Exact three-dimensional reduction of the Bethe-Salpeter equation

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The partial-wave Bethe-Salpeter equation in the ladder approximation is converted into a set of two one-dimensional equations. The resulting reduced equations determine two off-energy-shell, on-mass-shell amplitudes, one for the scattering of two positive-energy and the other for two negative-energy particles. The connection between the potentials of the reduced equations and the many-particle intermediate states is displayed by their series expansion. Two versions of the equations which are convenient for numerical computations are also obtained, and one of them is a one-channel equation which involves only the physical amplitude for the scattering of two positive-energy particles. The potentials are determined by auxiliary equations which are two dimensional but nonsingular and exactly solvable by standard numerical methods. One of the results of the derivation of the reduced equations is a set of reduction relations which express the off-mass-shell amplitude as a functional of a restricted shell amplitude.

I. INTRODUCTION

The Bethe-Salpeter (BS) equation[1-4] in momentum space[5] possesses two main advantages over dispersion-theoretic methods, in quantitative calculations with field-theory models for strongly interacting particles. These are its off-shell nature and the inelastic effects which are built into it.[6] Unfortunately these two properties also constitute the main difficulties of the equation. Since it is an off-shell equation it depends explicitly on four variables (or, when partial-wave equations are used, on two variables) and, to incorporate the effect of many-particle states on the scattering amplitude, the off-shell quantities involved in the solution of the equation have complicated singular dependence on the relative energy variables.

The complication of the equation initiated many works on approximate solutions, the main line of attack being the use of three-dimensional approximations[7-16] (for other approaches see Refs. 17–21). In these approximations the dependence on the fourth variable is approximated in some way so that the BS equation can be replaced by a relativistic analog of the Lippmann-Schwinger (LS) equation. Most of these reduction schemes are complemented by an auxiliary equation for the potential to be inserted in the three-dimensional equation.[11,17,22,23] Solution of the two simultaneous equations would render the reduction procedure exact, however, owing to the fact that the kernel of the auxiliary equation is still singular and four dimensional, it presents the same order of complexity as the original BS equation.[22]

Judging by published works, it seems that, although some progress towards exact numerical solutions in realistic models has recently been achieved,[25-32] one of the main trends in strong-interaction dynamical models is to use some version of the three-dimensional approximations.[33-41]

Some checks on the reliability of these approximations have been carried out,[22,24,36] but, because of the complexity of the neglected terms in the potential, the study is limited to numerical checks so that any conclusion is dependent on the model used and the specific parameters of that model.

In this paper we describe an exact reduction procedure leading to equations which share most of the advantages in common with the approximation schemes described above and yet can be considered to be exactly equivalent to the BS equation. As in every three-dimensional reduction, this reduction procedure replaces the BS equation by a three-dimensional nonsingular equation with a "quasipotential" which can be derived from an auxiliary integral equation. But, in contrast with these schemes, the last equation is also nonsingular and exactly solvable. The exact potential in the ladder approximation is presented as a series whose terms represent the contributions of the many-particle singularities, in the relative energy variable, of the scattering matrix. The equations obtained are very convenient for numerical computation, and the ability to obtain exact numerical solutions is briefly demonstrated.

Our reduction procedure utilizes the energy analytic representation[42,43] (EAR) of the scattering matrix and the generalized Wick rotation, and thus complements some works on this subject which has presented so many difficulties from the numerical point of view.[29,44,45] A detailed discussion on these methods was given by Broido and Taylor.[43]

In its present form, this work is restricted to the ϕ^3 model in the ladder approximation, but we think that it contributes on the one hand to a better understanding of the three-dimensional approximations and on the other hand to the efforts of finding new ways for computational applications of the Broido-Taylor approach. We chose this simplified model for the sake of clear representation of the results, but we intend to generalize the results to more realistic models of nucleon-nucleon and pion-nucleon interactions. Indeed, the ϕ^3 model has often been used as a testing model for more complicated interactions because the analytic properties in the relative energy variable are similar.

Sections II and III introduce the BS equation and the analytic properties of the kernel in the ladder approximation. Section IV serves as a review and introduction to the three-dimensional approach, and the main results of this paper are derived in Secs. V and VI. Section V introduces the basic procedure for deriving reduction relations which are based on a new form of the BS equation, and the three-dimensional equations are derived in Sec. VI.

II. THE BETHE-SALPETER EQUATION

In this paper we shall be concerned with the scattering amplitude for two nonidentical spinless particles of equal mass m (which we shall call nucleons) interacting via the field of another spinless particle ("meson") of mass μ.

In momentum space, the off-shell scattering amplitude depends on the relative momenta of the nucleons in the initial state $q_\nu = \frac{1}{2}(q_{1\nu} - q_{2\nu})$, the relative momenta of the nucleons in the final state $k_\nu = \frac{1}{2}(k_{1\nu} - k_{2\nu})$, and on the total momentum of each of these states $p_\nu = k_{1\nu} + k_{2\nu} = q_{1\nu} + q_{2\nu}$. The on-shell values of these momenta in the c.m. frame are $k_\nu = (0, \vec{k})$, $q_\nu = (0, \vec{q})$, and $p_\nu = (2E, \vec{0})$, where $k = |\vec{k}| = |\vec{q}| = q$, $E = (q^2 + m^2)^{1/2}$, and $2E = s^{1/2}$ is the total c.m. energy of the system.

To obtain the scattering amplitude by the BS equation one needs only the "half" off-shell amplitude, with the "leg" representing the initial state being on-shell, which has the following partial-wave expansion:

$$T(\vec{k}, k_0) = \sum_{l=0}^{\infty} \frac{2l+1}{2kq} T_l(k, k_0) P_l(\lambda) ,$$

where

$$\lambda = \frac{\vec{k} \cdot \vec{q}}{kq}$$

(here and throughout this paper the dependence of T and T_l on E is omitted for the sake of convenience).

The partial-wave BS equation in the ladder approximation is

$$T_l(k, k_0) = V_l(k, k_0, q, 0) + \int_0^\infty dk' \int_{-\infty}^{\infty} dk_0' V_l(k, k_0, k', k_0') \times G(k', k_0', E) T_l(k', k_0') , \tag{2.1}$$

where

$$V_l(k, k_0, k', k_0') = g^2 Q_l(z),$$
$$z = [k^2 + k'^2 + \mu^2 - (k_0 - k_0')^2]/2kk' .$$

The renormalized coupling constant of the interacting fields is g, and Q_l is the Legendre function of the second kind of order l. $G(k', k_0', E)$ is the two-nucleon propagator:

$$G(k', k_0', E) = \frac{-i}{(2\pi)^3} \{[k'^2 + m^2 - (E + k_0')^2] \times [k'^2 + m^2 - (E - k_0')^2]\}^{-1} .$$

The on-shell scattering amplitude is $t_l(s) = T_l(q, 0)$, and in the elastic scattering energy region $t_l(s) = \rho e^{i\delta_l} \sin\delta_l$, where δ_l is real and $\rho = 32\pi Eq$.

III. ANALYTIC PROPERTIES OF THE KERNEL

The two-nucleon propagator can be written as a sum of pole terms:

$$G(k, k_0, E) = \frac{-i}{2\pi} \frac{E}{E(k)} \left[g_1(k) \left(\frac{1}{k_0 + D_1(k)} - \frac{1}{k_0 - D_1(k)} \right) - g_2(k) \left(\frac{1}{k_0 + D_2(k)} - \frac{1}{k_0 - D_2(k)} \right) + g_3(k) \left(\frac{1}{k_0 + D_1(k)} - \frac{1}{k_0 + D_2(k)} \right) + g_4(k) \left(\frac{1}{k_0 - D_2(k)} - \frac{1}{k_0 - D_1(k)} \right) \right] , \tag{3.1}$$

where $E(k) = (k^2 + m^2)^{1/2} - i\epsilon$, $D_1(k) = E(k) - E$, $D_2(k) = E(k) + E$, $g_1(k) = N/2D_1(k)$, $g_2(k) = -N/2D_2(k)$, $g_3(k) = g_4(k) = N/2E$, and $N = [16\pi^2 E E(k)]^{-1}$.

The reason for presenting $G(k, k_0, E)$ in this form will become clear later, but it is obvious that its terms can be rearranged to obtain a more practical decomposition:

$$G(k,k_0,E)=\frac{-i}{2\pi}\left[g_1(k)\left(\frac{1}{k_0+D_1(k)}-\frac{1}{k_0-D_1(k)}\right)+g_2(k)\left(\frac{1}{k_0+D_2(k)}-\frac{1}{k_0-D_2(k)}\right)\right]. \tag{3.2}$$

According to the $i\epsilon$ prescription two of the four poles of $G(k,k_0,E)$ are in the upper half of the complex k_0 plane at $k_0=-E(k)\pm E$ and two are in the lower half at $k_0=E(k)\pm E$. The poles at $k_0=\pm D_1(k)$ are called positive-energy poles and those at $k_0=\pm D_2(k)$ are called negative-energy poles.

For further use we define here the following functions:

$$G_1(k,k_0,E)=g_1(k)\left(\frac{1}{k_0+D_1(k)}-\frac{1}{k_0-D_1(k)}\right),\qquad G_2(k,k_0,E)=g_2(k)\left(\frac{1}{k_0+D_2(k)}-\frac{1}{k_0-D_2(k)}\right),$$

and

$$G_3(k,k_0,E)=G_1(k,k_0,E)+G_2(k,k_0,E).$$

The potential of the partial-wave BS equation in the ladder approximation is proportional to the Legendre transform of the meson propagator. It can be decomposed according to its analyticity as follows (see Appendix A):

$$V_l(k,k_0,k',k_0')=V_l^+(k,k',k_0'-k_0)+V_l^-(k,k',k_0'-k_0), \tag{3.3}$$

where

$$V_l^+(k,k',k_0)=\frac{g^2}{2}kk'\int_{-1}^{1}d\lambda\,P_l(\lambda)\{[\omega(\vec{k}-\vec{k}')][\omega(\vec{k}-\vec{k}')+k_0-i\epsilon]\}^{-1},$$

$$\omega(\vec{k})=(k^2+\mu^2)^{1/2}\quad\text{and}\qquad\lambda=\frac{\vec{k}\cdot\vec{k}'}{kk'},$$

and

$$V_l^-(k,k',k_0)=V_l^+(k,k',-k_0).$$

$V_l^+(k,k',k_0'-k_0)$ is analytic in the lower half of the complex k_0' plane (with real k_0) and contains the left-hand singularities of V_l at $k_0'=-\omega_+(k,k')+k_0$ and V_l^- is analytic in the upper half of the k_0' plane and contains the singularities at $k_0'=\omega_+(k,k')-k_0$, where $\omega_\pm(k,k')=[(k\pm k')^2+\mu^2]^{1/2}-i\epsilon$.

IV. THREE-DIMENSIONAL REDUCTIONS

It has been shown by Woloshyn and Jackson[22] that a wide class of three-dimensional approximations to the BS equation can be formally obtained by the following common procedure: The k_0' dependence of $T_l(k',k_0')$ and of $V_l(k,k_0,k',k_0')$ in the homogeneous term of Eq. (2.1) is approximated by assigning a fixed value to k_0' as argument of V_l and T_l. The k_0' integral, with the remaining part of the kernel, is then evaluated by contour integration closing the contour in the upper half of the complex k_0' plane. In this way the only singularities which contribute to the reduced kernel are the poles of $G(k',k_0',E)$, and the equation obtained is a relativistic analog of the LS equation:

$$T_l(k)=V_l(k,q)+\frac{1}{16\pi^2}\int_0^\infty\frac{dk'}{E(k')}\,V_l(k,k')\times g(k')T_l(k'). \tag{4.1}$$

The form of the reduced propagator $g(k')$ depends on the way in which each approximation treats the nucleon poles. Gross's[13] and Thompson's[11] approximations can be obtained by neglecting the negative-energy poles; thus in these approximations

$$g_A(k')=\frac{1}{2E}\frac{1}{E(k')-E}. \tag{4.2}$$

The Blankenbecler-Sugar[10] (BBS) and Erkelenz-Holinde[22] approximations take into account both positive- and negative-energy poles and use the propagator

$$g_B(k')=\frac{1}{E(k')^2-E^2}. \tag{4.3}$$

The propagator in the Kadyshevsky[12] equation can be obtained by considering decomposition (3.1), neglecting there all but the first term which contains two positive-energy poles:

$$g_C(k')=\frac{1}{2E(k')}\frac{1}{E(k')-E}. \tag{4.4}$$

The imaginary part of all these approximations is $(\pi/2q)\delta(k'-q)$ so that they produce the correct relativistic unitarity cut in the scattering amplitude.

The potential in Eq. (4.1) is determined by the meson propagator and its general form is

$$V_l(k,k') = g^2 Q_l\left(\frac{k^2+k'^2+\mu^2-t^2}{2kk'}\right). \tag{4.5}$$

The precise form depends on the choice for k_0' in the reduction procedure. In the Thompson, Kadyshevsky, and the BBS equations $t=0$, while in the Gross and Erkelenz-Holinde equations $t = E(k') - E(k)$.

Our treatment of the BS equation may be considered as an extension of these reduction procedures. We shall perform the k_0' contour integration without approximations, taking into account the full dependence of the kernel and of T_l on the relative energy variable. This will be done in the following two sections where we present an exact derivation of the reduced equations and of the potentials. At this stage, however, we hope to obtain more insight by commenting on some aspects of relativistic scattering which are present in the BS equation and the form in which they appear in the exact three-dimensional reduction.

The form (4.4) for the propagator is obtained from the residues of the positive-energy poles. Since it is inversely proportional to the difference between the relativistic energy of two free nucleons with linear momentum $\vec{k}$ and $-\vec{k}$ and the energy of the initial state in the c.m. frame, it seems to be the correct analog of the propagator in the LS equation as long as we are considering only positive-energy nucleons. The first step in including all relativistic effects in the propagation of the two nucleons in momentum space is to consider also intermediate states with one or two nucleons in negative-energy states. In principle, this could have been accomplished by adding three more amplitudes[28] and the corresponding propagators which should have been inversely proportional to $2E$ and $-2[E(k')+E]$. This possibility is indeed implied by the decomposition (3.1) of the two-nucleon Green's function, but as a result of decomposition (3.2) of the same function, the final equations contain only two amplitudes (the situation is different in the case of nucleons with spin). Later on, in Sec. VI, we shall show how to obtain the following equations:

$$\begin{aligned} T_1(k) &= V_{11}(k,q) + \int_0^\infty dk' [V_{11}(k,k') g_1(k') T_1(k') + V_{12}(k,k') g_2(k') T_2(k')], \\ T_2(k) &= V_{21}(k,q) + \int_0^\infty dk' [V_{21}(k,k') g_1(k) T_1(k') + V_{22}(k,k') g_2(k') T_2(k')], \end{aligned} \tag{4.6}$$

where the subscript of the partial wave has been omitted.

This set of coupled two-channel equations does not present any new difficulties from either a theoretical or numerical point of view, as all the results of the one-channel equation can be generalized in a straightforward manner.

The positive-energy reduced propagator $g_1(k')$ has the correct imaginary part to produce the unitary cut of $T_1(k)$. As a result, the usual unitarity relation $T_1(q) = t_l(s) = \rho e^{i\delta_l} \sin\delta_l$ holds true, in the elastic energy region, with $\rho = 32\pi Eq$. For the numerical evaluation of the phase shifts, the set of equations (4.6) can be replaced by a similar set of equations where $g_1(k')$ is replaced by its principal value. The quantities obeying these equations are the off-shell continuations of the physical K matrix and $K_1(q) = \rho \tan\delta_l$. An alternative for avoiding the singularity of $g_1(k')$ is offered by a generalization of the kernel subtraction method of Levine *et al.*,[46] designed for many-channel equations by Gammel *et al.*[28]

So far nothing has been said about the effect of the mesons in the intermediate states. As the equations are essentially two-particle equations, it would be a convenient feature of the equations if this effect would appear only through the potentials. This indeed must be the case if Eqs. (4.6) are to be equivalent to the BS equation. We have been able to obtain explicit expressions for the potentials, and one of the results of using the two-channel equations is to effect a clear separation between the contribution of the many-meson states and the contribution of the negative-energy states.

Because of some complications which arise in the general case (see Sec. VI), the best way to demonstrate the many-meson content of the potential is to consider a one-channel approximation to Eqs. (4.6). This approximation can be obtained by obtained by neglecting the negative-energy poles in decomposition (3.2) of $G(k', k_0', E)$, and the resulting equation is

$$T_l(k) = V_l(k,q) + \int_0^\infty dk' V_l(k,k') g_1(k') T_l(k'). \tag{4.7}$$

The potential is given by a series, the nth term of which is the n-meson contribution:

$$V_l(k,k') = \sum_{n=1}^{\infty} V_l^{(n)}(k,k').$$

As we shall see in Secs. V and VI the contribution of the first term is

$$V_l^{(1)}(k,k') = V_l^{+}(k,k',E(k') - E(k)) + V_l^{+}(k,k',E(k') + E(k) - 2E).$$

On the energy shell this potential coincides with the potential (4.5). The difference in the off-shell continuation of this potential is due to the distinction between the roles of the positive-energy and the negative-energy mesons in the intermediate states (exchange mesons).

The general n-meson contribution to the potential is

$$V_l^{(n)}(k,k') = (g^2)^n \int_0^\infty dk_1 k' k_1 \int_{-1}^{1} d\lambda_1 \frac{p_l(\lambda_1)}{2\omega(\vec{k}' - \vec{k}_1)} G_1^{(1)}(k', k_1, \lambda_1) \int_0^\infty dk_2 k_1 k_2 \int_{-1}^{1} d\lambda_2 \frac{p_l(\lambda_2)}{2\omega(\vec{k}_1 - \vec{k}_2)} G_1^{(2)}(k', k_1, k_2, \lambda_1, \lambda_2) \cdots$$

$$\times \int_0^\infty dk_{n-1} k_{n-2} k_{n-1} \int_{-1}^{1} d\lambda_{n-1} \frac{p_l(\lambda_{n-1})}{2\omega(\vec{k}_{n-2} - \vec{k}_{n-1})} G_1^{(n-1)}(k', k_1 \cdots k_{n-1}, \lambda_1 \cdots \lambda_{n-1})$$

$$\times k_{n-1} k \int_{-1}^{1} d\lambda \frac{p_l(\lambda)}{2\omega(\vec{k}_{n-1} - \vec{k})} \{[W(k', k_1 \cdots k_{n-1}, k, \lambda_1 \cdots \lambda_{n-1}) + E(k') + E(k) - 2E]^{-1} + [W(k', k_1 \cdots k_{n-1}, k, \lambda_1 \cdots \lambda_{n-1}) + E(k') - E(k)]^{-1}\},$$

where

$$\lambda_i = \frac{\vec{k}_{i-1} \cdot \vec{k}_i}{k_{i-1} k_i} \text{ for } 1 < i \leq n-1, \quad \lambda_1 = \frac{\vec{k}' \cdot \vec{k}_1}{k' k_1}, \quad \lambda = \frac{\vec{k} \cdot \vec{k}_{n-1}}{k k_{n-1}},$$

$$G_1^{(m)}(k', k_1 \cdots k_m, \lambda_1 \cdots \lambda_m) = g_1(k_m)\{[W(k', k_1 \cdots k_m, \lambda_1 \cdots \lambda_m) + E(k') + E(k_m) - 2E]^{-1} - [W(k', k_1 \cdots k_m, \lambda_1 \cdots \lambda_m) + E(k') - E(k_m)]^{-1}\},$$

and

$$W(k', k_1 \cdots k_m, \lambda_1 \cdots \lambda_m) = \omega(\vec{k}' - \vec{k}_1) + \omega(\vec{k}_1 - \vec{k}_2) + \cdots + \omega(\vec{k}_{m-1} - \vec{k}_m).$$

V. ANALYTIC PROPERTIES OF THE SCATTERING AMPLITUDE AND THE REDUCTION RELATIONS

The analytic dependence of the scattering amplitude on the relative energy variable was first investigated by Wick[47] in the bound-state energy region and by Kemmer and Salam[48] in the scattering region. These authors derived the analytic structure from principles of field theory. Allcock and Leigh[49] discussed the assumption that the analyticity structure implied by field theory is valid also in truncated theories. Saenger[50] proved this assumption for the ladder-approximated partial-wave scattering matrix in his discussion of the Wick rotation in the scattering energy region, and Pagnamenta and Taylor[42] gave in detail the singularities of this amplitude.

The energy analytic representation[42,43] of the partial-wave scattering amplitude $T_l(k,k_0)$ exhibits its analytic properties in the k_0 complex plane:

$$T_l(k,k_0) = \int_{\alpha(k)}^\infty dt \frac{\rho_l^{+}(k,t)}{t + k_0 - i\epsilon} + \int_{\alpha(k)}^\infty dt \frac{\rho_l^{-}(k,t)}{t - k_0 - i\epsilon}. \tag{5.1}$$

The lowest branch point on the right-hand-side cut is at $k_0 = \alpha(k) - i\epsilon = -E + [k^2 + (m+\mu)^2]^{1/2} - i\epsilon$ and the highest one on the left-hand-side cut is at $k_0 = -\alpha(k) + i\epsilon$. This representation also allows the separation of T_l into two parts:

$$T_l(k,k_0) = T_l^{+}(k,k_0) + T_l^{-}(k,k_0),$$

where

$$T_l^{+}(k,k_0) = \int_{\alpha(k)}^\infty dt \frac{\rho_l^{+}(k,t)}{t + k_0 - i\epsilon}, \qquad T_l^{-}(k,k_0) = \int_{\alpha(k)}^\infty dt \frac{\rho_l^{-}(k,t)}{t - k_0 - i\epsilon}. \tag{5.2}$$

So that T_l^{+} is analytic in the lower half of the k_0 plane and T_l^{-} is analytic in the upper half of the k_0 plane.

We return now to Eq. (2.1). By decomposing the potential of this equation into V_l^{+} and V_l^{-} as in (3.3), the kernel is split into two parts. The k_0' integration is performed by contour integration, continuing the contour along the real k_0' line into the upper half plane with the part containing V_l^{-} and into the lower half plane with the part containing V_l^{+}.

The contours in the respective half planes are completed by describing a semicircle with center at the origin and returning to the real k_0' line avoiding the respective cuts of T_l (generalized Wick rotation). By taking the limit of infinitely large radius

(no contribution, in this limit, of the integral along the arcs of the semicircles), the k_0' integral along the real line can be evaluated as a sum of two contributions:

A. the contribution of the residues of the nucleon poles,

B. the contribution of the discontinuity along the cuts of T_l.

The resulting equation is

$$T_l(k,k_0)=V_l^+(k,q,k_0)+V_l^+(k,q,-k_0)$$
$$+\int_0^\infty dk'\{g_1(k')[V_l^+(k,k',D_1(k')-k_0)T_l(k',D_1(k'))+V_l^+(k,k',D_1(k')+k_0)T_l(k',-D_1(k'))]$$
$$+g_2(k')[V_l^+(k,k',D_2(k')-k_0)T_l(k',D_2(k'))+V_l^+(k,k',D_2(k')+k_0)T_l(k',-D_2(k'))]\}$$
$$+\int_0^\infty dk'\int_{\alpha(k')}^\infty dk_0'G_3(k',k_0',E)[V_l^+(k,k',k_0'-k_0)\rho_l^-(k,k_0')+V_l^+(k,k',k_0'+k_0)\rho_l^+(k,k_0')]. \quad (5.3)$$

By some rearrangement of its terms the last equation can be cast into a more convenient form. It is a known fact that Eq. (2.1) admits only symmetric solutions in k_0 in the scattering region, i.e., $T_l(k,k_0)=T_l(k,-k_0)$, which in turn implies

$$\rho_l^-(k,t)=\rho_l^+(k,t) \quad \text{and} \quad T_l^-(k,k_0)=T_l^+(k,-k_0).$$

A second point worth mentioning is that in the scattering region $T_l(k,D_2(k))$ and $T_l(k,-D_2(k))$ which appear in the equation have nonzero imaginary parts as $D_2(k)$ and $-D_2(k)$ lie along the right-hand cut and the left-hand cut, respectively, for any k, but this imaginary part is exactly canceled by a similar contribution in the second part of the equation. To see this explicitly note that on the one hand

$$T_l(k,D_2(k))=\mathrm{Re}T_l(k,D_2(k))+i\pi\rho_l^-(k,D_2(k)),$$
$$T_l(k,-D_2(k))=\mathrm{Re}T_l(k,-D_2(k))+i\pi\rho_l^+(k,D_2(k)),$$

whereas on the other hand

$$\frac{1}{k_0-D_2(k)-i\epsilon}=\mathrm{P}\frac{1}{k_0-D_2(k)}+i\pi\delta(k_0-D_2(k)),$$

where P denotes the principal-value part.

Taking into account the above considerations, the equation for $T_l(k,k_0)$ can now be rewritten as follows:

$$T_l(k,k_0)=V_l^+(k,q,k_0)+V_l^+(k,q,-k_0)+\int_0^\infty dk'\{g_1(k')[V_l^+(k,k',D_1(k')+k_0)+V_l^+(k,k',D_1(k')-k_0)]M_1(k')$$
$$+g_2(k')[V_l^+(k,k',D_2(k')+k_0)+V_l^+(k,k',D_2(k')-k_0)]M_2(k')\}$$
$$+\int_0^\infty dk'\int_{\alpha(k')}^\infty dk_0'G_4(k',k_0',E)[V_l^+(k,k',k_0'+k_0)+V_l^+(k,k',k_0'-k_0)]\rho_l(k',k_0'), \quad (5.4)$$

where

$$M_1(k)=T_l(k,D_1(k))=T_l(k,-D_1(k)),$$
$$M_2(k)=\mathrm{Re}T_l(k,D_2(k))=\mathrm{Re}T_l(k,-D_2(k)),$$

and

$$G_4(k,k_0,E)=G_1(k,k_0,E)+g_2(k)\left[\frac{1}{k_0+D_2(k)}-\mathrm{P}\frac{1}{k_0-D_2(k)}\right].$$

Inspection of the singularities of the last equation (see Appendix B) (see Ref. 51) confirms the singularity structure on which the EAR [Eq. (5.1)] is based and also admits the following equation for the spectral function:

$$\rho_l(k,k_0)=\sigma_l(k,q,k_0)R(k,q,k_0,0)+\int_0^\infty dk'\sum_{i=1}^2 g_i(k')\sigma_l(k,k',k_0-D_i(k'))R(k,k',k_0,D_i(k'))M_i(k')$$
$$+\int_0^\infty dk'\int_{\alpha(k')}^\infty dk_0'G_4(k',k_0',E)\sigma_l(k,k',k_0-k_0')R(k,k',k_0,k_0')\rho_l(k',k_0'), \quad (5.5)$$

where

$$\sigma_l(k,k',k_0)=\tfrac{1}{2}g^2P_l\left(\frac{k^2+k'^2+\mu^2-k_0^2}{2kk'}\right)$$

and

$$R(k_1,k_2,k_0,a)=\theta(k_0-\omega_-(k_1,k_2)-a)\theta(\omega_+(k_1,k_2)+a-k_0),$$

where $\theta(x)$ is the usual step function

$$\theta(x)=\begin{cases}1 & \text{if } x\geq 0\\ 0 & \text{if } x<0.\end{cases}$$

Equations for the spectral function ρ_l were obtained by Ito *et al.*[29] and for ρ_l^+ and ρ_l^- by Pagnamenta and Taylor.[42] These authors used relations (5.1) and (5.2) for $M_i(k')$ in Eq. (5.5) to obtain closed equations for the spectral functions. However, Ito *et al.* found difficulties in obtaining numerical iterative solutions for ρ_l owing to the appearance of the step functions (but see Ref. 45). Mugibayashi, in a work on a very simplified model,[44] used mixed equations of the type of our Eqs. (5.4) and (5.5) but he failed to obtain numerical solutions for the spectral function.

Our program will utilize both Eqs. (5.4) and (5.5) to obtain closed equations for M_1 and M_2. This aim will be accomplished by obtaining reduction relations which will express $T_l(k,k_0)$ for any k_0 in terms of M_1 and M_2.

To proceed, Eq. (5.5) can be used to define a series expansion for $\rho_l(k,k_0)$:

$$\rho_l(k,k_0)=\sum_{n=1}\rho_l^{(n)}(k,k_0), \tag{5.6}$$

where

$$\rho_l^{(1)}(k,k_0)=\sigma_l(k,q,k_0)R(k,q,k_0,0)+\int_0^\infty dk'\sum_{i=1}^{2}g_i(k')\sigma_l(k,k',k_0-D_i(k'))R(k,k',k_0,D_i(k'))M_i(k'), \tag{5.7}$$

and for $n\geq 2$

$$\rho_l^{(n)}(k,k_0)=\int_0^\infty dk'\int_{\alpha(k')}^\infty dk_0'G_4(k',k_0',E)\sigma_l(k,k',k_0-k_0')R(k,k',k_0,k_0')\rho_l^{(n-1)}(k',k_0'). \tag{5.8}$$

The convergence of this series is obvious since (see Appendix C)

$$\begin{aligned}\rho_l(k,k_0)-\sum_{j=1}^{n}\rho_l^{(j)}(k,k_0)&=\int_0^\infty dk_1\cdots\int_0^\infty dk_n\int_{\alpha(k_1)}^\infty dt_1\int_{s_1^-}^{s_1^+}dt_2\cdots\int_{s_{n-1}^-}^{s_{n-1}^+}dt_nG_4(k_1,t_1,E)\sigma_l(k_2,k_1,t_2-t_1)\cdots G_4(k_{n-1},t_{n-1},E)\\&\quad\times\sigma_l(k_n,k_{n-1},t_n-t_{n-1})G_4(k_n,t_n,E)\sigma_l(k,k_n,k_0-t_n)R(k,k_n,k_0,t_n)\rho_l(k_1,t_1),\end{aligned} \tag{5.9}$$

where the limits of the integrals over t_i were determined by the θ functions: $s_i^\pm=\omega_\pm(k_{i+1},k_i)+t_i$.

Expression (5.9) is zero for $k_0<-E+\{k^2+[m+(n+1)\mu]^2\}^{1/2}$ so that, for a particular k_0, only a finite number of terms in the series (5.6) contribute to $\rho_l(k,k_0)$.

The reason for using expansion (5.6) is that each term of the series is determined by M_1 and M_2. This is clear for $\rho_l^{(1)}$, while for the other terms it can be seen by successive use of (5.8) followed by an appropriate change in the order of integration:

$$\rho_l^{(n)}(k,k_0)=K_1^{(n)}(k,k_0,q)+\int_0^\infty dk'\sum_{i=1}^{2}g_i(k')K_i^{(n)}(k,k_0,k')M_i(k'), \tag{5.10}$$

where

$$\begin{aligned}K_i^{(n)}(k,k_0,k')&=\int_0^\infty dk_n\cdots\int_0^\infty dk_2\int_{s_n^-}^{s_n^+}dt_n\int_{s_{n-1}^-}^{s_{n-1}^+}dt_{n-1}\cdots\int_{s_2^-}^{s_2^+}dt_2G_4(k_n,t_n,E)\\&\quad\times\sigma_l(k_n,k',t_n-D_i(k'))G_4(k_{n-1},t_{n-1},E)\sigma_l(k_{n-1},k_n,t_{n-1}-t_n)\cdots G_4(k_2,t_2,E)\\&\quad\times\sigma_l(k_2,k_3,t_2-t_3)\sigma_l(k,k_2,k_0-t_2)R(k,k_2,k_0,t_2);\end{aligned} \tag{5.11}$$

here $s_n^{\pm}=\omega_{\pm}(k_n,k')+D_i(k')$ and $s_j^{\pm}=\omega_{\pm}(k_{j-1},k_j)+t_{j+1}$ for $j<n$.

Inserting (5.11) into the last part of Eq. (5.4) we obtain

$$\int_0^\infty dk' \int_{\alpha(k_1)}^\infty dt_1 G_4(k_1,t_1,E)[V_l^+(k,k_1,t_1+k_0)+V_l^+(k,k_1,t_1-k_0)]\rho_l^{(n)}(k_1,t_1)$$

$$=A_1^{(n)}(k,k_0,q)+\int_0^\infty dk' \sum_{i=1}^{2} g_i(k')A_i^{(n)}(k,k_0,k')M_i(k'), \quad (5.12)$$

where

$$A_i^{(n)}(k,k_0,k')=\int_0^\infty dk_1 \int_{\alpha(k_1)}^\infty dt_1 G_4(k_1,t_1,E[V_l^+(k,k_1,t_1+k_0)+V_l^+(k,k_1,t_1-k_0)]K_i^{(n)}(k_1,t_1,k')$$

$$=\int_0^\infty dk_n \cdots \int_0^\infty dk_1 \int_{s_n^-}^{s_n^+} dt_n \cdots \int_{s_1^-}^{s_1^+} dt_1 G_4(k_n,t_n,E)\sigma_l(k_n,k',t_n-D_i(k'))G_4(k_{n-1},t_{n-1},E)$$

$$\times \sigma_l(k_{n-1},k_n,t_{n-1}-t_n)\cdots G_4(k_1,t_1,E)\sigma_l(k_1,k_2,t_1-t_2)[V_l^+(k,k_1,t_1+k_0)+V_l^+(k,k_1,t_1-k_0)]. \quad (5.13)$$

With the aid of (5.13), (5.6) and Eq. (5.4) we finally obtain the reduction relation

$$T_l(k,k_0)=A_1(k,k_0,q)+\int_0^\infty dk' \sum_{i=1}^{2} g_i(k')A_i(k,k_0,k')M_i(k'), \quad (5.14)$$

where

$$A_i(k,k_0,k')=V_l^+(k,k',D_i(k')+k_0)+V_l^+(k,k',D_i(k')-k_0)+\sum_{n=2}^{\infty} A_i^{(n-1)}(k,k_0,k'). \quad (5.15)$$

By considering expressions (5.13) and (5.15) it can be seen that A_1 and A_2 can both be obtained from a single function $X(k,k_0,k',k_0')$ as follows: $A_i(k,k_0,k')=X(k,k_0,k',D_i(k'))$, where X is determined by the integral equation

$$X(k,k_0,k',k_0')=V_l^+(k,k',k_0'+k_0)+V_l^+(k,k',k_0'-k_0)$$

$$+\int_0^\infty dk_1 \int_{\omega_-(k',k_1)+k_0'}^{\omega_+(k',k_1)+k_0'} dt_1 \sigma_l(k',k_1,t_1-k_0')G_4(k_1,t_1,E)X(k,k_0,k_1,t_1). \quad (5.16)$$

This equation may be written in another form by a change of the variable of integration in the homogeneous term:

$$X(k,k_0,k',k_0')=V_l^+(k,k',k_0'+k_0)+V_l^+(k,k',k_0'-k_0)$$

$$+\frac{g^2}{2}\int_0^\infty dk_1 \int_{-1}^{1} d\lambda\, P_l(\lambda)\frac{k'k_1}{\omega(\vec{k}_1-\vec{k}_1)}G_4(k_1,k_0'+\omega(\vec{k}'-\vec{k}_1),E)X(k,k_0,k_1,k_0'+\omega(\vec{k}'-\vec{k}_1)). \quad (5.17)$$

For the construction of the three-dimensional equations we shall need also a reduction relation for $\mathrm{Re}T_l(k,k_0)$. Examination of (5.14) and the equations leading to (5.14) shows that this relation is of the form

$$\mathrm{Re}T_l(k,k_0)=B_1(k,k_0,q)+\int_0^\infty dk' \sum_{i=1}^{2} g_i(k')B_i(k,k_0,k')M_i(k'), \quad (5.18)$$

where $B_i(k,k_0,k')=Y(k,k_0,k',D_i(k'))$ and $Y(k,k_0,k',k_0')$ is determined by

$$Y(k,k_0,k',k_0')=U_l(k,k',k_0'+k_0)+U_l(k,k',k_0'-k_0)$$

$$+\int_0^\infty dk_1 \int_{\omega_-(k',k_1)+k_0'}^{\omega_+(k',k_1)+k_0'} dt_1 \sigma_l(k',k_1,t_1-k_0')G_4(k_1,t_1,E)Y(k,k_0,k_1,t_1). \quad (5.19)$$

The general form of the functions $U_l=\mathrm{Re}V_l^+$ is derived in Appendix A.

VI. THREE-DIMENSIONAL EQUATIONS

In this section we shall derive the equations for the determination of $M_1(k)$, defined in Sec. V. This function was identified in Sec. IV as the three-dimensional off-shell continuation of the partial-wave physical amplitude, and it corresponds to the scattering of two positive-energy nucleons from the initial state to the intermediate state of two positive-energy nucleons with momenta $\vec{k}$ and $-\vec{k}$ in the c.m. system. On the same lines we identify $M_2(k)$ as the partial-wave amplitude corresponding to the scattering from the initial state to an intermediate state of two negative-energy nucleons with momenta $\vec{k}$ and $-\vec{k}$. As already mentioned the advantage of the two-channel equations for M_1 and M_2 is the neat separation of the effect of many-meson intermediate states on the potentials. But on the other hand, these equations will suffer a serious disadvantage from the numerical point of view: The potentials will be singular, and although the singularities are only logarithmic and thus integrable, this fact will cause numerical complications. To avoid the numerical problems we shall present in this section another version of two-channel partial-wave three-dimensional equations involving only regular potentials (in the elastic scattering region) which are very convenient for numerical computation. Finally, as a demonstration of the flexibility of the reduction procedure (a common feature of all reduction procedures) we shall present also a one-channel equation for $M_1(k)$.

The equations in this section will be limited to the elastic scattering energy region. By slight generalizations, all the equations can be adapted for use in higher-energy regions but this will not be done here.

The first form of the three-dimensional equations is a straightforward result of the reduction relations (5.14) and (5.18). By substituting the definitions of $M_1(k)$ and $M_2(k)$ in (5.14) and (5.18), respectively, we obtain closed three-dimensional equations for these functions:

$$M_i(k) = V_{i1}(k,q) + \int_0^\infty dk' \sum_{j=1}^{2} g_j(k') V_{ij}(k,k') M_j(k') , \quad (6.1)$$

where

$$V_{1j}(k,k') = A_j(k, D_1(k), k')$$

and

$$V_{2j}(k,k') = B_j(k, D_2(k), k') ,$$

and $M_i(k)$ follow Eq. (5.4).

The first term in the series for the $V_{21}(k,k')$ element of the potential assumes infinite values in some parts of the domain of the physical momenta. For example, in the S-wave equation this term is

$$V_0^+(k,k', D_1(k') + D_2(k)) + U_0(k,k', D_1(k') - D_2(k)) = \tfrac{1}{2} g^2 \ln\left[\frac{\omega_+(k,k') + E(k') + E(k)}{\omega_-(k,k') + E(k') + E(k)}\right] + \tfrac{1}{2} g^2 \ln\left|\frac{\omega_+(k,k') + E(k') - E(k) - 2E}{\omega_-(k,k') + E(k') - E(k) - 2E}\right| .$$

By deriving equation (6.1) we actually demonstrated only one way of obtaining three-dimensional equations. The procedure begins with Eq. (5.4) and depends on the separation of this equation into two parts, one part containing $M_1(k')$ and $M_2(k')$ and the other part containing $\rho_l(k', k_0')$. This separation is, to some extent, arbitrary because by using relations (5.2) the terms of this equation can be rearranged. As a result, a reduction procedure which will begin with the modified equation will lead to different three-dimensional equations. We shall now present a new set of equations which are obtained by using the relations

$$T_l^-(k', D_2(k')) = T_l^+(k', -D_2(k')) = \int_{\alpha(k')}^\infty dt \frac{\rho_l(k', t)}{t - D_2(k')}$$

in Eq. (5.4). The modified form of the equation is then

$$T_l(k,k_0) = V_l^+(k,q,k_0) + V_l^+(k,q,-k_0) + \int_0^\infty dk' \{ g_1(k') [V_l^+(k,k', D_1(k') + k_0) + V_l^+(k,k', D_1(k') - k_0)] M_1(k') + g_2(k') [V_l^+(k,k', D_2(k') + k_0) + V_l^+(k,k', D_2(k') - k_0)] M_2'(k') \} + \int_0^\infty dk' \int_{\alpha(k')}^\infty dk_0' \{ G_3(k', k_0', E) [V_l^+(k,k', k_0' + k_0) + V_l^+(k,k', k_0' - k_0)] + G_2^-(k', k_0', E) [V_l^+(k,k', D_2(k') + k_0) + V_l^+(k,k', D_2(k') - k_0)] \} \rho_l(k', k_0') , \quad (6.2)$$

where

$$M_2'(k) = T_l^+(k, D_2(k))$$

and

$$G_2^-(k', k_0', E) = g_2(k') \frac{1}{k_0' - D_2(k')}.$$

The modified equation for the spectral function can be obtained by an analysis of the singularities of Eq. (6.2):

$$\rho_l(k, k_0) = \sigma_l(k, q, k_0) R(k, q, k_0, 0) + \int_0^\infty dk' [g_1(k') \sigma_l(k, k', k_0 - D_1(k')) R(k, k', k_0, D_1(k')) M_1(k') + g_2(k') \sigma_l(k, k', k_0 - D_2(k')) R(k, k', k_0, D_2(k')) M_2'(k')] + \int_0^\infty dk' \int_{\alpha(k')}^\infty dk_0' [G_3(k', k_0', E) \sigma_l(k, k', k_0 - k_0') R(k, k', k_0, k_0') + G_2^-(k', k_0', E) \sigma_l(k, k', k_0 - D_2(k')) R(k, k', k_0, D_2(k'))] \rho_l(k', k_0'). \quad (6.3)$$

New reduction relations can now be obtained on the basis of Eqs. (6.2) and (6.3). These will relate $T_l(k, k_0)$ to M_1 and M_2' and will be of the form

$$T_l(k, k_0) = A_1'(k, k_0, q) + \int_0^\infty dk' [g_1(k') A_1'(k, k_0, k') M_1(k') + g_2(k') A_2'(k, k_0, k') M_2'(k')], \quad (6.4)$$

where now

$$A_i'(k, k_0, k') = X'(k, k_0, k', D_i(k'))$$

and

$$X'(k, k_0, k', k_0') = V_l^+(k, k', k_0' + k_0) + V_l^+(k, k', k_0' - k_0) + \int_0^\infty dk' \int_{\omega_-(k', k_1) + k_0'}^{\omega_+(k', k_1) + k_0'} dt_1 \sigma_l(k', k_1, t_1 - k_0') [G_3(k_1, t_1, E) X'(k, k_0, k_1, t_1) + G_2^-(k_1, t_1, E) X'(k, k_0, k_1, D_2(k_1))]. \quad (6.5)$$

Because of the inclusion of $M_2'(k)$ in the equations, a similar expression relating $T_l^+(k, k_0)$ to M_1 and M_2' is required. This can be obtained by considering Eq. (6.3) and the following equation for $T_l^+(k, k_0)$:

$$T_l^+(k, k_0) = V_l^+(k, q, k_0) + \int_0^\infty dk' [g_1(k') V_l^+(k, k', D_1(k') + k_0) M_1(k') + g_2(k') V_l^+(k, k', D_2(k') + k_0) M_2'(k')] + \int_0^\infty dk' \int_{\alpha(k')}^\infty dk_0' [G_4(k', k_0', E) V_l^+(k, k', k_0' + k_0) + G_2^-(k', k_0', E) V_l^+(k, k', D_2(k') + k_0)] \rho_l(k', k_0'). \quad (6.6)$$

Thus the relation will be of the form

$$T_l^+(k, k_0) = B_1'(k, k_0, q) + \int_0^\infty dk' [g_1(k') B_1'(k, k_0, k') M_1(k') + g_2(k') B_2'(k, k_0, k') M_2'(k')], \quad (6.7)$$

where

$$B_i'(k, k_0, k') = Y'(k, k_0, k', D_i(k'))$$

and

$$Y'(k, k_0, k', k_0') = V_l^+(k, k', k_0' + k_0) + \int_0^\infty dk_1 \int_{\omega_-(k', k_1) + k_0'}^{\omega_+(k', k_1) + k_0'} dt_1 \sigma_l(k', k_1, t_1 - k_0') [G_3(k_1, t_1, E) Y'(k, k_0, k_1, t_1) + G_2^-(k_1, t_1, E) Y'(k, k_0, k_1, D_2(k_1))]. \quad (6.8)$$

Equations (6.4) and (6.7) lead finally to the new three-dimensional equations

$$M_1(k) = V_{11}(k, q) + \int_0^\infty dk' [g_1(k') V_{11}(k, k') M_1(k') + g_2(k') V_{12}(k, k') M_2'(k)],$$
$$M_2'(k) = V_{21}(k, q) + \int_0^\infty dk' [g_1(k') V_{21}(k, k') M_1(k') + g_2(k') V_{22}(k, k') M_2'(k')], \quad (6.9)$$

where

$$V_{1j}(k,k') = A'_j(k, D_1(k), k')$$

and

$$V_{2j}(k,k') = B'_j(k, D_2(k), k').$$

The last version of the reduction relations is simply

$$T_l(k,k_0) = C(k,k_0,q) + \int_0^\infty dk' g_1(k') C(k,k_0,k') M_1(k'), \tag{6.10}$$

where

$$C(k,k_0,k') = W(k,k_0,k',D_1(k')),$$

and

$$W(k,k_0,k',k'_0) = V_l^+(k,k',k'_0+k_0) + V_l^+(k,k',k'_0-k_0) + \int_0^\infty dk_1 \int_{\omega_-(k',k_1)+k'_0}^{\omega_+(k',k_1)+k'_0} dt_1 \sigma_l(k',k_1,t_1-k'_0)[G_3(k_1,t_1,E)W(k,k_0,k_1,t_1) + G_2(k_1,t_1,E)W(k,k_0,k_1,D_2(k_1))]. \tag{6.11}$$

This relation can be obtained by using (5.2) to eliminate $M'(k')$ in Eq. (6.2) and then carrying out the same procedure as that demonstrated in the former cases.

A relation quite similar to the reduction relation (6.10) was assumed by early workers (see Ref. 52 and references quoted therein) in the field of three-dimensional equations, to hold for the BS wave function in coordinate space. It does indeed lead to the most accepted form of three-dimensional equation (one-channel equation) which is a relativistic analog of the LS equation. In the present treatment the one-channel equation has the form of Eq. (4.7). The potential here is determined by taking $k_0 = D_1(k)$ in Eq. (6.10) and Eq. (6.11), and is given by

$$V_l(k,k') = C(k, D_1(k), k').$$

In practice, to calculate the potential $V_l(k,k')$ of the one-channel equation, one has to solve Eq. (6.11) for each k with the appropriate value of k_0 [i.e., $k_0 = D_1(k)$]. The solution should be obtained over the whole domain of the momentum k' and over an interval of k'_0 which includes the point $k'_0 = D_1(k')$. A convenient choice for the k'_0 interval is $[D_1(k'), \infty)$, because by this choice the equation is kept nonsingular.

Equation (6.11) poses a special numerical problem, because the limits of integration in the right-hand side of the equation depend on the point in which the function W, in the left-hand side is calculated. This difficulty can be overcome by employing an interpolation scheme for the calculation of the unknown function in the region of integration. A detailed description of this method for the treatment of similar equations is given in Ref. 45.

The S-wave phase shifts for $m = \mu = 1$ have been calculated by the one-channel equation in the elastic energy region for several values of the coupling constant and the results agree quite accurately with those of Schwartz and Zemach and of others.[27,53]

Equivalent results were obtained by the use of the two-channel equation and Eqs. (6.5) and (6.8) which can be treated in the same way as Eq. (6.11).

The results are summarized in Table I (see Refs. 27, 45, 54), to which we have added the results of two approximate calculations which were carried out by using the two types of equations, Eq. (4.7) and Eq. (6.9). The potentials inserted in these equations are the [1/1] Padé approximants constructed from the leading two iterations of the corresponding equations for the potentials with the appropriate arguments.

The numerical integrations were performed by Gaussian integration, after transforming all intervals to the interval $[-1,1]$. The transformation used for the k' integral was $k' = 0.5(1+t)/(1-t)$, and twelve integration points were used. The number of k'_0 points was eight and the finite intervals were transformed by linear transformations. The k'_0 integration employed polynomial interpolation with eight fixed points.

VII. SUMMARY AND DISCUSSION

We have presented here a set of three-dimensional equations which are equivalent to the four-dimensional BS equation. In the derivation of the various equations we tried to remain as general as possible, expressing in this way our opinion that no specific type of equation should be preferred from a theoretical point of view as long as exact potentials are used.

TABLE I. Computed values of S-wave phase shifts for $m=\mu=1$ (the listed values are those for δ_0/π). The results in the first row were obtained by use of the one-channel equation (4.7) and those in the second row by use of the two-channel equations (6.9). In both cases the potentials were approximated by [1/1] Padé approximants. The results in the last row are those of Schwartz and Zemach.[27,45,54] The coupling constant g is related to λ by $\lambda=g^2/(4\pi)^2$.

Potentials and equations	$\lambda=1$ $k^2=0.1$	$\lambda=1$ $k^2=0.4$	$\lambda=3$ $k^2=0.3$	$\lambda=1$ $k^2=0.2$	$\lambda=3$ $k^2=0.2$	$\lambda=1$ $k^2=0.8$	$\lambda=3$ $k^2=0.8$
[1/1] Padé approximant one-channel equation	0.5008	0.3464	0.7505	0.4229	0.8015	0.2731	0.5974
[1/1] Padé approximant two-channel equation	0.5045	0.3473	0.7412	0.4251	0.7911	0.2732	0.5914
Exact	0.4982	0.3436	0.7292	0.4200	0.7785	0.2708	0.5853
Schwartz and Zemach	0.49865	0.34354	0.72906	0.42004	0.77787	0.27081	0.58470

The possibility of generating approximation schemes based on the three-dimensional reduction procedure and the possible relationship between the present approach and existing ones has only been touched upon in this paper. This question becomes relevant when treating models which are more spohisticated and more complicated than the one-meson-exchange model, and will be dealt with in future works.

The numerical results given at the end of Sec. VI demonstrate the degree of accuracy which can be obtained from our equations despite the lack of rigorous justification for some of the steps taken at different stages of the work. The most obvious of these being the extensive use of change in the order of integrations during the derivation of the equations. The problem of the convergence of the series expansions of the potentials was bypassed by using exact numerical solutions. It should be pointed out, however, that the existence of the solutions in the general case remains an open problem.

APPENDIX A

By decomposing the potential of the BS equation in the ladder approximation

$$V(k,k_0,k',k_0')=kk'\frac{g^2}{(\vec{k}-\vec{k})^2-(k_0-k_0')^2+\mu^2}=\frac{g^2}{2}\frac{kk'}{\omega(\vec{k}-\vec{k}')}\left(\frac{1}{\omega(\vec{k}-\vec{k}')+k_0-k_0'}+\frac{1}{\omega(\vec{k}-\vec{k}')-k_0+k_0'}\right),$$

where $\omega(\vec{k})=(k^2+\mu^2)^{1/2}$, the partial-wave potential can be written

$$V_l(k,k_0,k',k_0')=V_l^+(k,k',k_0-k_0')+V_l^+(k,k',k_0'-k_0),$$

where

$$V_l^+(k,k',k_0'-k_0)=\frac{g^2}{2}\int_{-1}^{1}d\lambda\,P_l(\lambda)\frac{kk'}{\omega(\vec{k}-\vec{k}')}\frac{1}{\omega(\vec{k}-\vec{k}')+k_0'-k_0},\tag{A1}$$

and

$$\lambda=\frac{\vec{k}\cdot\vec{k}'}{kk'}.$$

The integral in (A1) as a function of k_0' can be analytically continued to the complex k_0' plane, and it represents an analytic function in the cut plane, the cut extending from $k_0-\omega_+(k,k')$ to $k_0-\omega_-(k,k')$.

By changing the integration variable to $z=\omega(\vec{k}-\vec{k}')$, we obtain

$$V_l^+(k,k',k_0'-k_0)=\int_{\omega_-(k,k')}^{\omega_+(k,k')}dz\,\sigma_l(k,k',z)\frac{1}{z-k_0+k_0'},\tag{A2}$$

where

$$\sigma_l(k,k',z)=\tfrac{1}{2}g^2P_l\left(\frac{k^2+k'^2+\mu^2-z^2}{2kk'}\right).$$

By (A2) the discontinuity of V_l^+ across the cut is easily seen to be $2\pi i\sigma_l(k,k',k_0-k_0')$.

Expanding $\sigma_l(k,k',z)$ around $z=k_0-k_0'$, (A2) can be written as

$$V_l^+(k,k',k_0-k_0')=\sigma_l(k,k',k_0-k_0')\ln\left[\frac{\omega_+(k,k')+k_0'-k_0}{\omega_-(k,k')+k_0'-k_0}\right]+u_l(k,k',k_0-k_0'),$$

where

$$u_l(k,k',t)=\sum_{n=1}^{2l}\frac{1}{n!}\frac{d^n}{dy^n}\sigma_l(k,k',y)\Big|_{y=t}\int_{\omega_-(k,k')}^{\omega_+(k,k')}dz(z-t)^{n-1}.$$

$u_l(k,k',k_0-k_0')$ is obviously a polynomial in k_0' so that the real part of V_l^+ (with k_0 real and k_0' along the cut) is simply

$$U_l(k,k',k_0-k_0')=\mathrm{Re}V_l^+(k,k',k_0-k_0')=\sigma_l(k,k',k_0-k_0')\ln\left|\frac{\omega_+(k,k')+k_0'-k_0}{\omega_-(k,k')+k_0'-k_0}\right|+u_l(k,k',k_0-k_0').$$

APPENDIX B

Analysis of the singularities in the relative energy variable of $T_l(k,k_0)$ is needed both for proving the consistency of the EAR and for the construction of the equations for the spectral functions [Eqs. (5.5) and (6.3)]. The analysis will be based on the right-hand side of Eq. (5.4). The inhomogenous term was considered in Appendix A, and here we first consider those parts of the homogenous term involving only one integration. These consist of terms of the form

$$\int_0^\infty dk' g_i(k')\,V_l^+(k,k',D_i(k')\pm k_0)M_i(k'), \qquad \text{(B1)}$$

where $i=1$ or $i=2$.

The singularity of $g_1(k')$ at $k'=q$ does not depend on k_0 and is treated in the usual way (see the end of Sec. IV). The integrand is regular at $k'=0$ and $k'=\infty$ so that there are no end-point singularities.[51] The pinch singularities are located by solving the two equations

$$S_i(k,k',k_0)=0, \qquad \text{(B2)}$$

$$\frac{\partial}{\partial k'}S_i(k,k',k_0)=0, \qquad \text{(B3)}$$

where $S_i=0$ are the singularities of the integrand. The singularities in this case are determined by

$$S_1=\omega_+(k,k')+E(k')-E+k_0,$$

$$S_2=\omega_+(k,k')+E(k')-E-k_0,$$

$$S_3=\omega_+(k,k')+E(k')+E+k_0,$$

$$S_4=\omega_+(k,k')+E(k')+E-k_0,$$

$$S_5=\omega_-(k,k')+E(k')-E+k_0,$$

$$S_6=\omega_-(k,k')+E(k')-E-k_0,$$

$$S_7=\omega_-(k,k')+E(k')+E+k_0,$$

$$S_8=\omega_-(k,k')+E(k')+E-k_0.$$

The four singularities of the integrand determined by S_1, S_2, S_3, and S_4 do not cause singularities in the integral as Eq. (B3) does not have solutions when $i=1,2,3,4$. When $S_i=S_5$, S_6, S_7, or S_8, Eq. (B3) implies $k'=km/(m+\mu)$ or $\omega_-(k,k')+E(k')=[k^2+(m+\mu)^2]^{1/2}$. This result together with (B1) determines the singularities generated by each of integrand singularities (see Table II). The parts of the homogenous term which involve two integrations can be written as

$$\int_0^\infty dk'\sum_{i=1}^{2}C_i(k,k',k_0), \qquad \text{(B4)}$$

TABLE II. Singularities of the scattering amplitude in the complex relative energy plane. Singularities in the lower half plane are listed in the first column and those in the upper half plane are listed in the left column.

Singularities of the integral	Singularities of the scattering amplitude	
	Lower half plane	Upper half plane
S_5		$k_0=E-[k^2+(m+\mu)^2]^{1/2}$
S_6	$k_0=-E+[k^2+(m+\mu)^2]^{1/2}$	
S_7		$k_0=-E-[k^2+(m+\mu)^2]^{1/2}$
S_8	$k_0=E+[k^2+(m+\mu)^2]^{1/2}$	
S_{11}		$k_0=E-[k^2+(m+2\mu)^2]^{1/2}$
S_{12}	$k_0=-E+[k^2+(m+2\mu)^2]^{1/2}$	

where

$$C_1(k,k',k_0)=\int_{\alpha(k')}^{\infty}dk_0' G_3(k',k_0',E) \times V_l^+(k,k',k_0'+k_0)\rho_l(k',k_0')\,, \quad \text{(B5)}$$

$$C_2(k,k',k_0)=\int_{\alpha(k')}^{\infty}dk_0' G_3(k',k_0',E) \times V_l^+(k,k',k_0'-k_0)\rho_l(k',k_0')\,. \quad \text{(B6)}$$

As G_3 is nonsingular in the integration domain, the singularities of C_1 and C_2 are end-point singularities which are generated by the four singularities of the integrand

$$S_9=\omega_+(k,k')+\alpha(k')+k_0\,,$$
$$S_{10}=\omega_+(k,k')+\alpha(k')-k_0\,,$$
$$S_{11}=\omega_-(k,k')+\alpha(k')+k_0\,,$$
$$S_{12}=\omega_-(k,k')+\alpha(k')-k_0\,.$$

Again, the singularities of C_1 and C_2 which correspond to S_9 and S_{10} do not cause singularities of the integral, but Eq. (B3) when applied to S_{11} and S_{12} determines the relation $\omega_-(k,k')+\alpha(k')=-E+[k^2+(m+2\mu)^2]^{1/2}$ which, in turn, determines together with (B2), the two corresponding singularities in k_0 which are summarized in Table II.

The list of Table II confirms the consistency achieved by taking $\alpha(k)$ as the lowest limit of integration in the EAR, Eq. (5.1), but it does not exhaust all the singularities of $T_l(k,k_0)$ (for a complete list see Ref. 42). By considering the lowest limit of the k_0' integration in (B5) and (B6) only the lowest (and highest) singularities in k_0 can be found, as the other singularities are caused by the discontinuity of $\rho_l(k',k_0')$ at points higher than $\alpha(k')$.

APPENDIX C

In this appendix we prove Eq. (5.9). To this purpose we begin by defining the functions $F^{(n)}(k,k_0)$:

$$F^{(n)}(k,k_0)=\int_0^{\infty}dk_1\int_{\alpha(k_1)}^{\infty}dt_1\sigma_l(k,k_1,k_0-t_1)G_4(k_1,t_1,E)R(k,k_1,k_0,t_1)F^{(n-1)}(k_1,t_1) \quad \text{(C1)}$$

and

$$F^{(1)}(k,k_0)=\int_0^{\infty}dk_1\int_{\alpha(k_1)}^{\infty}dt_1\sigma_l(k,k_1,k_0-t_1)G_4(k_1,t_1,E)R(k,k_1,k_0,t_1)\rho_l(k_1,t_1)\,, \quad \text{(C2)}$$

and note that the right-hand side of Eq. (5.9) is just $F^{(n)}(k,k_0)$. This is true for $n=1$ simply by the definition of $F^{(1)}(k,k_0)$. For $n\neq 1$, assume that $F^{(n-1)}(k,k_0)$ has the correct form, i.e.,

$$F^{(n-1)}(k,k_0)=\int_0^{\infty}dk_n\cdots\int_0^{\infty}dk_2\int_{\alpha(k_n)}^{\infty}dt_n\int_{s_{n-1}^-}^{s_{n-1}^+}dt_{n-1}\cdots\int_{s_2^-}^{s_2^+}dt_2 G_4(k_n,t_n,E)\sigma_l(k_{n-1},k_n,t_{n-1}-t_n) \times\cdots G_4(k_2,t_2,E)\sigma_l(k,k_2,k_0-t_2)R(k,k_2,k_0,t_2)\rho_l(k_n,t_n)\,,$$

then by using (C1), changing the order of integration, and relabeling the variables of integration we see that $F^{(n)}(k,k_0)$ has also the right form.

It remains to show that $F^{(n)}(k,k_0)$ is indeed equal to the left-hand side of Eq. (5.9). The proof is established by induction. By Eq. (5.5) and the definitions of $\rho_l^{(1)}$ and $F^{(1)}$ we can write

$$F^{(1)}(k,k_0)=\rho_l(k,k_0)-\rho_l^{(1)}(k,k_0)\,, \quad \text{(C3)}$$

which justifies Eq. (5.9) for $n=1$. For $n>1$, assume Eq. (5.9) to hold true for $n-1$, which implies

$$\rho_l^{(n-1)}(k,k_0)=\rho_l(k,k_0)-F^{(n-1)}(k,k_0)-\sum_{i=1}^{n-2}\rho_l^{(i)}(k,k_0)\,. \quad \text{(C4)}$$

Using the definition (5.8) of $\rho_l^{(n)}(k,k_0)$ we obtain from (C4)

$$\rho_l^{(n)}(k,k_0)=F^{(1)}(k,k_0)-F^{(n)}(k,k_0)-\sum_{i=2}^{n-1}\rho_l^{(i)}(k,k_0)\,.$$

By substituting in the last identity the value of $F^{(1)}(k,k_0)$ as given by (C3), and rearranging the terms we establish the proof for general n.

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