# Contribution to the eighth-order anomalous magnetic moment of the muon*

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The eighth-order contribution to the muon anomaly from second-order vacuum polarization insertions into photon-photon scattering diagrams is accurately determined. The result is $a_\mu^{(8)}(\gamma\gamma) = (117.4 \pm 0.5)(\alpha/\pi)^4$. The coefficients of the $\ln^2(m_\mu/m_e)$ and $\ln(m_\mu/m_e)$ terms are also evaluated. The coefficient of $\ln^2(m_\mu/m_e)$ is found to be just one half the value expected from naive application of renormalization-group methods, and it is shown how this arises.

## I. INTRODUCTION

The dominant contribution in eighth order to the anomalous magnetic moment of the muon is associated with 18 Feynman diagrams of the type shown in Fig. 1, obtained by inserting a single electron loop in all possible ways into the sixth-order photon-photon scattering graphs.

In the case of the sixth-order photon-photon scattering contribution, we found[1] that accurate computation was limited primarily by the singularity structure of the integrand used in the multidimensional numerical integration. This had the effect of causing the contribution to be systematically underestimated. The problem was overcome by changes of variables and the introduction of an $\epsilon$ cutoff on the limits of integration near the singularity. Careful study of the dominant behavior showed that as a function of the cutoff we could write the contribution as follows: For small $\epsilon$,

$$I(\epsilon) \sim I_0 - A\sqrt{\epsilon}\,. \tag{1}$$

Having evaluated $I(\epsilon)$ accurately for several values of $\epsilon$, we then extrapolated to $\epsilon = 0$ by using Padé approximants to obtain the result

$$a_\mu^{(6)}(\gamma\gamma) = I_0 = (21.32 \pm 0.05)\left(\frac{\alpha}{\pi}\right)^3. \tag{2}$$

In the case of the eighth-order contribution, we will find a similar technique to be effective in refining the previous numerical estimate of Calmet and Peterman[2]:

$$a_\mu^{(8)}(\gamma\gamma) = (111.1 \pm 8.1)\left(\frac{\alpha}{\pi}\right)^4. \tag{3}$$

## II. THE METHOD

We may determine the contribution of these eighth-order graphs to the muon anomaly by the replacement of the photon propagators of the sixth-order diagrams with the modification due to vacuum polarization:

$$\frac{1}{k^2} \to \frac{-\mathrm{Re}\,\pi^{(2)}(k^2)}{k^2} = \int_0^\infty \frac{dt}{t}\,\frac{\mathrm{Im}\,\pi^{(2)}(t)/\pi}{k^2 - t}. \tag{4}$$

As is known, the contribution to the muon anomaly from the sixth-order graphs may be written as an integral over a 7-dimensional simplex,

$$I^{(6)}(\rho) = \frac{a_\mu^{(6)}(\gamma\gamma)}{(\alpha/\pi)^3} = \int dz\, F(z, U, W)\delta(1 - z_t)\,, \tag{5}$$

where

$$z_t = \sum_{i=1}^{8} z_i\,, \quad dz = \prod_{i=1}^{8} dz_i\,,$$

and

$$\rho = \left(\frac{m_e}{m_\mu}\right)^2.$$

The integrand $F$ is given by Aldins *et al.*[3] and may be expressed as a sum of terms

$$F = \sum_{n=1}^{4} \sum_{k=1}^{3} \frac{C_{nk}}{U^n W^k}\,, \tag{6}$$

where $U$, $W$, and the $C_{nk}$ are homogeneous functions of the $z_i$. $W$ and some of the $C_{nk}$ also depend upon the square of the mass ratio $\rho$.

The propagator replacement Eq. (4) is made into

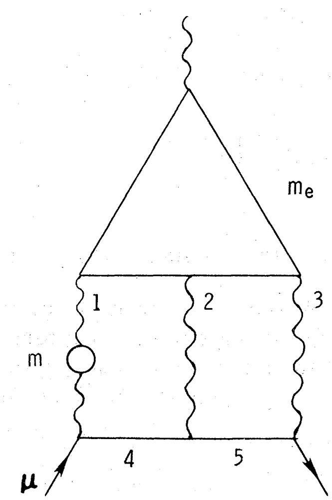


FIG. 1. Eighth-order photon-photon scattering diagram with vacuum polarization insertion.

each of the photon lines labeled 1, 2, 3 of Fig. 1. Subsequent expression of the anomaly in terms of an integral over Feynman parameters is effected by application of the double parametric representation of Feynman amplitudes to the sixth-order diagrams, using $\lambda^2=t$ for the squared mass of the photon. Thus we find upon considering an insertion into internal lines $(1,4)\equiv$ chain $\alpha$ that the "mass" of the chain $\alpha$ is modified as follows:

$$V_\alpha \to V'_\alpha = V_\alpha + x_1 t. \tag{7}$$

The functions $V(x,z)$ and $W$ are similarly modified:

$$V(x,z)\to V'(x,z)=V(x,z)+z_1 t, \tag{8}$$

$$W\to W' = m_\mu^{-2}UV'(x,z) = W+\frac{tUz_1}{m_\mu^2}. \tag{9}$$

We arrive at similar results for internal lines $(3,5)\equiv$ chain $\beta$ and $(2)\equiv$ chain $\gamma$ so that generally we can write

$$W\to W'_i = W+\frac{tUz_i}{m_\mu^2},\quad i=1,2,3. \tag{10}$$

In this way, we obtain from Eqs. (4), (5) and (10) the expression for the eighth-order contribution:

$$I^{(8)}(\rho,\rho') = \frac{a_\mu^{(8)}(\gamma\gamma)}{(\alpha/\pi)^4} = \sum_{i=1}^{3}\int_0^1 dy\,\frac{y^2(1-y^2/3)}{1-y^2} \times \int dz\, F(z,U,W'_i)\delta(1-z_t), \tag{11}$$

where we have made the change of variables $t=4m^2/(1-y^2)$ and defined $\rho'=(m/m_\mu)^2$. The integral on $y$ may be readily evaluated. Using Eq. (6) the result may be written as

$$I^{(8)}(\rho,\rho')=\tfrac{1}{3}\sum_{i=1}^{3}\int dz\sum_{nk}\frac{C_{nk}}{U^nW^k}J_k(\delta), \tag{12}$$

where

$$J_k(\delta)=\int_0^1 \frac{y^2(3-y^2)(1-y^2)^k}{(1-y^2+\delta)^k},$$

$$=\begin{cases} -\frac{5}{3}+\delta+r\left(1-\frac{\delta}{2}\right)l, & k=1\\ -\frac{8}{3}+\frac{5\delta}{2}+\left(1-\frac{\delta}{2}-\frac{5\delta}{4}\right)\frac{l}{r}, & k=2\\ -\frac{19}{6}+\left(35-\frac{3}{r^2}\right)\frac{\delta}{8}+(16+24\delta-30\delta^2-35\delta^3)\frac{l}{16r^3}, & k=3,\end{cases}$$

$$\delta=\frac{4\rho' Uz_i}{W},$$

$$r=(1+\delta)^{1/2},$$

$$l=\ln\left|\frac{1+r}{1-r}\right|.$$

## III. CALCULATION OF THE LN² AND LN COEFFICIENTS

Before obtaining an accurate value for Eq. (11) at $\rho=\rho'$, we calculate the leading terms that depend logarithmically on the mass ratios $\rho$ and $\rho'$. The reduction is simplified if we make the changes of variables

$$z_4=uv \text{ and } z_5=v(1-u). \tag{13}$$

In terms of these variables, we isolate the essential dependence on the variable $v$ in the integrand:

$$W=\Sigma v^2+\rho\Delta,$$

$$C_{nk}=G_{nk}v^{2k+n-5}\times\begin{cases}\rho, & n=1,2\\ 1, & n=3,4\end{cases} \tag{14}$$

where in addition to factoring out the overall $v$ dependence in the $C_{nk}$, we have also factored out the $\rho$ dependence.

Extraction of the logarithmic dependence on $\rho$ and $\rho'$ proceeds most simply by considering the limits

$$\lim_{\rho\to 0} I^{(8)}(\rho,\rho') \text{ and } \lim_{\rho'\to 0} I^{(8)}(\rho,\rho'),$$

keeping only terms that diverge as $\rho\to 0$ or $\rho'\to 0$. In this manner one finds that

$$I^{(8)}(\rho,\rho)=\tfrac{1}{3}\sum_{i=1}^{3}\int dz\sum_{nk}\frac{C_{nk}}{U^n W^k}\times\begin{cases}-\frac{5}{3}+E, & k=1\\ -\frac{8}{3}+E, & k=2\\ -\frac{19}{6}+E, & k=3\end{cases}$$

$$+(\text{Nondivergent terms})_{\rho'\to 0},\quad \rho'\lesssim\rho\ll 1\,, \tag{15}$$

where $E=-\ln(Uz_i)+\ln W-\ln\rho'$. A straightforward reduction of this expression, similar to that discussed in the Appendix for $I^{(6)}(\rho)$, leads to

$$I^{(8)}(\rho,\rho')=A^{(6)}(\tfrac{1}{2}\ln^2\rho-\ln\rho\ln\rho')+O^{(6)}(1)(\ln\rho-\ln\rho')+B\ln\rho+O^{(8)}(1)+\cdots,\quad \rho'\lesssim\rho\ll 1 \tag{16}$$

where $B=B_3+B_3'+B_4+B_5$,

$$B_3=\tfrac{1}{6}\int\frac{dz''}{U_0^3}\left(G_3^0\equiv\frac{G_{31}^0}{\Sigma_0}+\frac{G_{32}^0}{\Sigma_0^2}+\frac{G_{33}^0}{\Sigma_0^3}\right)\times\ln\left[z_1z_2z_3\left(\frac{U_0}{\Sigma_0K^2}\right)^3\right],$$

$$B_3'=-\int\frac{dvdz''}{v}\left(\frac{G_3}{U^3}-\frac{G_3^0}{U_0^3}\right),$$

$$B_4=-\int\frac{dvdz''}{U^4}\left(G_4\equiv\frac{G_{41}}{\Sigma}+\frac{G_{42}}{\Sigma^2}+\frac{G_{43}}{\Sigma^3}\right),$$

$$B_5=\tfrac{1}{6}\int\frac{dz''}{U_0^3}\left(G_5^0\equiv\frac{5G_{31}^0}{\Sigma_0}+\frac{8G_{32}^0}{\Sigma_0^2}+\frac{\frac{19}{3}G_{33}^0}{\Sigma_0^3}\right),$$

and

$$A^{(6)}\equiv-\int dz''\frac{G_3^0}{2U_0^3},$$

where (all expressions are evaluated at $z_3=1-z_1-z_2-v-z_6-z_7-z_8$)

$$dz''=dz_1dz_2dz_6dz_7dz_8du\,\theta(K)\,,$$

$$K=z_3(v=0)\,,$$

$$\Sigma_0=\Sigma(v=0)\,,\quad G_{31}^0=G_{31}(v=0)\,,\quad\text{etc.}$$

The factors $A^{(6)}$ and $O^{(6)}(1)$ can be shown (see Appendix) to be the coefficients of the $\ln\rho$ and the $\rho$-independent terms, respectively, in the expansion of the sixth-order photon-photon scattering contribution:

$$I^{(6)}(\rho)=\frac{a_\mu^{(6)}(\gamma\gamma)}{(\alpha/\pi)^3}=A^{(6)}\ln\rho+O^{(6)}(1)+\cdots. \tag{17}$$

Previous values for $A^{(6)}$ are (see Ref. 3) $-3.19\pm0.04$ and (see Ref. 4) $-3.145\pm0.028$. We have calculated an improved value for $A^{(6)}$. The difficulty of obtaining an accurate value is identical to that for evaluating Eq. (5). [We use a cutoff here as in Eq. (1); here $A^{(6)}(\epsilon)\sim A^{(6)}-(\pi^2/6)\sqrt{\epsilon}$.] Our result is

$$-A^{(6)}=3.29\pm0.01\,. \tag{18}$$

(This is tantalizingly close to $\pi^2/3$.) We have also evaluated $O^{(6)}(1)$ numerically from Eq. (A11) with the result

$$O^{(6)}(1)=-13.52\pm0.17\,. \tag{19}$$

Alternatively, using Eqs. (2), (17), and (18), we obtain the result $-13.76\pm0.12$, which includes terms which vanish in the $\rho\to0$ limit. This result is consistent with Eq. (19).

For the $\ln\rho$ terms we numerically evaluated each of the $B_i$ obtaining

$$\left.\begin{aligned}B_3&=2.21\pm0.09\\ B_3'&=1.30\pm0.08\\ B_4&=-1.87\pm0.04\\ B_5&=5.81\pm0.06\end{aligned}\right\}\to B=7.55\pm0.15\,. \tag{20}$$

Setting $\rho=\rho'$ we obtain the leading logarithmic terms to $I^{(8)}(\rho)$:

$$I^{(8)}(\rho)=(1.645\pm0.005)\ln^2\rho+(7.55\pm0.15)\ln\rho+O^{(8)}(1)\,, \tag{21}$$

which gives a contribution of

$$I^{(8)}=106.5\pm1.7+O^{(8)}(1) \tag{22}$$

for the physical mass ratio.

As a check on these results we numerically evaluated $I^{(8)}(\rho)$ for several vaules of $\rho$. In Fig. 2 we plot

$$I'(\rho)=I^{(8)}(\rho)/\ln(1/\rho)$$

versus $\ln(1/\rho)$ and find that the results are consistent with a curve that is asymptotic to a line of slope 1.645 and intercept $-7.55$.

## IV. THE KINOSHITA METHOD

In Eq. (21) we find that the coefficient of $\ln^2\rho$ is just $\frac{1}{2}$ the result obtained by naive application of renormalization-group methods to this class of diagrams[5]. We consider now how we can account for this.

The application of the Kinoshita method[6] to the diagrams of Fig. 1 yields the following equation for the partially renormalized moment:

$$I_{\mathrm{pr}}^{(8)}(m_e,m,m_\mu,\Lambda)=3Z_3^{(2)}(m,\Lambda)I^{(6)}(\rho)+I^{(8)}(\rho,\rho')\,, \tag{23}$$

where

$$Z_3^{(2)}(m,\Lambda)\equiv-\left(\tfrac{2}{3}\ln\frac{\Lambda}{m}-\tfrac{5}{9}\right)$$

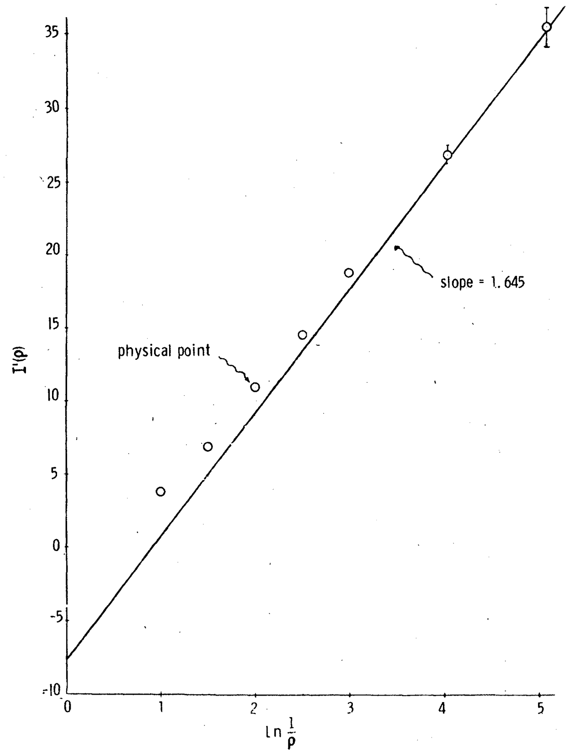


FIG. 2. $I'(\rho) \equiv I^{(8)}(\rho)/\ln(1/\rho)$ versus $\ln(1/\rho)$ [in units of $\ln(m_\mu/m_e)$]. This confirms the coefficients of $\ln^2\rho$ and $\ln\rho$ to be 1.645 and 7.55, respectively.

and $I^{(6)}(\rho)$ is given by the expansion in Eq. (17). From the theory of mass singularity[7] we know that

$$\lim_{m\to 0} I^{(8)}_{\mathrm{pr}}(m_e, m, m_\mu, \Lambda) \tag{24}$$

exists. Using this fact we solve for $I^{(8)}(\rho,\rho')$ in terms of an unknown function $f(\rho)$:

$$I^{(8)}(\rho,\rho') = -\ln\rho'[A^{(6)}\ln\rho + O^{(6)}(1)] + f(\rho), \quad \rho' \lesssim \rho \ll 1. \tag{25}$$

Owing to the absence of an auxiliary condition, such as a symmetry relation, we are unable to determine $f(\rho)$ without a direct calculation. From Eq. (16), $f(\rho)$ is found to be

$$f(\rho) = \tfrac{1}{2}A^{(6)}\ln^2\rho + [O^{(6)}(1) + B]\ln\rho + O^{(8)}(1). \tag{26}$$

We can see in another way how the $\frac{1}{2}A^{(6)}\ln^2\rho$ term arises by utilizing a formula due to Lautrup and de Rafael[8,9]:

$$a_\mu^{(n+2)} = \frac{\mathrm{Im}\,\pi^{(2)}(t=\infty)}{\pi}\int_{4m^2}^{\infty}\frac{dt}{t}a_\mu^{(n)}(t) + \int_{4m^2}^{\infty}\frac{dt}{t}\left[\frac{\mathrm{Im}\,\pi^{(2)}(t) - \mathrm{Im}\,\pi^{(2)}(t=\infty)}{\pi}\right]a_\mu^{(n)}(t) \tag{27}$$

to identify the contributing terms. For $a_\mu^{(8)}(\gamma\gamma)$ the second term of Eq. (27) contributes terms $O(\ln\rho)$ as $\rho' \to 0$ or $O(\ln\rho')$ as $\rho \to 0$:

$$a_\mu^{(8)}(\gamma\gamma) = \frac{\alpha}{3\pi}\sum_{i=1}^{3}\int_{4m^2}^{\infty}\frac{dt}{t}a_\mu^{(6)}(t) + O(\ln\rho \text{ or } \ln\rho'), \tag{28}$$

where $a_\mu^{(6)}(t)$ is given by Eqs. (5) and (9).

To understand in a simple way the origin of the $\ln^2\rho$ term, we consider the contribution to Eq. (28) from an interval $4m^2 \le t \le 4\lambda^2$, where $(m \text{ or } m_e) \ll \lambda \ll m_\mu$. We notice first that the introduction of $tUz_i/m_\mu^2$ into the denominators via $W \to W'$ effectively changes the mass of the photon-photon scattering electron loop

$$m_e^2 \to m_e^2 + \frac{tUz_i}{\Delta} \equiv m_{\mathrm{eff}}^2. \tag{29}$$

Let us now examine the terms in $a_\mu^{(6)}(t)$ which contribute to the $\ln\rho$ dependence in $a_\mu^{(6)}(0)$. Similar to Eq. (A3), the leading $(n=3)$ term of $a_\mu^{(6)}(t)$ is

$$\left(\frac{\alpha}{\pi}\right)^3 \int \frac{dz}{(W + tUz_i/m_\mu^2)}\Sigma G_3. \tag{30}$$

For small $\rho_{\mathrm{eff}} \equiv (m_{\mathrm{eff}}/m_\mu)^2$, the dominant contribution to Eq. (30) arises in the same way as for the corresponding terms of $a_\mu^{(6)}(0)$, that is, in a neighborhood of $v=0$. We obtain

$$a_\mu^{(6)}(t) = \left(\frac{\alpha}{\pi}\right)^3 \int dz''\,\frac{G_3^0}{2U_0^3}\ln\frac{\Sigma_0K^2 + \rho_{\mathrm{eff}}\Delta_0}{\rho_{\mathrm{eff}}\Delta_0} + \cdots. \tag{31}$$

For $t=0$ in Eq. (31) we recover the first term in Eq. (17). To further isolate the dominant behavior of $a_\mu^{(6)}(t)$, it is reasonable to replace $\Sigma_0K^2$, $\Delta_0$, and $U_0z_i$ by average values since they will not introduce much variation in the logarithm. With the definitions

$$a = \langle U_0z_i\rangle_{\mathrm{av}}, \quad b = \langle\Delta_0\rangle_{\mathrm{av}}, \quad \text{and } c = \langle\Sigma_0K^2\rangle_{\mathrm{av}}, \tag{32}$$

we obtain an expansion for Eq. (31) of the form

$$a_\mu^{(6)}(t) = \left(\frac{\alpha}{\pi}\right)^3\left[A^{(6)}\ln\frac{m_e^2 + (a/b)t}{m_\mu^2} + \cdots\right]. \tag{33}$$

From Eq. (33) it is clear how the introduction of virtual photons of squared mass $t$ into the photon lines of Fig. 1 leads to an effective modification of the electron mass $m_e$ in Eq. (17). Upon performing the integration over $t$, $4m^2 \le t \le 4\lambda^2$, in Eq. (28) using Eq. (33), we readily obtain

$$I^{(8)}(\rho,\rho') \sim A^{(6)}(\tfrac{1}{2}\ln^2\rho - \ln\rho\ln\rho'), \quad \rho' \lesssim \rho \ll 1. \tag{34}$$

The effective increase in the mass $m_e$ of the photon-photon scattering electron loop $(m_{\mathrm{eff}} \ge m_e)$ could correspond to a reduced current, and the fact that the contribution $\frac{1}{2}A^{(6)}\ln^2\rho$ in Eq. (34) is negative suggests an analogy to Lenz's law effect.[10]

For fixed $\rho' > 0$, $I^{(8)}(\rho, \rho')$ is convergent at $\rho = 0$ and the leading contribution to $I^{(8)}(\rho, \rho')$ is[11]

$$I^{(8)}(\rho, \rho') \sim -\frac{A^{(6)}}{2}\ln^2\rho' + (B_3 + B_3' + B_4 - \tfrac{5}{3}A^{(6)})\ln\rho', \quad \rho \ll \rho' \ll 1.$$

## V. NUMERICAL EVALUATION

We proceed now to the accurate numerical evaluation of $I^{(8)}(\rho)$. The difficulty of this computation is similar to that encountered in evaluating Eq. (5), since we again have factors $C_{nk}/U^nW^k$, but now modified by the $J_k$. We expect, therefore, that the method of evaluation described for the sixth-order photon-photon scattering will also improve the convergence here. We again introduce a cutoff on $T = z_6 + z_7 + z_8$, and in addition to the changes of variables defined in Eq. (13) of Ref. 1, we also let $T = T'^2$. This has the effect of changing the $\sqrt{\epsilon}$ dependence noted earlier in Eq. (1) to an $\epsilon$ dependence. Defining $D(\epsilon, \epsilon_1)$ to be the contribution to $I^{(8)}(\rho)$ from the interval $\epsilon \leq T' \leq \epsilon_1$, we find that for $\epsilon$ small enough

$$D(\epsilon, 1) \sim D(0, 1) - M\epsilon, \tag{35}$$

where $M \sim 55$. Hence as a method of obtaining $D(0, 1)$, we evaluate $D(\epsilon, 1)$ accurately for small enough values of $\epsilon$ and extrapolate to $\epsilon = 0$. The results of the $\epsilon$ cutoff shown in Fig. 3 confirm a linear dependence for small $\epsilon$. Extrapolating to $\epsilon = 0$ we obtain

$$D(0, (0.1)^{1/2}) = 28.7 \pm 0.2. \tag{36}$$

Combining this with

$$D((0.1)^{1/2}, 1) = 88.7 \pm 0.4, \tag{37}$$

we have

$$I^{(8)}(\rho) = D(0, 1) = 117.4 \pm 0.5. \tag{38}$$

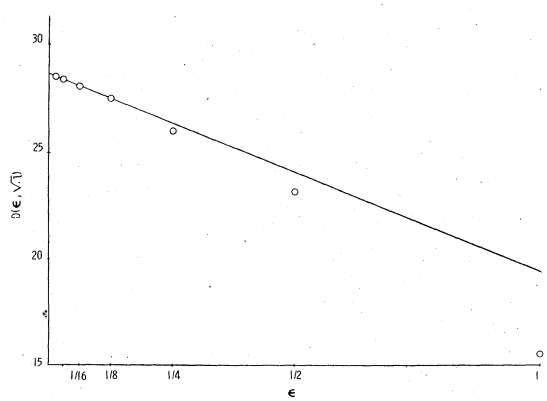


FIG. 3. $D(\epsilon, (0.1)^{1/2})$ versus $\epsilon$ [in units of $(0.1)^{1/2}$].

Finally, we note that this result is consistent with the expectation that the logarithmic terms dominate the contribution. Taking into account the contribution of 106.5 from Eq. (22), the order-one term is estimated to be

$$O^{(8)}(1) \sim 10.9 \pm 1.8. \tag{39}$$

A recent review of all the contributions to the theoretical muon anomaly is given by Calmet *et al.*[12]

*Note added in proof.* The closeness of the result in Eq. (18) to $\pi^2/3$ was so tantalizing that it has now been evaluated analytically by B. E. Lautrup and M. A. Samuel [Niels Bohr Institute Report No. NBI-HE-77-32 (unpublished)]. The result is indeed $\pi^2/3$.

## ACKNOWLEDGMENT

It is a pleasure to thank Stan Brodsky for stimulating and helpful discussions.

## APPENDIX

We now examine in detail the determination of the $O^{(6)}(1)$ term and the identification of $A^{(6)}$ as the coefficient of $\ln\rho$ in Eq. (17). We begin with the expression in Eq. (5) for $I^{(6)}(\rho)$. Making use of Eqs. (6) and (13), we can write $I^{(6)}(\rho)$ as

$$I^{(6)}(\rho) = I_A^{(6)}(\rho) + I_B^{(6)}(\rho), \tag{A1}$$

where

$$I_B^{(6)}(\rho) = \int \frac{v\,dv\,dz'}{U^3}\left(\frac{C_{31}}{W} + \frac{C_{32}}{W^2} + \frac{C_{33}}{W^3}\right),$$

$$I_A^{(6)}(\rho) = \sum_{\substack{nk \\ n\neq 3}} \int v\,dv\,dz'\,\frac{C_{nk}}{U^nW^k},$$

$$dz' = \delta(1 - z_t)dz_1dz_2dz_3dz_6dz_7dz_8du.$$

It is easily seen that $\lim_{\rho\to 0} I_A^{(6)}(\rho)$ exists. Evaluating the integral over $z_3$ using the $\delta$ function, and letting $v = \sqrt{\rho}\,x$ for terms $n = 1, 2$, we find that

$$O_A^{(6)}(1) \equiv \lim_{\rho\to 0} I^{(6)}(\rho) = \int \frac{dz''}{2U_0\Sigma_0\Delta_0}\left(G_{12}^0 + \frac{G_{13}^0}{2\Sigma_0}\right) + \int dz''dv\frac{G_4}{U^4}, \tag{A2}$$

where $dz''$, $\Sigma_0$, $G_4$, etc. are defined in Eq. (16).

To extract the $O(1)$ part of $I_B^{(6)}$ we first expand

$$I_B^{(6)} = \int \frac{dz''v\,dv}{U^3W}\Sigma G_3 + \int \frac{dz''v\,dv}{U^3W}\left[-\frac{G_{32}}{\Sigma}\frac{\rho\Delta}{W} + \frac{G_{33}}{\Sigma^2}\left(\frac{-2\rho\Delta}{W} + \frac{\rho\Delta}{W}\right)\right]. \tag{A3}$$

The $\lim_{\rho\to 0}$ of the second term in Eq. (A3) exists and is

$$O_{B2}^{(6)} = -\int \frac{dz''}{2U_0^3}\left(\frac{G_{32}^0}{\Sigma_0^2} + \frac{3}{2}\,\frac{G_{33}^0}{\Sigma_0^3}\right). \qquad (A4)$$

We now consider the extraction of the underlying $O(1)$ term, which we call $O_{B1}^{(6)}(1)$, in the first term of Eq. (A3). As is known, $I_B^{(6)}$ is logarithmically divergent. The coefficient of $\ln\rho$ is

$$A^{(6)} = \lim_{\rho\to 0}\rho\frac{d}{d\rho}\int dz''\int_0^K \frac{vdv}{U^3W}\Sigma G_3 = -\int dz''\frac{G_3{}^0}{2U_0^3}. \qquad (A5)$$

To obtain $O_{B1}^{(6)}(1)$ consider

$$\int dz''\frac{\Sigma_0 G_3^0}{U_0^3}\int_0^K\frac{vdv}{W_0} = \int dz''\frac{G_3^0}{2U_0^3}\left(\ln\frac{\Sigma_0K^2+\rho\Delta_0}{\Delta_0} - \ln\rho\right), \qquad (A6)$$

where $W_0 = \Sigma_0 U^2 + \rho\Delta_0$.

Now consider

$$D = \lim_{\rho\to 0}\int dz''\int_0^K vdv\left(\frac{\Sigma G_3}{U^3W} - \frac{\Sigma_0G_3^0}{U_0^3W_0}\right); \qquad (A7)$$

$D$ exists and is given by

$$D = \int dz''\int_0^K\frac{dv}{v}\left(\frac{G_3}{U^3} - \frac{G_3^0}{U_0^3}\right). \qquad (A8)$$

On the other hand, using Eqs. (A5) and (A6), we can expand Eq. (A7) to obtain

$$D = O_{B1}^{(6)}(1) - \int dz''\frac{G_3^0}{2U_0^3}\ln\frac{\Sigma_0K^2}{\Delta_0}, \qquad (A9)$$

where

$$O_{B1}^{(6)}(1) \equiv \lim_{\rho\to 0}\int dz''\left(\int_0^K\frac{vdv\Sigma G_3}{U^3W} + \frac{G_3^0}{2U_0^3}\ln\rho\right). \qquad (A10)$$

Combining Eqs. (A2), (A4), and (A10), we obtain the $O(1)$ term of $I^{(6)}(\rho)$:

$$O^{(6)}(1) = \int dz''\left(\frac{G_{12}^0 + G_{13}^0/2\Sigma_0}{2U_0\Sigma_0\Delta_0} - \frac{G_{32}^0/\Sigma_0{}^2 + \frac{3}{2}G_{33}^0/\Sigma_0{}^3}{2U_0^3} + \frac{G_3^0}{2U_0^3}\ln\frac{\Sigma_0K^2}{\Delta_0}\right) - B_4 - B_3'. \qquad (A11)$$

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