Note on classical solution of the Yang-Mills equations in Minkowski space*†

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A real solution of the Yang-Mills equations in Minkowski space is constructed by means of a conformal transformation.

There has been great interest recently in explicit solutions of the SU(2) Yang-Mills equation in Euclidean four-space. It is also desirable to find solutions with finite action and energy for these equations in Minkowski space:

$$\partial_\mu F^{\mu\nu}+[A_\mu,F^{\mu\nu}]=0\,,$$
$$A_\mu\equiv g\frac{\sigma^a}{2i}A^a_\mu, \tag{1}$$
$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]\equiv g\frac{\sigma^a}{2i}F^a_{\mu\nu}\,.$$

Solutions to (1) may, for example, serve as the starting point of a semiclassical approximation to the quantum Yang-Mills theory.

An ansatz for A_μ analogous to the one used to derive multipseudoparticle configurations,[1] i.e.,

$$A^\mu(x)=i\sigma^{\mu\nu}\partial_\nu\ln\phi(x)\ , \tag{2}$$

with antisymmetric matrices $\sigma^{\mu\nu}$ satisfying the O(3,1) commutation relations

$$i[\sigma^{\mu\alpha},\sigma^{\nu\beta}]=g^{\alpha\nu}\sigma^{\mu\beta}+g^{\mu\nu}\sigma^{\beta\alpha}+g^{\alpha\beta}\sigma^{\nu\mu}+g^{\mu\beta}\sigma^{\alpha\nu}\,, \tag{3}$$

reduces Eq. (1) to

$$\sigma^{\nu\alpha}\left[\phi^2\partial_\alpha\left(\frac{1}{\phi^3}\quad\partial_\mu\partial^\mu\phi\right)\right]=0\,. \tag{4}$$

I.e., any solution of the equation

$$\partial_\mu\partial^\mu\phi+\lambda\phi^3=0 \tag{5}$$

leads, via (2), to a solution of the Yang-Mills equations. The solution given by De Alfaro, Fubini, and Furlan[2] is generated by $\phi=2[\lambda(1+t_+^2)\times(1+t_-^2)]^{-1/2}$. [A more general solution to Eq. (5) ($\lambda\neq 0$) is known[3,4] which, after an appropriate adjustment of the integration parameters, leads to a finite action and energy in Minkowski space.[4]] Let us note that, if $\lambda\neq 0$, the potentials of the form (2) have action and stress-tensor density, respectively (without loss of generality we set $\lambda=1$), given by

$$\mathcal{L}(x)=\frac{-1}{2g^2}\operatorname{Tr}F_{\mu\nu}F^{\mu\nu}=\frac{-1}{2g^2}(\partial^2\partial^2\ln\phi-3\phi^4)\,, \tag{6}$$

$$\theta_{\mu\nu}(x)=\frac{-2}{g^2}\operatorname{Tr}\left[F_{\mu\alpha}F_\nu{}^\alpha-\frac{g^{\mu\nu}}{4}F_{\alpha\beta}F^{\alpha\beta}\right]$$
$$=\frac{-2}{g^2}\Big[-2\partial_\mu\phi\partial_\nu\phi+\phi\partial_\mu\partial_\nu\phi+\frac{g_{\mu\nu}}{4}(\phi^4+2\partial_\alpha\phi\partial^\alpha\phi)\Big]\,, \tag{7}$$

and the energy is

$$E=\int d^3x\,\theta_{00}=\frac{6}{g^2}\int d^3x[\tfrac{1}{2}(\partial_0\phi)^2+\tfrac{1}{2}(\vec\nabla\phi)^2+\tfrac{1}{4}\phi^4]\,. \tag{8}$$

If $\lambda=0$, i.e., if ϕ is a solution of the scalar-wave equation, then the corresponding potential A^μ has a vanishing stress tensor $\theta_{\mu\nu}$. [Notice that for $\lambda=0$, potentials of the form (2) are also solutions of the self-duality equations in Minkowski space: $F_{\mu\nu}=\pm{}^*F_{\mu\nu}$.] There exists, of course, an infinite class of solutions to the wave equation which leads to finite action. For example, any function $\phi(x)=\sum_i\tanh P_i(\eta^{(i)}\cdot x)$ ($\eta^{(i)}$ denotes lightlike four-vectors independent of x, the P_i are even polynomials in $\eta^{(i)}\cdot x$ such that $P_i\gtrless 0$ for all i and x_μ) leads to an integrable Yang-Mills action density

$$\mathcal{L}(x)=-(1/2g^2)\partial^2\partial^2\ln\phi(x).$$

But all the solutions discussed so far are complex, in Minkowski space, as a consequence of the ansatz (2) which can be seen in the following way: We want to deal with an SU(2), i.e., compact, gauge theory. If we denote, as usual, $\sigma^{ij}=\epsilon^{ijk}\mathfrak{F}_k$, $\sigma^{i0}=K^i$ and if we define, furthermore, $\vec A=\frac{1}{2}(\vec{\mathfrak{F}}+i\vec K)$, $\vec B=\frac{1}{2}(\vec{\mathfrak{F}}-i\vec K)$ then A_i and B_i satisfy SU(2) commutation relations and $[A_i,B_j]=0$ showing the well-known isomorphism $O(3,1)_C\cong SU(2)\times SU(2)$. It means that we have to choose

$$\sigma^{ij}=\tfrac{1}{2}\epsilon^{ijk}\sigma_k,\quad \sigma^{i0}=\pm\frac{i}{2}\sigma^i\,, \tag{9}$$

which, in turn, means that the Yang-Mills potentials $A^a_\mu=(i/g)\operatorname{Tr}\sigma^a A_\mu$ are complex for any function $\phi(x)$:

$$A_0^a = \pm i \partial_a \ln\phi\,,$$
$$A_i^a = -\epsilon_{ija}\partial_j \ln\phi \mp i\delta_{ia}\partial_0 \ln\phi\,. \tag{10}$$

For real ϕ, of course, the action and stress-tensor densities (6) and (7) are always real and the energy (8) is always positive. One might suspect, therefore, that there exist (complex) gauge transformations which transform solutions (10) into real functions. However, this cannot be the case in general. A gauge-invariant quantity such as $\mathrm{Tr}F_{\mu\nu}F_{\alpha\beta}$, evaluated for potentials of the form (2), is complex in general. But physical solutions of Eq. (1) should correspond to classical limits of self-adjoint operators, i.e., they should be real functions.

One way of finding real solutions is the following: The Yang-Mills equations in Minkowski space are transformed into Euclidean equations by means of a conformal transformation. For these equations real solutions are known. Let us define variables y_α and potentials $\hat{A}_\alpha$ [in the following we use the metric convention $g_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ and the notations $\mu,\nu = 0,1,2,3$; $\alpha,\beta = 1,2,3,4$; $i,j = 1,2,3$] as follows:

$$y_i = x_i\,,$$
$$y_4 = \tfrac{1}{2}(1 + x_0^2 - \vec{x}^2)\,, \tag{11}$$

$$A_0(x) = -x_0 \hat{A}_4(y)\,,$$
$$A_i(x) = \hat{A}_i(y) - x_i \hat{A}_4(y)\,. \tag{12}$$

Equation (12) corresponds to a conformal transformation of the potentials $A_\mu(x)$.[5] Reexpressing the field-strength tensor $F_{\mu\nu}(x)$ in terms of the new potentials $\hat{A}_\alpha$ and derivatives $\hat{\partial}_\alpha$ with respect to the new variables y_α we find

$$F_{0i}(x) = -x_0 \hat{F}_{4i}(y)\,,$$
$$F_{ij}(x) = \hat{F}_{ij}(y) - x_i \hat{F}_{4j}(y) - x_j F_{i4}(y)\,, \tag{13}$$
$$\hat{F}_{\alpha\beta} \equiv \hat{\partial}_\alpha \hat{A}_\beta - \hat{\partial}_\beta \hat{A}_\alpha + [\hat{A}_\alpha, \hat{A}_\beta]\,.$$

Now, if

$$y_\beta \hat{A}_\beta = 0\,, \tag{14}$$

and if the potentials $\hat{A}_\alpha$ are homogenous functions of degree minus one, i.e., if [the scalar products in (14) and (15) are Euclidean]

$$y_\beta \hat{\partial}_\beta \hat{A}_\alpha = -\hat{A}_\alpha\,, \tag{15}$$

we can rewrite the Yang-Mills equations as follows:

$$0 = g^{\mu\nu}(\partial_\mu F_{\nu 0} + [A_\mu, F_{\nu 0}])$$
$$= -x_0 \delta^{\alpha\beta}(\hat{\partial}_\alpha \hat{F}_{\beta 4} + [\hat{A}_\alpha, \hat{F}_{\beta 4}])\,,$$

$$0 = g^{\mu\nu}(\partial_\mu F_{\nu i} + [A_\mu, F_{\nu i}])$$
$$= \delta^{\alpha\beta}(\hat{\partial}_\alpha \hat{F}_{\beta i} + [\hat{A}_\alpha, \hat{F}_{\beta i}])$$
$$- x_i \delta^{\alpha\beta}(\hat{\partial}_\alpha \hat{F}_{\beta 4} + [\hat{A}_\alpha, \hat{F}_{\beta 4}])\,.$$

This means that any solution of the Euclidean Yang-Mills equations

$$\hat{\partial}_\alpha \hat{F}_{\alpha\beta} + [\hat{A}_\alpha, \hat{F}_{\alpha\beta}] = 0\,, \tag{16}$$

which satisfies conditions (14) and (15), leads, via (11) and (12), to a solution of the equations in Minkowski space. All the nontrivial solutions of Eq. (16) which are known are of the form $A_\alpha = i\hat{\sigma}_{\alpha\beta}\hat{\partial}_\beta \ln\phi$, where $\hat{\sigma}_{\alpha\beta}$ are O(4) matrices ($\hat{\sigma}_{ij} = \frac{1}{2}\epsilon_{ijk}\sigma_k$, $\hat{\sigma}_{i4} = \frac{1}{2}\sigma_i$) [i.e., $A_\mu^a(x)$ is real if $\phi(y)$ is real] and ϕ is a solution of

$$\Box\phi = \lambda\phi^3\,. \tag{17}$$

Solutions of this equation, such that conditions (14) and (15) are valid (note that the homogeneity requirement is very restrictive), are $\phi_i = b\cdot y^{-2}$ if $\lambda = 0$, and $\phi_2 = (\lambda y^2)^{-1/2}$ if $\lambda \neq 0$. The function ϕ_1 generates a potential $\hat{A}_\alpha$ which is a pure gauge. Therefore the field strengths $F_{\mu\nu}(x)$ vanish. The potentials $A_\mu(x)$ corresponding to ϕ_2 are

$$A_0(x) = \pm i \frac{x_0}{y^2}\vec{\sigma}\cdot\vec{x}\,,$$

$$A_i(x) = \frac{-1}{2y^2}[\epsilon_{ijk}x_j\sigma_k \pm \tfrac{1}{2}(1 + x_0^2 - \vec{x}^2)\sigma_i \pm x_i\vec{\sigma}\cdot\vec{x}]\,, \tag{18}$$

and

$$y^2 = \tfrac{1}{4}(1 + t_+^2)(1 + t_-^2)\,, \quad t_\pm = x_0 \pm |\vec{x}|\,.$$

The solution (18), leading to real potentials $A_\mu^a(x)$, is related to the solution given by De Alfaro, Fubini, and Furlan by a conformal transformation.[6] The Lagrange densities coincide (the definitions of our action and energy densities differ from those given by De Alfaro, Fubini, and Furlan by a factor of 2):

$$\mathcal{L}(x) = \frac{24}{g^2}\left[\frac{1}{(1 + t_+^2)(1 + t_-^2)}\right]^2,$$
$$A = \int \mathcal{L}\, d^4x = 3\pi^3/g^2\,. \tag{19}$$

The energy density of (18) is given by

$$\theta_{00} = \frac{24}{g^2}\left[\frac{1}{(1 + t_+^2)(1 + t_-^2)}\right]^2 - \frac{128}{g^2}x_0^2\vec{x}^2\left[\frac{1}{(1 + t_+^2)(1 + t_-^2)}\right]^3. \tag{20}$$

For both solutions the values of the total energy are also the same:

$$E=\int \theta_{00}(x_0=0)d^3x=3\pi^2/g^2.$$

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[5]For a general discussion of the conformal group, see, e.g., G. Mack and A. Salam, Ann. Phys. (N.Y.) 53, 174 (1969).

[6]After the completion of this work I learned that B. Schechter [Phys. Rev. D 16, 3015 (1977)] has found a similar form of solution (18) by projecting the Yang-Mills equations on a hypertorus.