+\sqrt{2}}{(2)^{1/4}}K^4\right)^{(D-2)/4}\frac{1}{M^{(D-2)/16}}\,. \tag{4.23}$$

This is the amplitude for a particular breaking time, 0, and we must sum over all such times. The dependence on the breaking time for eigenstates of the Hamiltonian is clearly

$$\exp[-La(P_2^- - P_1^-)]\,.$$

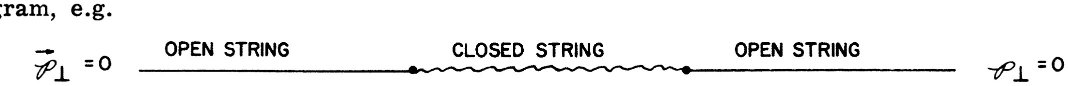
In the continuum limit $\sum_L \to (1/a)\int d\tau$, and when we go back to real times this will yield (as $\tau_2\to\infty$, $\tau_1\to-\infty$)

$$2\pi\delta(P_2^- - P_1^-)\,.$$

Thus in the continuum limit our transition amplitude is (for $\tau_2-\tau_1\to\infty$)

$$\int d\tau\langle G_2|\mathcal{V}_\tau|G_1\rangle = \lim_{M\to\infty}\frac{1}{a}\left(K^4\frac{1+\sqrt{2}}{(2)^{1/4}}\right)^{(D-2)/4}\frac{1}{M^{(D-2)/16}}2\pi\delta(P_2^- - P_1^-)\,. \tag{4.24}$$

To understand the critical dimension we must consider the effect of putting this vertex in a larger diagram, e.g.



Near the ground-state poles this amplitude is

$$g_0^2\sum\frac{1}{P^- - m_G^{2\,\text{open}}/2P^+}\langle G, \text{open}|\mathcal{V}_0|G, \text{closed}\rangle\frac{1}{P^- - M_G^{2\,\text{closed}}/2P^+}\langle G\ \text{closed}|\mathcal{V}_0|G\ \text{open}\rangle\frac{1}{P^- - M_G^{2\,\text{open}}/2P^+}\,,$$

where the sum is over all alternatives. It is clear that the closed string can in fact break at any point relative to the point at which the open string joined, and these relative positions must be summed over. Near the pole (corresponding to large times) this sum just gives a factor of M. If we convert to covariant propagators there will be a further factor of $2P^+$ for each propagator. To get a covariant result, these factors must be canceled by the vertex factors requiring

$$\langle G \text{ closed} | \mathcal{V}_0 | G \text{ open}\rangle \propto \frac{1}{2P^+}\frac{1}{\sqrt{M}} = \frac{1}{2aM^{3/2}} . \tag{4.25}$$

Comparing with Eq. (4.24) we must have $(D-2)/16 = \frac{3}{2}$ or $D-2=24$. Notice also that for general $D-2$ the above amplitude has the structure

$$(\text{covariant})\,\frac{1}{a^2}\left(\frac{T_0 a}{P^+}\right)^{(D-2)/8} MP^{+2}$$
$$= (\text{covariant})\, a^{-3+(D-2)/8}\, P^{+3-(D-2)/8} ,$$

so as $a \to 0$ it is only for the critical dimension that the continuum limit exists.

For the critical dimension

$$\langle G_2 | \mathcal{V}_0 | G_1\rangle = \frac{1}{\sqrt{M}}\,\frac{T_0}{P^+}\left(K^4\,\frac{1+\sqrt{2}}{(2)^{1/4}}\right)^6 ,$$

and the invariant coupling (in field theory language) is

$$g = 2T_0\left(K^4\,\frac{1+\sqrt{2}}{(2)^{1/4}}\right)^6 g_0$$
$$\cong 24\,T_0 g_0 .$$

It is to be noted that the coefficient of g_0 is in principle computable. If we make the natural choice $g_0 = 1$, the dual coupling is fixed in terms of T_0.[10]

We have at this point recovered the familiar results of the dual string model. We have set up a general lattice formalism for the interacting string model and have studied the continuum limit in the case of weak breaking interactions. The results we obtained are not new. Mandelstam has in fact demonstrated that the interacting string formalism yields a covariant S matrix in 26 space-time dimensions. Our treatment has the virtue that ultraviolet singularities are systematically regularized symmetrically in τ and σ. Our bare coupling constant is finite and calculable in 26 space-time dimensions. In the next section we shall consider the possibility of going beyond perturbation theory using our lattice formalism.

V. INSTABILITIES IN THE STRING MODEL AND A POSSIBLE SOLUTION

The well-known tachyon problem in the string model can be understood very simply using our lattice dynamics. In the absence of interactions, the possible states of an M site "string" include the states of a closed string with M sites, an open string with M sites, and multistring states (either open or closed) with the total number of sites equal to M. To understand the energetics of the instabilities, it is convenient to define a ground-state "energy" (really P^-) per site density $\mathcal{E}^{\text{open (closed)}}(K)$ for a single string with K sites. Thus

$$\mathcal{E}^{\text{open}}(K) = \frac{1}{K}\left(\sum_{k=1}^{K-1} \sinh^{-1}\sin\frac{k\pi}{2K} + \tfrac{1}{2}\sinh^{-1}1\right) , \tag{5.1}$$

$$\mathcal{E}^{\text{closed}}(K) = \frac{1}{K}\sum_{k=1}^{K-1} \sinh^{-1}\sin\frac{k\pi}{K} . \tag{5.2}$$

Thus, for example

$$\mathcal{E}^{\text{open}}(\infty) = \mathcal{E}^{\text{closed}}(\infty)$$
$$= \frac{2G}{\pi} \cong 0.583\,121\,807\,9 . \tag{5.3}$$

We plot $\mathcal{E}^{\text{closed}}$ and $\mathcal{E}^{\text{open}}$ as a function if K in Fig. 6.

$\mathcal{E}(K)$ is in both cases a monotonically increasing function of K, and it is also true that $\mathcal{E}^{\text{closed}}(K) < \mathcal{E}^{\text{open}}(K)$ for all K. Thus if a weak breaking interaction is turned "on," it will be energetically favorable for a single open or closed string to evaporate into many little "stringlets." In the continuum limit this instability manifests itself in the appearance of tachyons in the spectrum. The similarity of this situation to that arising in a field theory in which a symmetry is spontaneously

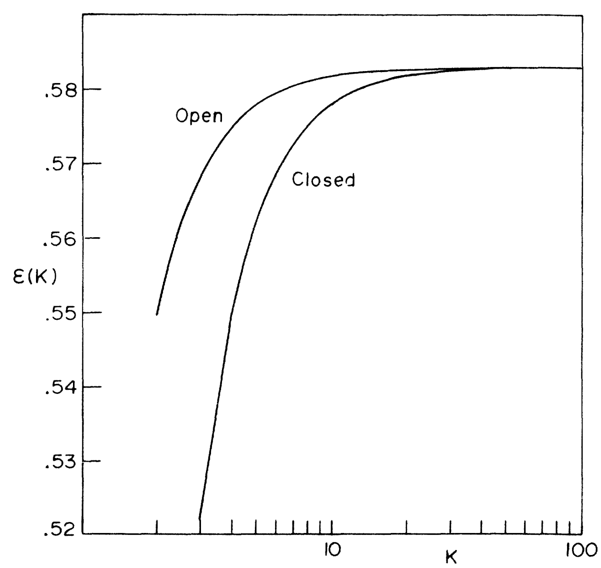


FIG. 6. The energy per site for closed and open strings.

broken has been emphasized by Bardakci and Halpern.[2] The importance of our result lies mainly in the recognition that the instabilities involved are just as evident on a finite lattice as they are in the continuum limit. Thus the question of stabilization may be confronted in our lattice formalism, where all calculations are well defined.

Just as in the field theory analogy, stabilization will depend critically on the nature of the interactions. For example, there is no stabilization in $g\phi^3$ theory, but there is in $\lambda\phi^4$ theory. In field theory the signal for stabilization is sometimes evident at the classical level from examination of the classical potential $V(\phi)$; however, quantum corrections can be important and one must, in general, use the effective potential.[10]

Before we can address the question of stabilization we need to set up a Lorentz-invariant theory of the mass spectrum in the presence of interactions. As a first step we confine our attention to those states which couple to the bare closed string. This is the sector with the lightest tachyon, namely the ground state of the bare closed string. To develop a Lorentz-invariant theory of the mass spectrum of this sector, we use the results of Green and Shapiro.[3] They were able to construct Lorentz-invariant off-shell Green's functions in terms of finite time transition amplitudes between closed string states for which $\vec{x}(\sigma)=\text{constant}$, i.e., states for which the whole string is concentrated at a single space-time point. For our purposes we only need the two-point function. Call the amplitude for a closed string in configuration $\vec{x}(\sigma)=0$ at $\tau=0$ to be observed in the configuration $\vec{x}(\sigma)=\vec{x}_0$ at $\tau=T$,

$$\langle\vec{x}_0, T\,|\,\vec{0}, 0\rangle .$$

(Note in our lattice formalism $\vec{x}_0=\vec{q}_0/\sqrt{M}$.) It is convenient to work in momentum space so we define

$$D(\vec{k}, T)=\int d^{D-2}\vec{x}_0 e^{i\vec{k}\cdot\vec{x}_0}\langle\vec{x}_0, T\,|\,\vec{0}, 0\rangle . \qquad (5.4)$$

Now since

$$\langle\vec{x}_0, T\,|\,\vec{0}, 0\rangle=\exp\left(-\frac{P^+}{2T}\vec{x}_0^{\,2}\right)\langle\vec{0}, T\,|\,\vec{0}, 0\rangle , \qquad (5.5)$$

we have

$$D(\vec{k}, T)=\left[\left(\frac{2\pi T}{P^+}\right)^{1/2}\right]^{D-2}\exp\left(-\frac{\vec{k}^2}{2P^+}T\right)\langle\vec{0}, T\,|\,\vec{0}, 0\rangle . \qquad (5.6)$$

It is presumably true, in the light of the work of Green and Shapiro, that the Fourier transform of D in T, i.e. $(k^+\equiv P^+)$

$$\tilde{D}(k)=N\int_0^\infty \frac{dT}{2P^+}e^{k^-T}D(\vec{k}, T) , \qquad (5.7)$$

is a *Lorentz scalar to all orders in perturbation theory*, with a suitable choice for N. N depends only on the normalization of the bare string states, and an examination of the closed-string propagator Eq. (3.15) reveals that the choice

$$1/N=\left(\frac{1+\sqrt{2}}{2\pi}\right)^{(M-1)(D-2)/2} \qquad (5.8)$$

is necessary. It should also be evident to the reader that (5.7) is a Lorentz scalar only if the renormalizations discussed in Sec. III are performed and space-time has the critical dimension. The spectrum of this sector of the theory is determined by the singularities in $D(k)=D(k^2)$ as a function of k^2. Lorentz covariance is obvious.

If there were no tachyons in the theory, (5.7) would converge for k^2 spacelike, i.e., $k^2>0$. The tachyons which appear in the weak-coupling limit make (5.7) ill defined because the thresholds associated with decay into multitachyon states occur at arbitrarily large positive k^2. In any case, $D(\vec{k}, \tau)$ is well defined and we may use our lattice techniques to attempt to discover its structure.

The singularities in (5.7) are determined by exponential time dependence of $D(\vec{k}, \tau)$,

$$D(\vec{k}, \tau)=\sum_n \exp[-P_n^-(\vec{k}^2)T]D_n(\vec{k}) , \qquad (5.9)$$

where n is also allowed to be a continuous parameter, in which case $\sum_n\to\int dn$, and we can obtain this dependence by diagonalizing the time transfer operator

$$\mathcal{T}(\{\vec{x}_i\}, L; \{\vec{x}_i'\}, L')\equiv\langle\{\vec{x}_i\}, L, a\,|\,\{\vec{x}_i'\}, L', 0\rangle$$
$$=\tau_{L,L'}g_0^{\,I_{L,L'}}\left[e^{2G/\pi}\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^{M(D-2)}\exp\left[-\tfrac{1}{2}T_0\sum_i(\vec{x}_i-\vec{x}_i')^2-\tfrac{1}{4}T_0(V_L(\vec{x})+V_{L'}(\vec{x}'))\right] , \qquad (5.10)$$

where L, L' label the link structure of the final and initial states, and $\tau_{L,L'}=1$ if the transition is allowed and zero if it is forbidden. Diagonalization corresponds to solving the integral-matrix equations:

$$\int\prod_i d^{D-2}x_i'\sum_{L'}\mathcal{T}(\{\vec{x}_i\}, L; \{x_i'\}, L')\Psi(\{x_i'\}, L')=t\Psi(\{x_i\}, L) . \qquad (5.11)$$

The P^- of a state is $-(1/a)\ln t$ so that the lowest-energy state corresponds to the maximum value of t. The condition for stability is that the lowest P^- state lie below the continuum. This criterion is obviously violated in the perturbative weak-coupling treatment.

Our analysis of Eq. (5.11) has only just begun, and we have little to say here except to indicate possible approaches. The two most obvious approaches are to study small numbers of sites or to study restricted link topologies. The kernel couples initial and final link configurations directly through $\tau_{L,L'}$ and $g_0{}^{I_{L,L'}}$ and indirectly through the x dependence of $V_{L'}(x')$ and $V_L(x)$.

If we restrict ourselves to planar topologies we have M links, each of which may be on or off independently, so that $\tau_{L,L'}=1$ in this sector. Further, if $g_0=1$ there remains no direct coupling between L and L'; it is in this sense that $g_0=1$ is particularly attractive. Thus, all eigenfunctions of $\mathcal{T}$ with $t\neq 0$ (finite energy) must have the form

$$\Psi(\{\vec{x}_i\},L)=\exp[-\tfrac{1}{4}T_0V_L(\vec{x})]\phi(\vec{x}_i)\,, \tag{5.12}$$

where $\phi(\vec{x}_i)$ obeys the equation

$$\int dx'\,T(x,x')\phi(x')=t\phi(x) \tag{5.13}$$

with

$$T(\vec{x},\vec{x}')=\left[e^{2G/\tau}\left(\frac{T_0}{2\pi}\right)^{1/2}\right]^{M(D-2)}\exp[-\tfrac{1}{2}T_0\sum_i(\vec{x}_i-\vec{x}_i')^2]\times\prod_j\left\{\exp[-\tfrac{1}{2}(D-2)\sinh^{-1}1]+\exp[-\tfrac{1}{2}T_0(\vec{x}_{j+1}'-\vec{x}_j')^2]\right\}. \tag{5.14}$$

This corresponds to the motion of M particles interacting through the nearest-neighbor potential in $x=|\vec{x}_{j+1}-\vec{x}_j|$,

$$V(x)=-\ln\{\exp[-\tfrac{1}{2}(D-2)\sinh^{-1}1]+\exp(-\tfrac{1}{2}T_0x^2)\}. \tag{5.15}$$

This potential is plotted in Fig. 7. The question of stability in this case is simply whether M sites in this potential form a bound state. This problem is not yet solved for general M.

In the special case, $M=2$, bound states do form. $M=2$ corresponds simply to two sites which may move either as free particles or form one link to move as a two-site open string.

VI. PUTTING BARYONS ON A LATTICE: HOPES FOR THE FUTURE

The problem of quantizing the baryon "three-string" has heretofore evaded an adequate solution.[12] We believe that our lattice dynamics can shed light on this problem. The classical equations of motion are consequences of minimizing the action

$$W=\sum_{i=1}^{3}\int_0^T d\tau\int_0^{P_i^+(\tau)/T_0}d\sigma\,\tfrac{1}{2}T_0(\dot{\vec{x}}_i^{\,2}-\vec{x}_i'^{\,2})$$

subject to the constraints that

$$\sum_{i=1}^{3}P_i^+(\tau)=P^+,\ \text{a constant}$$

and

$$\vec{x}_1(P_1^+(\tau)/T_0,\tau)=\vec{x}_2(P_2^+(\tau)/T_0,\tau)=\vec{x}_3(P_3^+(\tau)/T_0,\tau)\,.$$

The result is

$$\ddot{\vec{x}}_i-\vec{x}''=0\,,$$

with boundary conditions

$$\vec{x}_i'=0\ \text{at}\ \sigma_i=0\,,$$

$$\sum_i\left(\vec{x}_i'+\frac{\dot{P}_i^+\dot{\vec{x}}_i}{T_0}\right)=0\ \ \text{at}\ \sigma_i=P_i^+(\tau)\,,$$

and

$$\tfrac{1}{2}(\dot{\vec{x}}_i^{\,2}+\vec{x}_i'^{\,2})+\dot{P}_i^+\dot{\vec{x}}_i\cdot\vec{x}_i'=f(\tau)\ \ \text{at}\ \sigma_i=P_i^+(\tau)\,.$$

The obstacle to quantization is the adequate treatment of these difficult nonlinear boundary conditions.

However, if we quantize with path integrals we have the following very definite prescription. Set up a lattice in σ and τ. All sites except the junction site and the ends will have two links connected to them; the ends will have one; and the junction will have three. We neglect breaking interactions. For a particular choice of junction site the lattice

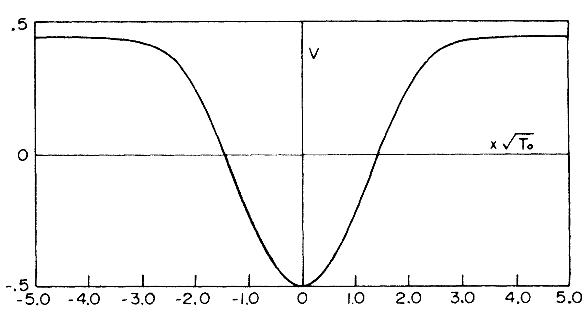


FIG. 7. The intersite potential $V(x)$.

version of the action is the obvious one. Now quantize by integrating e^{-W} over all x_i and summing over all possible paths the junction point can take as it hops from site to site on the lattice. To preserve causality we constrain the junction point to be able to hop only to a neighbor in a unit of time. It would also be consistent with classical mechanics to insist that there be a hop at each unit of time. A classical motion in which the junction point does not move would then correspond to a *Zitterbewegung* in which the junction point hops back and forth between a small number of sites. We prefer this last prescription as each step involves the same type of transition.

We only mention the three-string as a nice application of lattice quantization. The detailed analysis is a problem for the future. Another project would be to repeat our analysis for the Neveu-Schwarz-Ramond model, which is more realistic than the GVM. And of course the ultimate goal is to understand the origin of quarklike degrees of freedom in terms of a simple geometrical picture.

ACKNOWLEDGMENTS

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APPENDIX A: NORMAL MODES OF FREE STRINGS

The open-string potential

$$V^{\text{open}}=\sum_{j=1}^{M-1}(x_{j+1}-x_j)^2 \tag{A1}$$

can be diagonalized by a transformation to normal-mode coordinates $q_0^{(0)},\ldots,q_{M-1}^{(0)}$:

$$x_j=\frac{1}{\sqrt{M}}q_0^{(0)}+\sum_{m=1}^{M-1}\left(\frac{2}{M}\right)^{1/2}q_m^{(0)}\cos\frac{m\pi}{M}\left(j-\tfrac{1}{2}\right). \tag{A2}$$

The transformation to q's is orthogonal, and V^{open} may be reexpressed as

$$V^{\text{open}}(x_1,\ldots,x_M)=\sum_{m=1}^{M-1}\omega_m^{(0)2}q_m^{(0)2}, \tag{A3}$$

where

$$\omega_m^{(0)}=2\sin\frac{m\pi}{2M}. \tag{A4}$$

Similarly, the closed-string potential

$$V^{\text{closed}}(x_1,\ldots,x_M)=V^{\text{open}}(x_1,\ldots,x_M)+(x_1-x_M)^2 \tag{A5}$$

may be diagonalized by an orthogonal transformation. For M even

$$x_j=\frac{1}{\sqrt{M}}q_0+\sum_{m=1}^{M/2-1}\left(\frac{2}{M}\right)^{1/2}\left[q_m^c\cos\frac{2m\pi}{M}\left(j-\tfrac{1}{2}\right)+q_m^s\sin\frac{2m\pi}{M}\left(j-\tfrac{1}{2}\right)\right]+\frac{1}{\sqrt{M}}q_{M/2}(-1)^j. \tag{A6}$$

For M odd

$$x_j=\frac{1}{\sqrt{M}}q_0+\sum_{m=1}^{(M-1)/2}\left(\frac{2}{M}\right)^{1/2}\left[q_m^c\cos\frac{2m\pi}{M}\left(j-\tfrac{1}{2}\right)+q_m^s\sin\frac{2m\pi}{M}\left(j-\tfrac{1}{2}\right)\right]. \tag{A7}$$

Then

$$V^{\text{closed}}(x_1,\ldots,x_M)=\sum_m\omega_m^{(c)2}q_m^{\;2}, \tag{A8}$$

where the sum extends over all normal modes and

$$\omega_m^{(c)}=2\sin\frac{m\pi}{M}. \tag{A9}$$

For either the open or the closed string, the state dependence of the propagator may be expressed simply in terms of the corresponding normal modes, q_n. Indeed, the functional integral factorizes into the product of functional integrals for one free particle q_0 and $M-1$ harmonic oscillators q_n of frequency ω_n. For $\langle\{q_{m,N+1}\}|\{q_{m,0}\}\rangle$ the classical action is

$$iW_{\text{cl}}[\{x_{j,N+1}\},\{x_{j,0}\}]=\sum_{m=0}^{M-1}iW_{\text{cl},m}(q_{m,N+1};q_{m,0}) \tag{A10}$$

with

$$iW_{\text{cl},0}(q_{0,N+1};q_{0,0})=-\frac{T_0}{2}\,\frac{(q_{0,N+1}-q_{0,0})^2}{N+1}, \tag{A11}$$

$$iW_{\text{cl},m}(q_{m,N+1};q_{m,0})=-\tfrac{1}{2}T_0\sinh\lambda_m\left[(q_{m,0}^{\;2}+q_{m,N+1}^{\;2})\coth(N+1)\lambda_m-\frac{2}{\sinh(N+1)\lambda_m}q_{m,0}q_{m,N+1}\right], \tag{A12}$$

where $\lambda_m=2\sinh^{-1}(\tfrac{1}{2}\omega_m)$ is the lattice analog of the mth harmonic-oscillator level spacing.

In the normal-mode basis, the eigenfunctions of the transition amplitude are products:

$$\phi(x_1,\ldots,x_M)=\prod_m\phi(q_m), \tag{A13a}$$

$$\int dx_{1,0}\cdots dx_{M,0}\langle x_{i,N+1}|x_{i,0}\rangle\phi(x_{i,0};\ldots;x_{M,0})=e^{-TP^-}\phi(x_{i,N+1}) \tag{A13b}$$

with

$$P^-=\sum_m (P^-)_m\,. \tag{A13c}$$

The eigenfunctions and eigenvalues are as follows: For the translation mode $m=0$ we have

$$\phi_P(q_0)=e^{ipq_0}\,,\quad P_0^-(p)=\frac{P^2}{2aT_0}\,. \tag{A14}$$

For each harmonic-oscillator mode (q,ω)

$$\phi_L(q)=\left(\frac{\pi}{\sinh\lambda}\right)^{1/4}\left[\left(\frac{T_0\sinh\lambda}{2}\right)^{1/2}q-\frac{1}{(2T_0\sinh\lambda)^{1/2}}\frac{\partial}{\partial q}\right]^L e^{-(T_0\sinh\lambda/2)q^2} \tag{A15}$$

and

$$P^-(L)=\frac{1}{a}(L+\tfrac{1}{2})\lambda\,,$$

where $\lambda=2\sinh^{-1}(\frac{1}{2}\omega)$ as before, and $L=0,1,2,\ldots$ is the occupation number of the mode.

We note that these lattice results go over into the usual continuum theory. In the case of the open string, for fixed m as $M\to\infty$,

$$\lambda_m/a=(2/a)\sinh^{-1}\sin(m\pi/2M)\to m\pi/P^+\,, \tag{A16}$$

$$(1/\sqrt{a}\,)(2/M)^{1/2}\cos(m\pi/M)(j-\tfrac{1}{2})\to(2/P^+)^{1/2}\cos(m\pi/P^+)\sigma\,. \tag{A17}$$

These are the usual wave functions and excitation frequencies. It is important to note that the continuum theory arises from the low-frequency ($m\ll M$) part of the lattice theory. We expect that low-frequency effects will be cutoff-independent while high-frequency effects need not be.

The ground-state energies of the open and closed string are

$$(P^-)_{\rm open}=\sum_{m=1}^{M-1}(1/a)\sinh^{-1}\sin(m\pi/2M)\,, \tag{A18a}$$

$$(P^-)_{\rm closed}=\sum_{m=1}^{M-1}(1/a)\sinh^{-1}\sin(m\pi/M)\,. \tag{A18b}$$

Both are divergent in the continuum limit. Each has the form, as $M\to\infty$,

$$\frac{1}{a}\left[AM+B+\frac{C}{M}+O\left(\frac{1}{M^2}\right)\right], \tag{A19}$$

where A,B,C are finite constants which may be found using the Euler-Maclaurin summation formula.

If $F(x)$ is a bounded function for $0\leq x\leq 1$, we have

$$\sum_{m=0}^{M-1}F\left(\frac{m}{M}\right)=\int_0^M dm\,F\left(\frac{m}{M}\right)+\sum_{m=0}^{M-1}\left[F\left(\frac{m}{M}\right)-\int_0^1 dl\,F\left(\frac{m}{M}+\frac{l}{M}\right)\right].$$

Now

$$\int_0^1 dl\,F\left(\frac{m}{M}+\frac{l}{M}\right)=\int_0^1 dl\left[F\left(\frac{m}{M}\right)+\frac{l}{M}F'\left(\frac{m}{M}\right)+\frac{l^2}{2M^2}F''\left(\frac{m}{M}\right)+\cdots\right].$$

So

$$\sum_{m=0}^{M-1}F=M\int_0^1 dx\,F(x)+\sum_{m=0}^{M-1}\left[-\frac{1}{2M}F'\left(\frac{m}{M}\right)-\frac{1}{6M^2}F''\left(\frac{m}{M}\right)+\cdots\right].$$

Apply the procedure to the sums of F' and F''; we have

$$\sum_{m=0}^{M-1}F\left(\frac{m}{M}\right)=M\int_0^1 dx\,F(x)-\tfrac{1}{2}[F(1)-F(0)]+\frac{1}{12M}[F'(1)-F'(0)]\,. \tag{A20}$$

Applying this to the sums for P^- we have

$$(P^-)_{\text{open}} = \frac{2G}{\pi}\,\frac{M}{a} - \frac{1}{2a}\sinh^{-1}1 - \frac{\pi T_0}{24P^+}\,, \tag{A21}$$

$$(P^-)_{\text{closed}} = \frac{2G}{\pi}\,\frac{M}{a} - \frac{\pi T_0}{6P^+}\,, \tag{A22}$$

where

$$G = \int_0^{\pi/2} dx \sinh^{-1}\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \text{Catalan's constant}. \tag{A23}$$

APPENDIX B

We present some details of the calculation of the open-closed vertex function. For convenience, we consider the case in which M is odd.

The change of basis from open modes q_m to the closed modes q_m^s, q_m^c follows immediately from

$$q_m = \left(\frac{2}{M}\right)^{1/2} \sum_{j=1}^{M} x_j \cos\frac{m\pi}{M}(j-\tfrac{1}{2}), \tag{B1a}$$

$$x_j = \frac{1}{\sqrt{M}} q_0 + \sum_{m=1}^{(M-1)/2} \left(\frac{2}{M}\right)^{1/2} \left[q_m^c \cos\frac{2m\pi}{M}(j-\tfrac{1}{2}) + q_m^s \sin\frac{2m\pi}{M}(j-\tfrac{1}{2})\right], \tag{B1b}$$

whence

$$q_m = q_{m/2}^c, \text{ for } m \text{ even}$$

$$q_m = \frac{2}{M}\sum_{m'} U_{mm'} q_{m'}^s, \text{ for } m \text{ odd}$$

where

$$U_{mm'} = \sum_{j=1}^{M} \cos\frac{m\pi}{M}(j-\tfrac{1}{2}) \sin\frac{2m'\pi}{M}(j-\tfrac{1}{2}). \tag{B2}$$

This is a simple geometric series and gives the result in the text:

$$U_{mm'} = \frac{\sin(m'\pi/M)\cos(m\pi/2M)}{\sin^2(m'\pi/M) - \sin^2(m\pi/2M)}. \tag{B3}$$

Since m is odd, the denominator never vanishes.

We next consider the evaluation of the determinant, $\det(1+\mathcal{a})$, in the limit of large M. Recall that

$$\mathcal{a}_{m'm''} = \tfrac{1}{2}(\mathfrak{M}_{m'm''} - \delta_{m'm''})$$

with

$$\mathfrak{M}_{m'm''} = \frac{1}{(\sinh\lambda_{m'}^c \sinh\lambda_{m''}^c)^{1/2}} \left(\frac{4}{M^2}\sum_{m\ \text{odd}} \sinh\lambda_m^0 U_{mm'} U_{mm''} + \frac{4}{M}\sin\frac{m'\pi}{M}\sin\frac{m''\pi}{M}\right).$$

We shall compute the logarithm of the determinant using

$$\ln\det(1+\mathcal{a}) = \operatorname{tr}\ln(1+\mathcal{a}) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\operatorname{tr}\mathcal{a}^n.$$

We shall evaluate the sums in the trace and in the definition of $\mathcal{a}$ by converting them to contour integrals.

First consider the sum

$$S_{m'm''} = \frac{4}{M^2}\sum_{m\ \text{odd}} \sinh\lambda_m^0 U_{mm'} U_{mm''}$$

$$= \frac{4}{M^2}\sum_{P=0}^{(M-1)/2}\left\{2\sin(P+\tfrac{1}{2})\frac{\pi}{M}[1+\sin^2(P+\tfrac{1}{2})\pi/M]^{1/2}\right.$$

$$\left.\times\frac{\sin(m'\pi/M)\sin(m''\pi/M)\cos^2(P+\tfrac{1}{2})\pi/M}{[\sin^2(P+\tfrac{1}{2})\pi/M - \sin^2(m'\pi/M)][\sin^2(P+\tfrac{1}{2})\pi/M - \sin^2(m''\pi/M)]}\right\}. \tag{B4}$$

Define $Z=\sin\theta$ and $R_1(Z)=-(M/\cos\theta)\tan m\theta$. Then $R_1(Z)$ is meromorphic with poles of unit residue at $Z=\pm(p+\frac{1}{2})\pi/M$, $p=0,\ldots,(M-1)/2$. Denoting $y'=\sin(m'\pi/M)$, $y''=\sin(m''\pi/M)$ we have

$$S_{m'm''}=\frac{8}{M^2}y'y''\oint_{C_1}\frac{dZ}{2\pi i}\frac{Z(1+Z^2)^{1/2}(1-Z^2)}{(Z^2-y'^2)(Z^2-y''^2)}R_1(Z), \tag{B5}$$

where C_1 is a contour which encloses each of the points (see Fig. 8)

$$Z_P=\sin(P+\tfrac{1}{2})\frac{\pi}{M},\quad P=0,\ldots,(M-1)/2-1.$$

We choose the cuts from $(1+Z^2)^{1/2}$ to lie on the imaginary segments

$$(-i\infty,-i] \text{ and } [i,+i\infty).$$

We may extend the integration to a single contour C_2 enclosing the real line between 0 and 1 if we subtract off the extra terms coming from poles at $Z=y', Z=y''$. Since $R_1(y')=R_1(y'')=0$, these poles contribute only if $y'=y''$ (i.e., $m'=m''$). We find

$$S_{m'm''}=\frac{8}{M^2}y'y''\oint_{C_2}\frac{dZ}{2\pi i}\frac{Z(1+Z^2)^{1/2}(1-Z^2)}{(Z^2-y'^2)(Z^2-y''^2)}R_1(Z)+2y'(1+y'^2)^{1/2}\delta_{m',m''}. \tag{B6}$$

This gives

$$\mathfrak{M}_{m'm''}=\frac{1}{2(y'y'')^{1/2}(1+y'^2)^{1/4}(1+y''^2)^{1/4}}\left[\frac{8}{M^2}y'y''\oint_{C_2}\frac{dZ}{2\pi i}\frac{Z(1+Z^2)^{1/2}(1-Z^2)}{(Z^2-y'^2)(Z^2-y''^2)}R_1(Z)\right.$$
$$\left.+2y'(1+y'^2)^{1/2}\delta_{m'm''}+\frac{4}{M}y'y''\right] \tag{B7}$$

and

$$\mathfrak{a}_{m'm''}=\frac{(y'y'')^{1/2}}{M(1+y'^2)^{1/4}(1+y''^2)^{1/4}}I(y',y''), \tag{B8}$$

where we have defined

$$I(y',y'')=1+\frac{2}{M}\oint_{C_2}\frac{dZ}{2\pi i}\frac{Z(1+Z^2)^{1/2}(1-Z^2)}{(Z^2-y'^2)(Z^2-y''^2)}R_1(Z). \tag{B9}$$

We must compute

$$D_n=(-1)^{n+1}\operatorname{tr}a^n=\sum_{m_1\cdots m_n}\frac{y_{m_1}}{M(1+y_{m_1}^2)^{1/2}}I(y_{m_1},y_{m_2})\cdots\frac{y_{m_n}}{M(1+y_{m_n}^2)^{1/2}}I(y_{m_n},y_{m_1}), \tag{B10}$$

where $y_m=\sin(m\pi/M)$. $I(y,y')$ is analytic for y,y' inside the contour C_2. If $y\equiv\sin\theta$ the function $R_2(y)=(M/\cos\theta)\cos M\theta$ is meromorphic with poles of unit residue at $y=\sin(m\pi/M)$: $m=0,\pm1,\ldots,\pm(M-1)/2$. We may therefore replace the sums in (B10) by contour integrals,

$$\sum_{m=1}^{(m-1)/2}\frac{y_m}{M(1+y_m^2)^{1/2}}\to\oint_{C_3}\frac{dy}{2\pi i}\frac{y}{M(1+y^2)^{1/2}}R_2(y), \tag{B11}$$

where C_3 is a contour enclosing the poles $y_n=\sin(m\pi/M)$: $m=1,\ldots,(M-1)/2$ which lies wholly within C_2, (Fig. 8). Then

$$D_n=(-1)^{n+1}\oint_{C_3}\frac{dy_1}{2\pi i}\cdots\frac{dy_n}{2\pi i}\frac{y_1R_2(y_1)}{M(1+y_1^2)^{1/2}}I(y_1,y_2)\cdots\frac{y_nR(y_n)}{m(1+y_n^2)^{1/2}}I(y_n,y_1). \tag{B12}$$

We will now extract the leading ($\ln M$) behavior of D_n as $M\to\infty$. We shall see that the only parts of the contour integrals which contribute to this leading term come from the region where Z's and y's are small.

First consider $I(y,y')$. Explicitly,

$$I(y,y')=1-2\oint_{C_2}\frac{dZ}{2\pi i}\frac{Z(1+Z^2)^{1/2}(1-Z^2)^{1/2}}{(Z^2-y^2)(Z^2-y'^2)}\frac{[(1-Z^2)^{1/2}+iZ]^M-[(1-Z^2)^{1/2}-iZ]^M}{i\{[(1-Z^2)^{1/2}+iZ]^M+[(1-Z^2)^{1/2}-iZ]^M\}}. \tag{B13}$$

We note that there are no branch points arising from $(1-Z^2)^{1/2}$ in (B13).

We draw the contour C_2 as shown in Fig. 9. Because of the branch cuts (i,∞) and $(-i,-\infty)$, the contributions to (B13) from segments 1 and 2 cancel. The semicircle R, for $R\to\infty$, gives a constant term -1. So writing $Z=ir$ on the segment $[-i,i]$ we have

$$I(y,y')=-\frac{1}{\pi}\int_{-1}^{1}dr\,\frac{r(1-r^2)^{1/2}(1+r^2)^{1/2}}{(r^2+y^2)(r^2+y'^2)}\tanh[M\sinh^{-1}(r)]. \tag{B14}$$

To compute D_n, we must convolute n I's with the functions $h(y)$,

$$h(y)=\frac{yR_2(y)}{m(1+y^2)^{1/2}}=\frac{iy}{(1+y^2)^{1/2}(1-y^2)^{1/2}}\frac{[(1-y^2)^{1/2}+iy]^M+[(1-y^2)^{1/2}-iy]^M}{[(1-y^2)^{1/2}+iy]^M-[(1-y^2)^{1/2}-iy]^M}. \tag{B15}$$

We choose the contour C_3 as shown in Fig. 10. We have broken the contour into three regions. In region 1 $1/M\lesssim |y|\lesssim K/M$, where K is a large number fixed as $M\to\infty$. Region 2 consists of straight segments from $K/M\leq |y|\leq \delta$, where δ is fixed and $\delta\ll 1$. Region 3 comprises the remainder of the contour. We will show that the leading-log dependence of each D_n arises from integrals in region 2.

First, we estimate the order of magnitude (up to $\ln M$) of $h(y)$ and $I(y,y')$ in each region. From (B14), we have

$$\begin{aligned} &y,y'\in 1,\quad I(y,y')\sim O(M^2),\\ &y\in 1,\quad y'\in 2\text{ or }3,\quad I(y,y')\sim O(1),\\ &y,y'\in 2\text{ or }3,\quad I(y,y')\sim O(1), \end{aligned} \tag{B16}$$

while

$$\begin{aligned} &y\in 1,\quad h(y)\sim O\left(\frac{1}{M}\right),\\ &y\in 2\text{ or }3,\quad h(y)\sim O(1). \end{aligned} \tag{B17}$$

It follows immediately from (B16) and (B17) that the only contributions to D_n which are not down by one or more powers of $1/M$ are those which involve multiple integrations where either all y's are in region 1 or all y's are in regions 2 and 3.

In region 1, as $M\to\infty$, $I(y,y')$ is dominated by the integration over $r\sim O(1/M)$ in (B14). Defining $y=\eta/M$, $r=\rho/M$,

$$\begin{aligned} I(y,y')&=-\frac{M^2}{\pi}\int_{-M}^{M}d\rho\,\frac{\rho(1-\rho^4/M^4)^{1/2}}{(\rho^2+\eta^2)(\rho^2+\eta'^2)}\tanh(M\sinh^{-1}\rho/M)\\ &\approx-\frac{M^2}{\pi}\int_{-\infty}^{\infty}d\rho\,\frac{\rho\tanh\rho}{(\rho^2+\eta^2)(\rho^2+\eta'^2)}\\ &\equiv M^2J(\eta,\eta'), \end{aligned} \tag{B18}$$

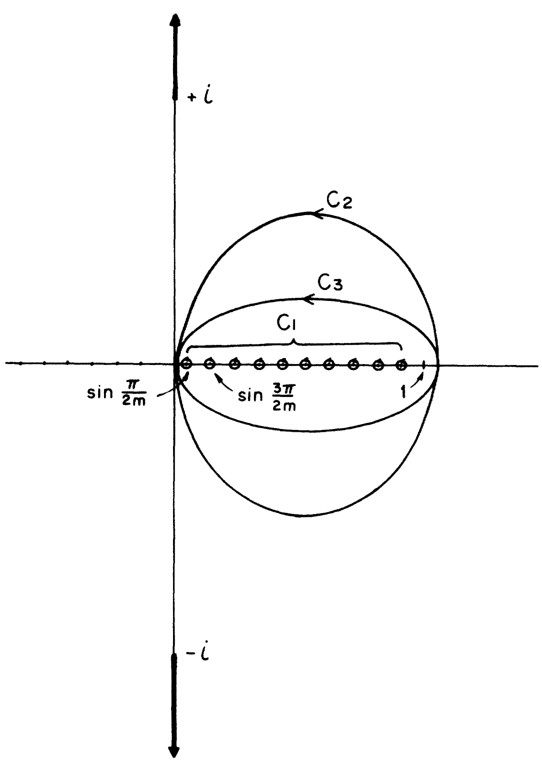

FIG. 8. Contours for evaluation of matrix products.

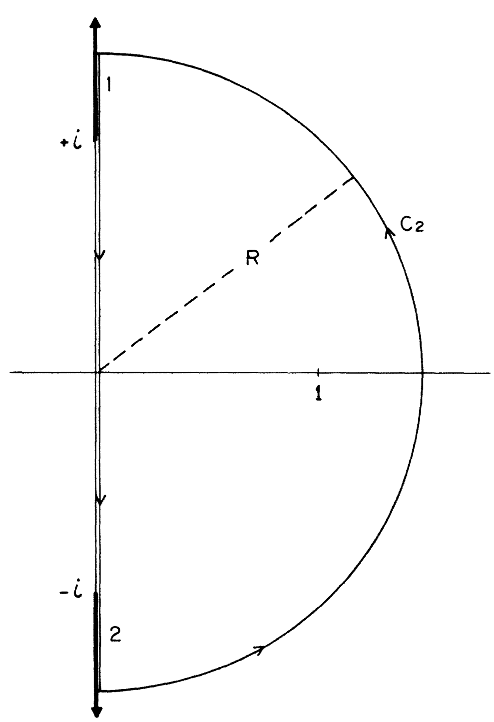

FIG. 9. The choice of contour for the evaluation of $\mathfrak{C}_{mn}$.

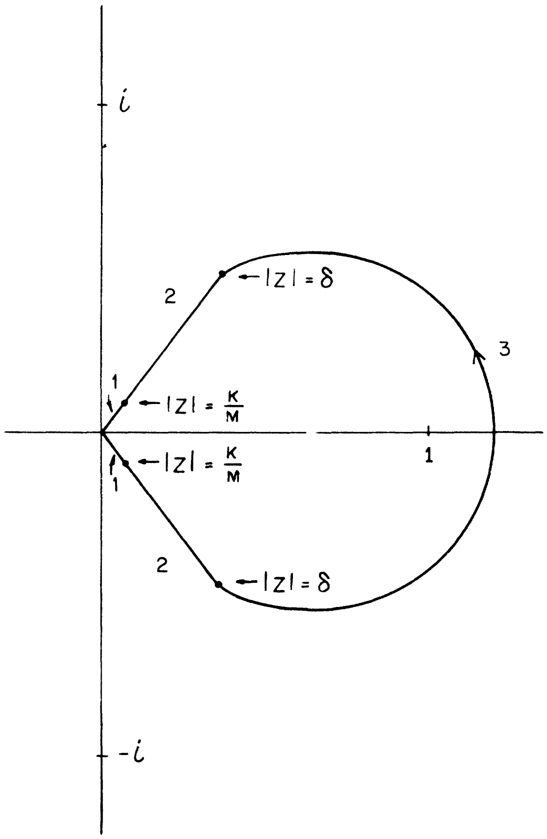

FIG. 10. The choice of contours for the evaluation of traces. Only sections 2 contribute the leading log M.

where we have extended the integral over ρ to ∞, which is allowable since the integral converges.

Similarly, the integrals over y can be rescaled:

$$\int_1 \frac{dy}{2\pi i} h(y) \sim \frac{1}{M^2} \int_{1'} \frac{d\eta}{2\pi i} \eta \cot\eta , \tag{B19}$$

where $1'$ is a contour which encloses $\eta = \pi/2$ on the real axis and extends out to $|\eta| = K$.

The overall contribution of region 1 to D_n is

$$(D_n)_1 = -\int_{1'} \frac{d\eta_1}{2\pi i} \cdots \frac{d\eta_n}{2\pi i} (\eta_1 \cot\eta_1) J(\eta_1, \eta_2) \cdots (\eta_n \cot\eta_n) \times J(\eta_n, \eta_1) . \tag{B20}$$

Expression (B20) is *finite* as $M \to \infty$, depending only on K. Thus (B20) does not contribute to the leading-log M dependence of D_n.

Next we consider integrals over regions 2 and 3. If y, y' remain finite as $M \to \infty$, we may take the limit inside the integral (B14), obtaining

$$I(y, y') = -\frac{1}{\pi} \int_{-1}^{1} dr \frac{r(1-r^4)^{1/2}}{(r^2+y^2)(r^2+y'^2)} \tag{B21}$$

and

$$h(y) = \begin{cases} \dfrac{-iy}{(1+y^2)^{1/2}(1-y^2)^{1/2}} , & \mathrm{Im}\, y > 0 \\[2ex] \dfrac{+iy}{(1+y^2)^{1/2}(1-y^2)^{1/2}} , & \mathrm{Im}\, y < 0 . \end{cases} \tag{B22}$$

Thus, any M dependence of D_n must come through the dependences of the integrals on their lower limit, K/M.

Now for small y, y' we have

$$I(y, y') \sim O\left(\frac{1}{y^2}\right) , \qquad h(y) \sim O(y) , \tag{B23}$$

while if y is small, but y' is not,

$$I(y, y') \sim O(1) . \tag{B24}$$

By the same sort of power counting used before, it follows that the leading dependence of D_n on the lower limit (K/M) is logarithmic and arises only when all y's are near their lower limit. Thus, we need consider the integrals only over regions 2: $K/M < |y| < \delta \ll 1$.

For small y, we may approximate the integral in (B21):

$$I(y, y') \sim -\frac{1}{\pi} \int_{-1}^{1} dr \frac{r}{(r^2+y^2)(r^2+y'^2)} \sim -\frac{1}{\pi} \frac{1}{y^2 - y'^2} \ln\frac{y^2}{y'^2} . \tag{B25}$$

Also,

$$h(y) \sim \begin{cases} -iy, & \mathrm{Im}\, y > 0 \\ iy, & \mathrm{Im}\, y < 0 . \end{cases} \tag{B26}$$

We have

$$D_n = -\int_2 \frac{dy_1}{2\pi i} \cdots \frac{dy_n}{2\pi i} h(y_1) \frac{\ln(y_1^2/y_2^2)}{\pi(y_1^2 - y_2^2)} \cdots h(y_n) \times \frac{\ln(y_n^2/y_1^2)}{y_n^2 - y_1^2} . \tag{B27}$$

There are no singular points in y between the two parts of each contour 2, so we can move both toward the real axis (Fig. 9). Both $h(y)$ and the direction of integration change sign in going from the upper to the lower contour, while the remainder of the integrand is symmetric. Thus we simply obtain, for each y, twice the integral over the upper part of 2,

$$D_n = -\frac{2^n}{\pi^{2n}} \int_{K/M}^{\delta} dy_1 \cdots dy_n\, y_1 \cdots y_n \times \frac{\ln(y_1/y_2)\cdots\ln(y_n/y_1)}{(y_1^2-y_2^2)\cdots(y_n^2-y_1^2)}$$
$$= -\frac{1}{2^n\pi^{2n}} \int_{K^2/M^2}^{\delta^2} du \circ\cdots du_n \frac{\ln(u_1/u_2)\cdots\ln(u_n/u_1)}{(u_1-u_2)\cdots(u_n-u_1)}\,, \tag{B28}$$

where $u_j = y_j^2$.

Next we define variables

$$u_1 = x_1\,,\quad u_2 = x_1x_2\,,\quad \ldots\quad u_n = x_1x_2\circ\cdots x_n\,. \tag{B29}$$

In the multiple integration x_1 ranges from $K^2/M^2 \to \delta^2$, while the ranges of the succeeding x's are rather complicated functions.

$$D_n = -\frac{1}{2^n\pi^{2n}} \int_{K^2/M^2}^{\delta^2} dx_1 \int dx_2 \circ\cdots dx_n (x_1^{n-1}x_2^{n-2}\circ\cdots x_{n-1}) \frac{\ln(1/x_2)\circ\cdots\ln(1/x_n)\ln(x_2\cdots x_n)}{x_1^n x_2^{n-2}\cdots x_{n-1}(1-x_2)\cdots(1-x_n)(x_2\cdots x_n-1)}$$
$$= -\frac{1}{2^n\pi^{2n}} \int_{K^2/M^2}^{\delta^2} \frac{dx_1}{x_1} \int dx_2\cdots dx_n \frac{\ln(1/x_2)\cdots\ln(1/x_n)\ln x_2\circ\cdots x_n}{(1-x_2)\circ\cdots(1-x_n)(x_2\cdots x_n-1)}\,. \tag{B30}$$

The integral over x_1 gives the leading log M while the remaining integrals may be extended to range from 0 to ∞ to give its coefficient,

$$D_n \sim -\frac{2\ln M}{2^n\pi^{2n}} \int_0^\infty dx_2\circ\circ\, dx_n \frac{\ln(1/x_2)\circ\cdots\ln(1/x_n)\ln(x_2\circ\circ\circ x_n)}{(1-x_2)\circ\circ\circ(1-x_n)(x_2\circ\circ\circ x_n-1)}\,, \tag{B31}$$

changing variables to $x_j = e^{-V_{j-1}}$,

$$D_n \sim -\frac{2\ln M}{4^n\pi^{2n}} \int_{-\infty}^\infty dV_1\circ\cdots dV_{n-1} \frac{V_1\circ\cdots V_{n-1}(V_1+\cdots+V_{n-1})}{\sinh(\frac{1}{2}V_1)\circ\circ\circ\sinh(\frac{1}{2}V_{n-1})\sinh\{[\frac{1}{2}(V_1+\cdots+V_{n-1})]\}} \tag{B32}$$

Introducing an nth V, and setting it equal to $-V_1-\circ\circ\circ-V_{n-1}$ with a δ function, we have

$$D_n \sim -\frac{2}{4^n\pi^{2n}}\ln M \int_{-\infty}^\infty \frac{dK}{2\pi} \int_{-\infty}^\infty dV_1\circ\cdots dV_n e^{iK(V_1+\cdots+V_n)} \prod_{K=1}^n \left(\frac{V_K}{\sinh\frac{1}{2}V_K}\right)$$
$$= -\frac{2}{4^n\pi^{2n}}\ln M \int_{-\infty}^\infty \frac{dK}{2\pi}\left(\int_{-\infty}^\infty dV \frac{Ve^{iKV}}{\sinh\frac{1}{2}V}\right)^n$$
$$= -2\ln M \int_{-\infty}^\infty \frac{dK}{2\pi}\left[\frac{1}{2\cosh^2(\pi K)}\right]^n\,. \tag{B33}$$

We may sum up these terms

$$\ln\det(1+a) = \sum_n \frac{1}{n} D_n \sim 2\ln M \int_{-\infty}^\infty \frac{dK}{2\pi} \ln\left[1-\frac{1}{2\cosh^2(\pi K)}\right]. \tag{B34}$$

Letting $u = e^{-\pi K}$, we have

$$\ln\det(1+a) \sim \frac{2\ln M}{\pi^2}\int_0^1 du \frac{1}{u}\ln\frac{1+u^4}{(1+u^2)^2} = -\frac{3}{2\pi^2}\ln M \int_0^1 \frac{dx}{x}\ln(1+x)$$
$$= -\frac{3}{2\pi^2}\ln M \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2}$$
$$= -\frac{3}{2\pi^2}(\ln M)\frac{\pi^2}{12}$$
$$= -\tfrac{1}{8}\ln M\,.$$

Thus

$$\det(1+a) \to \frac{1}{K^2M^{1/8}} \cong \frac{1.159}{M^{1/8}}\,.$$

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[1]For a collection of excellent reviews of dual string theory see, for example, *Dual Theory*, edited by M. Jacob (North-Holland, Amsterdam, 1974).

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[6]There is a caveat here. As pointed out by A. Patrascioiu, Nucl. Phys. B81, 525 (1974), there are solutions to the original equations of motion for which $P^+=0$ for a finite segment of σ. Such solutions will only be obtained in our description as (singular) limits of solutions for which $P^+\neq 0$ throughout the string. It should be carefully noted, however, that these solutions are *limit points* in our solution space, and it would be double-counting to include them as dynamically independent motions, except, of course, in zero transverse dimensions where our description certainly breaks down.

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[9]We remark here that the transverse spectrum is known to be compatible with Lorentz invariance (see Ref. 7), provided the first excited state is massless. With the preceding result for $m_G{}^{2\,\mathrm{open}}$, this yields the critical dimension. However, one can consistently restrict the spectrum to even-signature states, in which case the first excited state decouples, and the remaining spectrum is compatible with Lorentz invariance for any dimension. We shall see in Sec. IV how consistency of the interacting theory demands the critical dimension even in the even-signature sector.

[10]For an asymmetric lattice the strength of the physical coupling depends on the ratio of the lattice spacing, $\nu\equiv b/a$ as well as g_0. To study the dependence of our theory on the strength of the coupling, it may be simpler to keep $g_0=1$ and vary ν.

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