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Comments on a paper by Callan, Coote, and Gross*

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A recent work by Callan, Coote, and Gross has attempted to demonstrate quark binding in a two-dimensional gauge theory in light-cone coordinates. Among the assumptions employed in that paper are (a) that there exists a parity operator, (b) that the theory is covariant, (c) that anomalies in the theory present no essential complicatons, and (d) that "singular" cutoffs may be imposed in calculations without regard to considerations of gauge invariance or the equations of motion. Examined within the context of canonical field theory it is argued that none of these conditions is satisfied in the model which they consider.

In recent months probably no single topic has so captured the imagination of particle theorists as has the quark-binding problem. Significant progress in this field would doubtless do much to increase the potential for success of color gauge models as well as cast considerable light on the structure of quantum field theories. On the other hand precisely because quark-binding schemes could be such an important stimulus and directive in the development of strong-interaction theories, it is absolutely crucial that each alleged advance in this field be subjected to the most thorough scrutiny.

With this idea in mind we review here the status of a model recently proposed by 't Hooft[1] and subsequently expanded upon by a number of authors. The theory consists of a non-Abelian gauge field in a world of two space-time dimensions described by the U(N)-invariant Lagrangian

$$\mathcal{L} = \tfrac{1}{2}i\,\psi\alpha^{\mu}(\partial_{\mu} - ig_0T_aA_{\mu}^{a})\psi - \tfrac{1}{2}m\psi\beta\psi + \tfrac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^{a} - \tfrac{1}{2}F_a^{\mu\nu}(\partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a} + ig_0A_{\mu}t^{a}A_{\nu}),$$

where T_a and t_a refer, respectively, to the fundamental and adjoint representations of the group. Following 't Hooft, we may introduce light-cone coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^1 \pm x^0)$$

thereby implying the form

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for the metric tensor. Corresponding to this choice of coordinates and the selection of x^+ as the "time" coordinate, one has what may be termed the radiation gauge condition

$$A^+ = A_- = 0\ .$$

One of the principal conclusions obtained in Ref. 1 consisted in an argument which alleged to show that the quark mass m was renormalized to infinity by the effect of the interaction. Such a result, namely the disappearance of the quark from the realm of finite energy, would, of course, lend great support to current hopes that quarks could be freely used as building blocks without any accompanying embarrassment associated with their failure to be experimentally detected. However, this conclusion was shown not to be valid by the author[2] in a paper whose two principal results were the following:

(i) A careful consideration of the Coulomb potential in two dimensions, namely

$$-\tfrac{1}{2}|x^- - x^{-\prime}|, \tag{1}$$

shows that the quark mass acquires a finite mass renormalization equal to $-g^2/\pi$, where $g^2 = g_0^2N$. The infrared-divergent mass renormalization of 't Hooft was thus demonstrated to be the result of an incorrect attempt to introduce a regulator into the Fourier transform of (1).

(ii) The theory was found to contain an anomaly in the divergence of the current which was in direct contradiction to the equations of motion.

Subsequent to these developments the 't Hooft model was also discussed by Frishman,[3] who considered the properties of the operator

$$\sigma = \tfrac{1}{2}\psi\beta\psi$$

in the $m=0$ limit. His assertion of the existence of a second fermion of zero mass in the theory in order to explain the peculiar threshold of the σ operator was shown by the author[4] to be incorrect. This result is an immediate consequence of the fact that $\sigma = 0$ in the $m=0$ theory, as is most easily seen in the representation

$$\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \alpha^+ = \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \alpha^- = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

In such a representation the equation for ψ_2 becomes

$$\partial_-\psi_2 = 0,$$

thereby implying that $\psi_2 = 0$ and consequently $\sigma = 0$.

A more recent discussion of the 't Hooft model is a work by Callan, Coote, and Gross[5] (CCG), who allege to find substantial support for quark binding. That work relies heavily on four implicit assumptions, each of which is at variance with the principles of conventional canonical field theory. The remainder of this note will deal with each of these points in turn.

A. Absence of a parity operator

There are two distinct possibilities for the definition of a parity operation, namely,

(i) $x^{\pm} \to \pm x^{\pm}$,
(ii) $x^1 \to -x^1$ or $x^{\pm} \to -x^{\mp}$,

and we deal with each of these successively. In case (i) one requires the existence of a unitary operator U and a real matrix S_P such that

$$U\psi(x^+, x^-)U^\dagger = S_P\psi(x^+, -x^-)$$

with

$$S_P^T \alpha^{\pm} S_P = \pm\alpha^{\pm}.$$

These conditions can readily be shown[6] to yield

$$[(S_P)_{22}]^2 = -1 ,$$

thereby directly contradicting the reality of S_P.[7]

Definition (ii) can similarly be shown not to be unitarily implementable. In the latter case, $S_P = i\beta$ (up to an irrelevant overall sign), and

$$U\psi_{1,2}(x^+, x^-)U^\dagger = \pm\psi_{2,1}(-x^-, -x^+).$$

By considering the canonical commutator

$$\{\psi_1(x^+, x^-), \psi_1(x^+, x^{-\prime})\} = \frac{1}{\sqrt{2}}\,\delta(x^- - x^{-\prime})$$

and using the fact[4,6] that in the absence of coupling $\psi_2 = 0$ for $m = 0$, one readily shows

$$\{\psi_2(-x^-, -x^+), \psi_2(-x^{-\prime}, -x^+)\} = 0 = \frac{1}{\sqrt{2}}\,\delta(x^- - x^{-\prime}).$$

This clearly suffices to establish the absence of a parity operator in the massless case. The extension to $m \neq 0$ is accomplished by considering the conditions imposed on the coupled Dirac equation by the presumed existence of a parity operator. Since invariance is easily seen to require that

$$UA^{\pm}(x^+, x^-)U^\dagger = -A^{\mp}(-x^-, -x^+),$$

one has a contradiction of the equations

$$A^+ = 0 ,$$

$$\partial_-^{\,2} A^- = gj^+$$

for $g \neq 0$. There is consequently no parity operator in the case $m \neq 0$ for nonvanishing coupling.

B. Noncovariance of the massive two-dimensional fermion in light-cone coordinates

In Ref. 6 there was given the analog of the Dirac-Schwinger covariance condition[8] on the energy-momentum tensor $T^{\mu\nu}$

$$[T^{++}(x), T^{+-}(x')] = i[\partial_- T^{+-}(x)]\,\delta(x^- - x^{-\prime}) , \qquad (2)$$

which is sufficient to ensure covariance in light-cone coordinates. While the model of Ref. 6 satisfies (2), the 't Hooft model with $m \neq 0$ does not satisfy it even for the free-field case. Using the conventional form

$$T^{\mu\nu} = g^{\mu\nu}\mathcal{L} - \tfrac{1}{4} i\psi\beta(\gamma^\mu\partial^\nu + \gamma^\nu\partial^\mu)\psi$$

and the commutator of ψ_1 with itself, one finds an additional term on the right-hand side of (2) of the form

$$-\tfrac{1}{8} m\,\partial_-\partial_-^{\prime}[\psi_1(x)\epsilon(x - x')\psi_2(x')] , \qquad (3)$$

where we have written

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$$

and

$$\epsilon(x) = x/|x| .$$

From the result (3) there follows

$$[T^{+-}(x), J^{+-}] = -i(x^+\partial^- - x^-\partial^+)T^{+-}(x) + \frac{m^2}{8\sqrt{2}} \lim_{L\to\infty} L[\psi_1(L) - \psi_1(-L)]\,\psi_1(x),$$

where

$$J^{+-} = x^+P^- - \int x^- T^{++}\,dx^-$$

and

$$P^- = \int T^{+-}\,dx^- .$$

Using the fact that the two-point function of the free field falls off as $(x^- - x^{-\prime})^{-1}$ for $x^+ = x^{+\prime}$,[6] it is clear that the extra term in (4) cannot be made to vanish and that the Poincaré algebra consequently cannot be realized in the case $m \neq 0$. This establishes the asserted noncovariance of the theory.[9]

C. Contradiction of the equations of motion by anomalies

As already mentioned, the existence of an anomaly in the 't Hooft model was first pointed out in Ref. 2. Such an anomaly stands in direct contradiction to the equations of motion (as already observed in Ref. 2), and to dismiss it is essentially to sever the discussion of the 't Hooft model entirely from the realm of canonical field theory. It is essentially the same as arguing that the Maxwell equations

$$\partial_\nu F^{\mu\nu} = ej^\mu$$

and the associated consistency condition

$$\partial_\mu j^\mu = 0 \qquad (5)$$

required by the antisymmetry of $F^{\mu\nu}$ now need no longer be bound by the condition (5). That anomalies can be accommodated in certain theories is certainly true, but never in the sense of an outright violation of the operator equations of motion.[10]

D. Inadmissibility of the "singular" cutoff

In CCG the 't Hooft regularization (i.e., requiring $|q_-| > \lambda$) is employed at considerable length. It need only be repeated here that it introduces a spurious singularity into the quark propagator which implies an incorrect result for the quark mass. As shown in Ref. 2 the latter quantity is finite, and at no step of the calculation requires introduction of a cutoff. Thus the use of the "singular" cutoff which yields a divergent expression for the gauge-invariant quark mass is not allowable. To suggest as in CCG that, because the fermion two-point function is gauge variant, one need not be concerned about two different results for the quark mass is to ignore the fact that mass is a gauge-invariant concept which must have identical eigenvalues in all acceptable calculations.

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[1]G. 't Hooft, Nucl. Phys. B75, 461 (1974).

[2]C. R. Hagen, Nucl. Phys. B95, 477 (1975).

[3]Y. Frishman, Nucl. Phys. (to be published).

[4]C. R. Hagen (unpublished).

[5]C. G. Callan, N. Coote, and D. J. Gross, Phys. Rev. D 13, 1649 (1976).

[6]C. R. Hagen and J. H. Yee, Phys. Rev. D 13, 2789 (1976). The reader is referred to this work for detailed discussion of some of the more subtle aspects of two-dimensional theories in light-cone coordinates.

[7]The extension of this argument to the case of a nonlinear representation of the parity operator is readily carried out by allowing S_P to be a (Hermitian) operator.

[8]J. Schwinger, Phys. Rev. 127, 324 (1962); P. A. M. Dirac, Rev. Mod. Phys. 34, 592 (1962).

[9]It is worth mentioning here that Eq. (4) implies that the operator σ fails to transform as a scalar. This result is of crucial significance to the discussion in CCG of the "scalar" density.

[10]It is to be emphasized that no amount of discussion of regularization can circumvent such anomalies. As an example, one has in Ref. 6 a U(1) version of the 't Hooft model with a generalization to nonzero boson mass. The solution of that model by entirely straightforward functional methods leads unambiguously to a nonvanishing result for the current divergence.