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Relativistic eikonal expansion*

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The generalized ladder series of Feynman diagrams for scattering of two particles by scalar-meson exchange is expanded, using functional methods, to obtain the relativistic eikonal approximation and the next two terms of an expansion about the eikonal limit. The established similarity between nonrelativistic and relativistic eikonal approximations is shown to persist, in part, to the higher-order terms in the relativistic eikonal expansion. The leading-order correction to the eikonal limit differs only kinematically from its nonrelativistic counterpart. In second order, there is again much similarity with nonrelativistic results; however, a part of the second-order eikonal correction explicitly depends on the relative time coordinate of the scattering particles. An approximate relativistic Schrödinger equation is found to reproduce the leading corrections to the eikonal limit by means of a simple kinematic generalization of the nonrelativistic potential theory results; however, the relativistic time effect cannot be readily incorporated into a three-dimensional wave equation.

I. INTRODUCTION

A relativistic eikonal expansion method for high-energy two-particle scattering in quantum field theory was first outlined some years ago by Abarbanel and Itzykson.[1] This paper develops a similar expansion in detail and studies the persistent analogy with nonrelativistic potential theory which emerges in the eikonal limit.

Following the basic work of Cheng and Wu[2] and others[3] on the high-energy behavior of infinite sets of Feynman diagrams in quantum electrodynamics, Abarbanel and Itzykson showed that similar results could be derived by a straightforward application of functional-derivative techniques[4] together with the eikonal approximation. The basic result was that the relativistic eikonal approximation for the scattering amplitude reproduced in a transparent fashion the elaborate sum of QED leading terms from perturbation theory. Considerable interest in the relativistic eikonal approximation followed.

Whether or not the eikonal approximation was reliable for *scalar* field theory was not as clear, however. When individual Feynman diagrams were analyzed, it was found that delicate cancellations were responsible for the dominance of the eikonal contributions as $s\to\infty$. In addition, it was observed[5] that noneikonal leading terms must arise by eighth order for the scattering of two scalar particles via scalar-meson exchange (scalar-scalar theory). Hence it was doubtful[6] that the eikonal approximation continued to reproduce the high-energy behavior of perturbation theory beyond sixth order.

A recent analysis by Banerjee and Mallik[7] has reexamined the validity of the eikonal approximation in scalar field theory. The conclusion was that in scalar-scalar theory, in which the exchanged mesons are not identical to the scattering particles, the eikonal approximation fails to reproduce a part of the eighth-order amplitude which asymptotically behaves as $(\ln s)/s^3$, where s is the square of the center-of-mass energy. The eikonal estimate for $2n$th order of perturbation theory behaves asymptotically as s^{1-n}, i.e., $1/s^3$ for eighth order. However, in ϕ^3 theory, there is additional cancellation which eliminates the $(\ln s)/s^3$ term. Hence for ϕ^3 theory the question of validity of the eikonal approximation remains open while for the scalar-scalar theory the eikonal approximation does not incorporate the noneikonal routing of momenta which give rise to the asymptotically dominant $(\ln s)/s^3$ term in eighth order.

It should be emphasized, however, that the eikonal approximation still remains extremely useful. The approximation does reproduce the leading terms for QED and, in the scalar-scalar theory, the error made is generally very small at high energy and could, in principle, be corrected. However, the primary interest in high-energy scattering lies in the fermion-fermion case. Because there is a rather close connection between the scalar-scalar theory and the scalar limit of fermion-fermion scattering by vector-meson exchange, this paper considers in some detail a relativistic eikonal expansion for the scalar-scalar theory along lines proposed by Abarbanel and Itzykson.[1] The scalar-scalar theory is chosen for simplicity, keeping in mind that the theory contains most of the analytical complications which are present in the fermion-fermion case. A subsequent paper will deal with the differences between the scalar and fermion cases.

The paper is organized as follows: Section II discusses the generalized ladder diagrams which

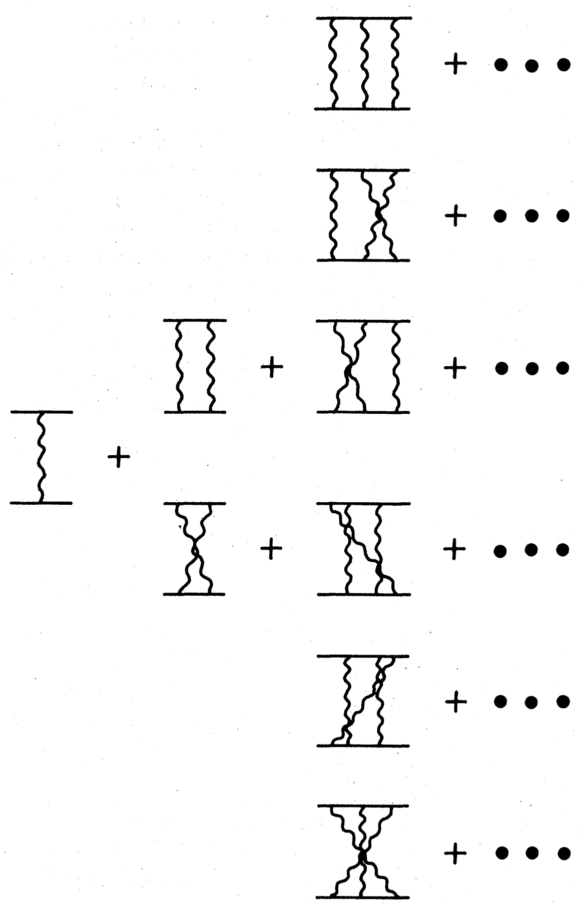

FIG. 1. The generalized ladder set of Feynman diagrams is depicted through sixth order. In the $2n$th order of perturbation theory, $n!$ distinct graphs account for all possible crossings of the meson lines (wiggly) exchanged between the two baryon lines (straight). The functional derivative formalism discussed in the text generates the full set of graphs.

are taken as a model for relativistic two-particle scattering. Section III reviews the functional-derivative approach to the relativistic eikonal approximation and establishes our notation in some detail. Section IV, in conjunction with the appendixes, details the relativistic eikonal expansion development of first- and second-order corrections to the eikonal limit and Sec. V presents a summary of our results in light of previous studies of the eikonal expansion. A relativistic wave equation suitable for two-particle scattering at high energy is suggested. Conclusions are presented in Sec. VI.

II. RELATIVISTIC TWO-PARTICLE SCATTERING

Consider the scattering of two massive, scalar baryons which interact via exchange of scalar mesons in the generalized ladder series of Feynman diagrams (Fig. 1). Following Abarbanel and Itzykson, we use an economical formalism which generates this set of diagrams by functional differentiation of external potential amplitudes as follows:

$$-i(2\pi)^{-4}T(s,t)\delta^{(4)}(k_1'+k_2'-k_1-k_2)$$
$$=\mathcal{K}\langle k_1'|T(A_1)|k_1\rangle\langle k_2'|T(A_2)|k_2\rangle|_{A_1=A_2=0}\,. \quad (1)$$

Here k_1, k_2 are four-momenta of the baryons before scattering and k_1', k_2' are those subsequent to scattering, $s=(k_1+k_2)^2$ and $t=(k_1-k_1')^2$.

The scattering of baryon 1 in a scalar external potential $A_1(x)$ is described by the matrix element of the operator

$$T(A_1)=A_1+A_1G_1(A_1)A_1\,, \quad (2)$$

where $G(A_1)$ is a Klein-Gordon propagator

$$G_1^{-1}(A_1)=p^2-m_1^2-A_1(x)+i\epsilon. \quad (3)$$

The external potential amplitude for baryon 2 in (1) is similarly defined.

Finally the functional-derivative operator[4] which generates the generalized ladder diagrams for meson exchange (see Fig. 1) is

$$\mathcal{K}=\exp\left[\int d^4y\int d^4y'\;\frac{\delta}{\delta A_1(y)}D(y-y')\frac{\delta}{\delta A_2(y')}\right]. \quad (4)$$

The notation of (1) means that after the functional derivatives are carried out, the fictitious external potentials A_1 and A_2 are set to zero.

The causal propagator of the exchanged meson is given by

$$D(y-y')=\int\frac{d^4l}{(2\pi)^4}\,e^{il\cdot(y-y')}\,\tilde{D}(l^2)\,, \quad (5)$$

where, for the exchange of a scalar meson of mass μ,

$$\tilde{D}(l^2)=\frac{ig_1g_2}{l^2-\mu^2+i\epsilon}\,. \quad (6)$$

It is clear that exchanges of nonelementary objects can be modeled by the phenomenological choice of $\tilde{D}(l^2)$.

The model of scattering represented by Eqs. (1)–(6) is incomplete as renormalization, vertex corrections, meson-meson interactions, and production channels are omitted. Nevertheless, the model is appropriate for scalar, relativistic, two-particle scattering in the same sense that the Schrödinger potential theory is appropriate for nonrelativistic two-particle scattering. We refer to the scattering particles as baryons to emphasize that all diagrams in the model have two continuous particle lines (solid lines in Fig. 1) between which mesons are exchanged in all possible orderings of emission and absorption. The role of the relative time coordinate of the baryons is completely specified by the relativistic dynamics.

Our present interest in this model centers on determining its structure for high-energy small-angle scattering. To this end, the amplitude $\mathcal{T}(s, t)$, for fixed t, is expanded about the eikonal limit through second order in the following relativistic eikonal expansion:

$$\mathcal{T}(s, t) = \mathcal{T}_E(s, t) + \mathcal{T}_E^1(s, t) + \mathcal{T}_E^2(s, t) + \cdots . \tag{7}$$

The series (7) is unambiguously defined by choosing $\mathcal{T}_E(s, t)$ to be a specific eikonal approximation to the generalized ladder diagrams and then perturbatively developing the corrections to this approximation. The correction terms $\mathcal{T}_E^1$, $\mathcal{T}_E^2$, etc., vanish increasingly rapidly as $s \to \infty$.

In general, the series (7) is more interesting in the case of fermion-fermion scattering via vector-meson exchange. Using the Block-Nordsieck approximation, $\gamma_1^\mu \simeq (k_1^\mu + k_1'^\mu)/2m_1$, the scalar limit of fermion-fermion scattering is easily related to the simple scalar scattering model described above. For either scalar baryons or fermions, we obtain a relativistic eikonal expansion (7) which has many similarities to a nonrelativistic eikonal expansion developed some years ago.[8-10]

III. RELATIVISTIC EIKONAL APPROXIMATION

The relativistic eikonal approximation is briefly reviewed to establish our notation for the subsequent development of explicit corrections to the eikonal limit.

To obtain the eikonal approximation for fixed momentum transfer q, the momentum operators of the scattering particles are expanded about the following on-mass-shell eikonal momenta:

$$\begin{aligned} \kappa_1 &= ((k^2 + m_1^2)^{1/2},\ \vec{0}, k), \\ \kappa_2 &= ((k^2 + m_2^2)^{1/2},\ \vec{0}, -k), \end{aligned} \tag{8}$$

where $k = |\vec{k}_1|$ is the c.m. momentum. The z direction along which κ_1 and κ_2 have components k and $-k$, respectively, is parallel to the average momentum of particle 1 (the projectile):

$$\hat{z} = \frac{\vec{k}_1 + \vec{k}_1'}{|\vec{k}_1 + \vec{k}_1'|} . \tag{9}$$

Thus the momenta κ_1 and κ_2 are orthogonal to the momentum transfer

$$q = (k_1 - k_1') = (0, \vec{q}, 0) , \tag{10}$$

where $\vec{q}$ is a two-dimensional vector in the (x, y) plane. The only departure here from the approach of Ref. 1 lies in our use of eikonal momenta (8) in place of the average of initial and final momenta $(k_1 + k_1')/2$ and $(k_2 + k_2')/2$. This difference shows up in the z components $\pm k$ in (8) in contrast with $\pm k \cos(\frac{1}{2}\theta)$ of the Abarbanel and Itzykson approach. The eikonal momenta (8) are preferred because they automatically yield a Fourier-Bessel representation in the relativistic eikonal expansion.

The eikonal approximation to the series of ladder diagrams is very simply obtained by making a standard eikonal approximation to the external potential amplitudes in (1). The propagator $G(A_1)$ is replaced by an eikonal propagator $g_1(A_1)$ which is linear in the momentum operator p^μ as follows:

$$g_1^{-1}(A_1) = 2\kappa_1 \cdot [p - \tfrac{1}{2}(k_1 + k_1')] - A_1(x) + i\epsilon , \tag{11}$$

$$g_2^{-1}(A_2) = 2\kappa_2 \cdot [p - \tfrac{1}{2}(k_2 + k_2')] - A_2(x) + i\epsilon. \tag{12}$$

The simplicity of the eikonal propagators is evident in the following relations:

$$\langle k_1' | [1 + A_1 g_1(A_1)] | x \rangle = e^{ik_1' \cdot x} \exp\left(-i\int_0^\infty d\tau A_1(x - 2\kappa_1\tau)\right) , \tag{13a}$$

$$\langle x | [g_1(A_1)A_1 + 1] | k_1 \rangle = e^{-ik_1 \cdot x} \exp\left(-i\int_{-\infty}^0 d\tau A_1(x - 2\kappa_1\tau)\right) \cdot \tag{13b}$$

The effect of the external potential is simply to phase-shift the initial or final plane wave, and similar results hold for particle 2.

It follows that the eikonal approximation to the relativistic external potential amplitude [Eq. (2) with $g_1(A_1)$ in place of $G_1(A_1)$] is

$$\begin{aligned} \langle k_1' | T_E^0(A_1) | k_1 \rangle &= (2\pi)^{-4} \int d^4x\, e^{-i(k_1 - k_1') \cdot x} \times \exp\left(-i\int_0^\infty d\tau A_1(x - 2\kappa_1\tau)\right) A_1(x) \\ &= (2\pi)^{-4} \int d^4x\, e^{-i(k_1 - k_1') \cdot x} \times \frac{1}{i}\frac{d}{d\alpha} e^{-iL_{+\alpha}A_1(x)} \Big|_{\alpha=0} , \end{aligned} \tag{14}$$

where we have written the line integral symbolically as

$$L_{+\alpha}A_1(x) \equiv \int_\alpha^\infty d\tau A_1(x - 2\kappa_1\tau). \tag{15}$$

The parameter α is introduced as a lower limit on the integral in the phase factor so that the factor $A_1(x)$ can be replaced by the operator $(1/i) \times d/d\alpha$ followed by the limit $\alpha = 0$.

Substituting the eikonal external-potential results into (1) and using the fact that the functional-derivative operator is a shifting operation leads directly to

$$-i\,(2\pi)^{-4}\,\mathcal{T}_E(s,t)\delta^{(4)}(k_1'+k_2'-k_1-k_2) = -(2\pi)^{-8}\int d^4x \int d^4x'\, e^{-i[(k_1-k_1')\cdot x+(k_2-k_2')\cdot x']} \times \frac{d}{d\alpha}\frac{d}{d\alpha'}\exp[-L_{+\alpha}L'_{+\alpha}D(x-x')]\Big|_{\alpha=\alpha'=0}. \quad (16)$$

In this expression the line integration symbols mean

$$L_{+\alpha}L'_{+\alpha}D(y)\equiv\int_\alpha^\infty d\tau\int_{\alpha'}^\infty d\tau' D(y-2\kappa_1+2\kappa_2\tau'). \quad (17)$$

Six of the eight integrations in (16) can be performed. Writing $y=x-x'$ and integrating over $x+x'$, one first extracts the energy-momentum-conserving δ function. By further writing

$$y=b-2\kappa_1\sigma+2\kappa_2\sigma', \quad (18a)$$

where

$$b=(0,\vec{b},0), \quad (18b)$$

and $\vec{b}$ is a two-dimensional impact vector, the equality reduces to

$$\mathcal{T}_E=-i\int d^4y\, e^{-iq\cdot y}\frac{d}{d\sigma}\frac{d}{d\sigma'} \times\exp\left(-\int_\sigma^\infty d\tau\int_{\sigma'}^\infty d\tau' D(b-2\kappa_1\tau+2\kappa_2\tau')\right). \quad (19)$$

Finally two more integrations over σ and σ' (in place of y_0, y_3) are carried out to obtain the relativistic eikonal approximation

$$f^0(s,t)=(8\pi\sqrt{s})^{-1}\mathcal{T}_E(s,t) = \frac{k}{2\pi i}\int d^2b\, e^{iq\cdot b}(e^{i\chi_0(\vec{b},s)}-1), \quad (20)$$

where the eikonal phase shift is

$$\chi_0(\vec{b},s)\equiv-\frac{m_1m_2}{k\sqrt{s}}\int_{-\infty}^\infty dz\, U(b,z), \quad (21a)$$

with

$$U(b,z)\equiv\frac{-i}{4\,m_1m_2}\int_{-\infty}^\infty dy_0 D(y_0,b,z). \quad (21b)$$

$U(b,z)$ is completely analogous to a nonrelativistic potential, which, in this case, is defined by the integral over relative time of the causal propagator D. For the scalar-meson exchange of Eq. (6), one easily verifies the Yukawa potential form $U(r) = -f^2 e^{-\mu r}/(4\pi r)$, $r=(b^2+z^2)^{1/2}$, provided we define $f^2\equiv g_1g_2/4m_1m_2$. The phase shift $\chi_0(\vec{b},s)$ is identical in form to its nonrelativistic counterpart, and since $m_1m_2/\sqrt{s}\to m_r$ is the reduced mass, Eq. (21a) represents only a kinematic generalization of the nonrelativistic eikonal approximation which involves m_r/k.

Invoking the Block-Nordsieck approximation, we can immediately determine the spin-nonflip amplitude appropriate to the scalar limit of fermion-fermion scattering via vector-meson exchange. The changes in (21a) and (21b) for the fermion-fermion case are

$$\chi_0(b,s)=-\frac{\kappa_1\cdot\kappa_2}{k\sqrt{s}}\int_{-\infty}^\infty dz\, U(b,z), \quad (21c)$$

where

$$U(b,z)\equiv-i\int_{-\infty}^\infty dy_0 D(y_0,b,z), \quad (21d)$$

and D is now the appropriate vector-meson propagator. Specifically in massive QED, the potential is $U(r)=(ee'/4\pi)\,e^{-\mu r}/r$, where e and e' are the charges of the fermions. Notice that the kinematic factor $\kappa_1\cdot\kappa_2/(k\sqrt{s})$ in (21c) is precisely E_L/k_L, i.e., the inverse of the laboratory velocity. Thus the scalar limit of fermion-fermion scattering corresponds to a much larger phase shift at high energy as is evident from comparison of the energy dependence of (21a) and (21c).

In the corrections to the eikonal limit which we consider, the simple replacement $m_1m_2\to\kappa_1\cdot\kappa_2$, which transforms (21a) to (21c), only partially accounts for the corrections to the scalar limit of fermion-fermion scattering.

IV. RELATIVISTIC EIKONAL EXPANSION

In the functional formalism, the corrections to the eikonal limit are calculated by improving upon the approximations used to represent the external potential amplitudes. Consider particle 1, for example. The neglected part of the two-particle propagator in the external potential A_1 is expressed as

$$g_1^{-1}(A_1)-G_1^{-1}(A_1)=N_1, \quad (22)$$

where

$$N_1=-(p-k_1')\cdot(p-k_1)-\lambda 2k[p_z-k\cos(\tfrac{1}{2}\theta)] \quad (23)$$

represents the defect of the eikonal propagator and where

$$\lambda=1-\cos(\tfrac{1}{2}\theta). \quad (24)$$

For small-angle scattering, $\lambda\approx\theta^2/8$ is quite small. The organization of Eq. (23) is motivated by a prior study of the nonrelativistic eikonal expansion[8] where λ-dependent terms canceled out.

Expanding $T_1(A_1)$ about the eikonal limit leads to a perturbation series in the neglected part as follows:

$$T_1(A_1)=(A_1+A_1g_1A_1)+A_1g_1N_1g_1A_1+A_1g_1N_1g_1N_1g_1A_1+\cdots=T_E^0(A_1)+T_E^1(A_1)+T_E^2(A_1)+\cdots . \quad (25)$$

The procedure is essentially the same as that of the nonrelativistic expansion. Using Eqs. (13), and deferring treatment of the λ terms of (23), the first order in N_1 correction to the external-potential matrix element is developed as follows:

$$\langle k_1'|T_E^1(A_1)|k_1\rangle=-\left\langle k_1'\left|\exp\left(-i\int_0^\infty dt\,A_1(x-2\kappa_1t)\right)(p-k_1')_\mu(p-k_1)^\mu\exp\left(-i\int_{-\infty}^0 dt\,A_1(x-2\kappa_1t)\right)\right|k_1\right\rangle .$$

Commuting $(p-k_1')_\mu$ leftward and $(p-k_1)^\mu$ rightward gives

$$\langle k_1'|T_E^1(A_1)|k_1'\rangle=\left\langle k_1'\left|\exp\left(-i\int_{-\infty}^\infty dt\,A_1(x-2\kappa_1t)\right)\partial_\mu\int_0^\infty dt_1A_1(x-2\kappa_1t_1)\partial^\mu\int_{-\infty}^0 dt_2A_2(x-2\kappa_2t_2)\right|k_1\right\rangle , \quad (26)$$

where $p^\mu=i\partial^\mu$ is used. The second order in the N_1 terms of (25) is developed in Appendix C. We have the following:

$$\begin{aligned}\langle k_1'|T_E^2(A_1)|k_1\rangle=\Big\langle k_1'\Big|\,&i\int_{-\infty}^\infty d\tau\exp\left(-i\int_{-\infty}^\infty dt\,A_1(x-2\kappa_1t)\right)\\
&\times\Big\{-\left[\partial^\nu\int_0^\infty dt_1A_1(x-2\kappa_1t_1)\right]\left[\partial_\nu\int_{-\infty}^0 dt_2A_1(x-2\kappa_1t_2)\right]\\
&\quad\times\left[\partial^\mu\int_0^\infty dt_3A_1(x-2\kappa_1(t_3-\tau))\right]\left[\partial_\mu\int_{-\infty}^0 dt_4A_1(x-2\kappa_1(t_4-\tau))\right]\\
&-i\left[\partial^\nu\int_0^\infty dt_1A_1(x-2\kappa_1t_1)\right]\left[\partial_\nu\partial^\mu\int_0^\infty dt_2A_1(x-2\kappa_1(t_2-\tau))\right]\left[\partial_\mu\int_{-\infty}^0 dt_3A_1(x-2\kappa_1(t_3-\tau))\right]\\
&-i\left[\partial^\nu\int_0^\infty dt_1A_1(x-2\kappa_1t_1)\right]\left[\partial_\nu\partial^\mu\int_{-\infty}^0 dt_2A_1(x-2\kappa_1(t_2-\tau))\right]\left[\partial_\mu\int_0^\infty dt_3A_1(x-2\kappa_1(t_3-\tau))\right]\\
&+i\left[\partial^\nu\int_0^\infty dt_1A_1(x-2\kappa_1t_1)\right]\left[\partial_\nu\partial^\mu\int_{-\infty}^\infty dt_2A_1(x-2\kappa_1(t_2-\tau))\right]\left[\partial_\mu\int_{-\infty}^0 dt_3A_1(x-2\kappa_1(t_3-\tau))\right]\\
&-i\left[\partial^\nu\int_{-\infty}^0 dt_1A_1(x-2\kappa_1t_1)\right]\left[\partial_\nu\partial^\mu\int_0^\infty dt_2A_1(x-2\kappa_1t_2)\right]\left[\partial_\mu\int_{-\infty}^0 dt_3A_1(x-2\kappa_1(t_3-\tau))\right]\\
&-i\left[\partial^\nu\int_0^\infty dt_1A_1(x-2\kappa_1t_1)\right]\left[\partial_\nu\partial^\mu\int_{-\infty}^0 dt_2A_1(x-2\kappa_1t_2)\right]\left[\partial_\mu\int_{-\infty}^0 dt_3A_1(x-2\kappa_1(t_3-\tau))\right]\\
&+\left[\partial^\mu\partial^\nu\int_0^\infty dt_1A_1(x-2\kappa_1t_1)\right]\left[\partial_\mu\partial_\nu\int_{-\infty}^0 dt_2A_1(x-2\kappa_1(t_2-\tau))\right]\Big\}\Big|\,k_1\Big\rangle . \end{aligned} \quad (27)$$

When (25) and a similar expansion for particle 2 are used in Eq. (1), the leading-order correction to the eikonal limit is seen to be

$$-\frac{i}{(2\pi)^4}\,\mathcal{T}_E^1(s,t)\delta^{(4)}(k_1'+k_2'-k_1-k_2)=\mathcal{K}[\langle k_1'|T_E^0(A_1)|k_1\rangle\langle k_2'|T_E^1(A_2)|k_2\rangle+\langle k_1'|T_E^1(A_1)|k_1\rangle\langle k_2'|T_E^0(A_2)|k_2\rangle]|_{A_1=A_2=0} . \quad (28)$$

The functional differentiation proceeds by standard rules as before. When the results (14) and (26) are substituted into (28), we find

$$\begin{aligned}\mathcal{T}_E^1=-i\int d^4y\,e^{-iq\cdot y}\Big\{&\Big[\frac{1}{i}\frac{d}{d\alpha}\exp\left(-\int_\alpha^\infty d\tau\int_{-\infty}^\infty d\tau'D(y-2\kappa_1\tau+2\kappa_2\tau')\right)\\
&\times\int_\alpha^\infty d\tau\int_{-\infty}^0 d\tau'\partial^\mu D(y-2\kappa_1\tau+2\kappa_2\tau')\int_\alpha^\infty d\tau\int_0^\infty d\tau'\partial_\mu D(y-2\kappa_1\tau+2\kappa_2\tau')\Big]\\
&+(\kappa_1\longleftrightarrow-\kappa_2,\ \alpha\longleftrightarrow\alpha')\Big\}\Big|_{\alpha=\alpha'=0} \end{aligned} \quad (29)$$

after the energy-momentum δ function has been eliminated as before. The eikonal correction involves two parts, of which only the first is explicitly shown, the second being obtained from it by the indicated substitutions. It is convenient to convert the integration over y_0, y_3 to integration over the variables σ, σ' in (19), with the result

$$\mathcal{T}_E^1 = -i\,4k\sqrt{s}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\sigma' \left\{\left[\frac{1}{i}\frac{d}{d\sigma}\exp\left(-i\int_{\sigma}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' D(b-2\kappa_1\tau+2\kappa_2\tau')\right)\right.\right.$$
$$\times \int_{\sigma}^{\infty} d\tau \int_{-\infty}^{\sigma'} d\tau' \partial^\mu D(b-2\kappa_1\tau+2\kappa_2\tau')$$
$$\left.\times \int_{\sigma}^{\infty} d\tau \int_{\sigma'}^{\infty} d\tau' \partial_\mu D(b-2\kappa_1\tau+2\kappa_2\tau')\right]$$
$$\left. + (\kappa_1 \longleftrightarrow -\kappa_2,\ \sigma \longleftrightarrow \sigma')\right\}. \tag{30}$$

Finally, the integrations over σ, σ' are performed to obtain the leading-order correction to the eikonal approximation in the simplified form

$$\mathcal{T}_E^1 = -i\,4k\sqrt{s}\int d^2b\, e^{i\vec{q}\cdot\vec{b}} e^{i\chi_0(b,s)} i\chi_1(b,s), \tag{31}$$

where $\chi_0(b,s)$ is as previously defined by (21) and the essence of the leading-order correction is

$$\chi_1(b,s) = \int_{-\infty}^{\infty} d\sigma' \left[\int_{-\infty}^{\infty} d\tau \int_{\sigma'}^{\infty} d\tau' \partial^\mu D(b-2\kappa_1\tau+2\kappa_2\tau')\right]\left[\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\sigma'} d\tau' \partial_\mu D(b-2\kappa_1\tau+2\kappa_2\tau')\right]$$
$$+ \int_{-\infty}^{\infty} d\sigma \left[\int_{-\infty}^{\infty} d\tau' \int_{\sigma}^{\infty} d\tau\, \partial^\mu D(b-2\kappa_1\tau+2\kappa_2\tau')\right]\left[\int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\sigma} d\tau \partial_\mu D(b-2\kappa_1\tau+2\kappa_2\tau')\right]. \tag{32}$$

In terms of the scalar-exchange potential, $U(\vec{r})$, defined by Eq. (21b), we show in Appendix A that the four-gradients in (32) simplify to three-gradients and at the same time relativistic effects in (32) largely cancel out. The result is that $\chi_1(b,s)$ reduces to

$$\chi_1(b,s) = \frac{m_1+m_2}{\sqrt{s}}\int_{-\infty}^{\infty} dz\, \vec{\nabla}\chi_+(\vec{b},z)\cdot\vec{\nabla}\chi_-(\vec{b},z)/2k, \tag{33}$$

where

$$\chi_+(\vec{b},z) = -\frac{m_1 m_2}{k\sqrt{s}}\int_{-\infty}^{z} dz_1 U(\vec{b},z_1), \tag{34a}$$

$$\chi_-(\vec{b},z) = -\frac{m_1 m_2}{k\sqrt{s}}\int_{z}^{\infty} dz_1 U(\vec{b},z_1). \tag{34b}$$

Apart from kinematical factors, the phase correction χ_1 is *identical* to its nonrelativistic counterpart $\tau_1(b)$ of Ref. 8. To leading order, our result is that the relative time plays no role and hence the only relativistic effects are kinematical.

For subsequent use, the first-order λ correction is next given. Using the functional techniques already outlined, we find (Appendix B)

$$\mathcal{T}_\lambda^1 = i\,4k\sqrt{s}\int d^2b\, e^{i\vec{q}\cdot\vec{b}} e^{i\chi_0(\vec{b},s)}\lambda[1-i\chi_0(\vec{b},s)]. \tag{35}$$

As in (33), the only hint of this being a relativistic calculation lies in the kinematics.

The second-order correction is considerably more involved. In the functional formalism, this correction is generated by

$$-\frac{i}{(2\pi)^4}\mathcal{T}_E^{(2)}\delta^4(k_1+k_2-k_1'-k_2')$$
$$= \mathcal{K}[\langle k_1'|T_E^{(2)}(A_1)|k_1\rangle\langle k_2'|T_E^{(0)}(A_2)|k_2\rangle$$
$$+\langle k_1'|T_E^{(0)}(A_1)|k_1\rangle\langle k_2'|T_E^{(2)}(A_2)|k_2\rangle$$
$$+\langle k_1'|T_E^{(1)}(A_1)|k_1\rangle\langle k_2'|T_E^{(1)}(A_2)|k_2\rangle]\,|_{A_1=A_2=0}. \tag{36}$$

After operating with $\mathcal{K}$ and setting $A_1=A_2=0$, the following result is obtained (see Appendix C for details):

$$\mathcal{T}_E^{(2)}(s,t) = -i\,4k\sqrt{s}\int d^2b\, e^{i\vec{q}\cdot\vec{b}} e^{i\chi_0(b,s)} \times\left[\frac{(i\chi_1)^2}{2}+\frac{W_2}{4k^2}+i\bar{\chi}_2\right], \tag{37a}$$

with χ_1 as defined by (33) and with

$$W_2(b,s) = \int_{-\infty}^{\infty} dz\, \nabla_l \nabla_m \chi_-(b,z)\int_{-\infty}^{z} dz' \nabla_l \nabla_m \chi_+(b,z'), \tag{37b}$$

and

$$\bar{\chi}_2(b,s)=-\left(\frac{m_1^2+m_2^2}{s}\frac{1}{4k^2}\right)\int_{-\infty}^{\infty}dz\,\frac{m_1m_2}{k\sqrt{s}}\nabla_l\nabla_m U(\mathbf{b},z)\left[\int_z^{\infty}dz'\nabla_l\chi_-(b,z')+\int_{-\infty}^{z}dz'\nabla_l\chi_+(b,z')\right]$$
$$\times\int_{-\infty}^{z}dz''\nabla_m\chi_+(b,z'')$$
$$+\left(\frac{2m_1m_2}{s}\right)\frac{1}{4k^2}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dz\,\partial_\mu\partial_\nu\frac{i}{4k\sqrt{s}}D(t,b,z)$$
$$\times\left[\partial^\mu\int_{z_1}^{\infty}dz'\chi_-(b,z')+\partial^\mu\int_{-\infty}^{z_1}dz'\chi_+(b,z')\right]\left[\partial^\nu\int_{-\infty}^{z_2}dz'\chi_+(b,z')\right],\qquad(37c)$$

where l and m are summed $1\to3$ and

$$z_1=(z-v_1t)(1-v_1^2)^{-1/2},\quad z_2=(z+v_2t)(1-v_2^2)^{-1/2}.\qquad(37d)$$

Equations (37) are very similar to the nonrelativistic results [Eqs. (2.44) to (2.51) of Ref. 8]. The nonrelativistic limit (v_1, $v_2\to0$) of Eq. (37c) corresponds to $\bar{\tau}_2(b)-U_2(b)/(4k^2)$ of Ref. 8 while Eq. (37b) is exactly the same $W_2(b)$ which is given by Eq. (2.51) of Ref. 8.

A partial cancellation of the λ correction Eq. (35) with part of the second-order correction has been demonstrated for the nonrelativistic case. Likewise, in the present relativistic case, a partial cancellation of λ to order $1/k^2$ can be shown. Following Ref. 8, the sum of the second-order eikonal correction (37) and first-order λ correction (35) can be written

$$\mathcal{T}^{(2)}+\mathcal{T}^{(1)}_\lambda=-i4k\sqrt{s}\int d^2b\,e^{i\vec{q}\cdot\vec{b}}\left[\frac{[i\chi_1(b,s)]^2}{2}-\omega_2(b,s)+i\chi_2(b,s)-(\lambda+\nabla^2/8k^2)(1-i\chi_0(b,s))\right]e^{i\chi_0(b,s)},\qquad(38)$$

where the Laplacian operator acts on everything to its right, and where

$$\omega_2(b,s)=\frac{b\chi_0'\nabla^2\chi_0}{8k^2}\qquad(39a)$$

and

$$\chi_2(b,s)=\bar{\chi}_2+\frac{\chi_0\chi_0'^2}{8k^2},\qquad(39b)$$

with primes denoting differentiation with respect to b. The partial cancellation now can be seen by expanding λ:

$$\lambda=[1-(1-q^2/4k^2)^{1/2}]=q^2/8k^2+O(1/k^4).$$

Thus to order k^{-2}, the combination $(\nabla^2+q^2)/8k^2$ appears in the integrand of the two-dimensional Fourier integral of (38). This part vanishes identically as its Fourier transform is zero. Equations (38) and (39) correspond to Eqs. (2.55) to (2.57) of Ref. 8 and nonrelativistically our $\omega_2(b,s)$ and $\chi_2(b,s)$ become identical to $\omega_2(b)$ and $\tau_2(b)$ of Ref. 8.

V. SUMMARY OF RESULTS

Our results are summarized and extended by the following statements. The scalar scattering amplitude at high energy and small t takes the Fourier-Bessel form

$$f(s,t)=k(2\pi i)^{-1}\int d^2b\,e^{i\vec{q}\cdot\vec{b}}[S_F(b,s)-1],\qquad(41a)$$

where the impact-parameter S matrix is written in exponentiated form as

$$S_F(b,s)=e^{i\chi(b,s)-\omega(b,s)},\qquad(41b)$$

and the high-energy expansion of the eikonal phase shift is

$$\chi(b,s)=\chi_0(b,s)+\chi_1(b,s)+\chi_2(b,s)+\cdots,\qquad(41c)$$

$$\omega(b,s)=\omega_2(b,s)+\cdots.\qquad(41d)$$

The exponential form (41b) is valid in either the relativistic or nonrelativistic case as discussed in Ref. 9.

Our results for the eikonal phases can be written in rather simple forms based on the meson-exchange potential

$$U(r)\equiv-i(4m_1m_2)^{-1}\int_{-\infty}^{\infty}dt\,D(t,\vec{r}),\qquad(42)$$

and the causal propagator D. Utilizing the nonrelativistic results of Ref. 8 [specifically, Eq. (2.41) and Eqs. (2.50) to (2.54)], expressions in Eqs. (33), (37b), and (37c) are reduced to the following:

$$\chi_0(b,s)=-2k\epsilon\int_0^{\infty}dz\,U(r),\qquad(43a)$$

$$\chi_1(b,s)=\left(\frac{m_1+m_2}{s^{1/2}}\right)\left[-k\epsilon^2\int_0^\infty dz\left(2+r\frac{d}{dr}\right)U^2(r)\right]. \tag{43b}$$

The second-order eikonal phase term consists of two parts as follows:

$$\chi_2(b,s)=[(m_1^2+m_2^2)/s]\,\chi_{2a}(b,s)+(2m_1m_2/s)\,\chi_{2b}(b,s)\,, \tag{43c}$$

where χ_{2a} is identical in form to the nonrelativistic result,

$$\chi_{2a}(b,s)=-k\epsilon^3\int_0^\infty dz\left(\tfrac{8}{3}+\tfrac{7}{3}r\frac{d}{dr}+\tfrac{1}{3}r^2\frac{d^2}{dr^2}\right)U^3(r)-b[\chi_0'(b,s)]^3/(24k^2)\,. \tag{43d}$$

However, χ_{2b} involves the relative time t and cannot be reduced to a form involving just the potential $U(r)$:

$$\chi_{2b}(b,s)=(4k^2)^{-1}\int_{-\infty}^\infty dt\int_{-\infty}^\infty dz\left\{\frac{i\partial_\mu\partial_\nu D(t,b,z)}{4k\sqrt{s}}\left[\partial^\mu\int_{z_1}^\infty dz'\chi_-(b,z')+\partial^\mu\int_{-\infty}^{z_1}dz'\chi_+(b,z')\right]\times\left[\partial^\nu\int_{-\infty}^{z_2}dz'\chi_+(b,z')\right]\right\}+(8k^2)^{-1}[\kappa_1\cdot\kappa_2/(m_1m_2)]\,\chi_0(b,s)[\chi_0'(b,s)]^2\,. \tag{43e}$$

The integration limits z_1 and z_2 depend on relative time as seen in Eqs. (37d). If the limits z_1 and z_2 are replaced by just z, then the time integration in (43e) can be performed using (42) and $\chi_{2b}(b,s)$ also can be reduced to the form given by Eq. (43d). However, in the general case, we have been unable to obtain a simplified form of $\chi_{2b}(b,s)$. The real part of the eikonal phase is given by

$$\omega_2(b,s)=b\chi_0'(b,s)\nabla^2\chi_0(b,s)/(8k^2)\,, \tag{43f}$$

just as in the nonrelativistic case.

The above expressions are valid for scalar scattering. The scalar limit of fermion-fermion scattering although similar, is not identical to the scalar case. Nevertheless, the major difference between the two cases can be accounted for by the substitution $m_1m_2\to\kappa_1\cdot\kappa_2$ as mentioned before. Hence the expansion parameter ϵ in Eqs. (43) has two values of interest. For the scalar case,

$$\epsilon=m_1m_2/(k^2s^{1/2})\,, \tag{44a}$$

while for the fermion-fermion case,

$$\epsilon=\kappa_1\cdot\kappa_2/(k^2s^{1/2})\,, \tag{44b}$$

either of which reduces to the correct nonrelativistic limit,

$$\epsilon_{\rm NR}=m_r/k^2\,, \tag{44c}$$

where $m_r=m_1m_2/(m_1+m_2)$ is the reduced mass. At high energy, the eikonal phase corrections $\chi_1(b,s)$ and $\chi_2(b,s)$ become small due to the expansion parameter ϵ and also due to the parameters $m_1/s^{1/2}$ and $m_2/s^{1/2}$ evident in Eqs. (43).

If one particle is much more massive than the other (e.g., $m_2\gg m_1$), then the contribution of χ_{2b} in Eq. (43c) becomes negligible since $m_1/s^{1/2}\to 0$ and $m_2/s^{1/2}\to 1$. In the static limit the eikonal expansion results are identical in form to the nonrelativistic eikonal expansion results based on Schrödinger potential theory, the only relativistic effect being the kinematic parameter ϵ of Eq. (44b).

For completeness, analytic expressions[8] for the eikonal phase shifts, except χ_{2b}, are given based on the simplest choice for $D(t,\vec{r})$ which corresponds to the Yukawa potential

$$U(r)=-\frac{f^2}{4\pi}e^{-\mu r}/r\,. \tag{45a}$$

A dimensionless expansion parameter is defined by

$$\epsilon_1\equiv-\frac{f^2\epsilon}{4\pi}\,, \tag{45b}$$

and then one has

$$\chi_0(b,s)=-2k\epsilon_1K_0(\mu b)\,, \tag{45c}$$

$$\chi_1(b,s)=[(m_1+m_2)/s^{1/2}][-2k\epsilon_1^2K_0(2\mu b)]\,, \tag{45d}$$

$$\chi_{2a}(b,s)=-3k\epsilon_1^3[K_0(3\mu b)-(3\mu b)^{-1}K_1(3\mu b)+(\mu b/9)K_1^3(\mu b)]\,, \tag{45e}$$

$$\omega_2(b,s)=-\tfrac{1}{2}(\mu\epsilon_1)^2\mu bK_1(\mu b)K_0(\mu b)\,, \tag{45f}$$

where $K_n(x)$ are modified Bessel functions. *Nonelementary exchanges can be modeled by other choices of the D function for which the results (41)–(43) provide a succinct summary of the high-energy expansion to the corresponding sum of generalized ladder graphs.*

An effective Schrödinger theory which produces the same eikonal phase expansion as the generalized ladder set of Feynman diagrams is readily deduced.

The relativistic Schrödinger equation is

$$\left[\frac{\nabla^2+k^2}{2E}-V(r)\right]\psi(\vec{r})=0\ , \tag{46a}$$

where k is the c.m. momentum

$$k=P_L m_2/s^{1/2}\ , \tag{46b}$$

and the energy parameter E is so determined that

$$E/k^2=\kappa_1\cdot\kappa_2/(k^2 s^{1/2})\ .$$

This condition equates the ϵ parameter of Ref. 8, using E in place of the two-particle reduced mass, to the ϵ parameter of Eq. (44b). The E parameter so deduced is expressed in terms of the laboratory energy, E_L, of particle 1 as follows:

$$E=E_L m_2/s^{1/2}\ . \tag{46c}$$

In the nonrelativistic limit, E becomes the reduced mass m_r and Eqs. (45) become the usual Schrödinger theory.

The potential $V(r)$ in (46a) is related to the meson-exchange potential $U(r)$ defined by (42) as follows:

$$V(r)\simeq U(r)+(\alpha-1)\epsilon\left(1+\tfrac{1}{2}r\frac{d}{dr}\right)U^2(r)+\cdots\ , \tag{46d}$$

where

$$\alpha\equiv(m_1+m_2)/s^{1/2}\ . \tag{46e}$$

In the relativistic limit or the limit $\alpha\to 1$, $V(r) = U(r)$.

Terms of order $\epsilon^2(\alpha-1)$ have been dropped in (45d), hence at high energy the potential $V(r)$ used in (46a) reproduces the relativistic eikonal expansion results only through the leading correction $\chi_1(b,s)$. Among the higher-order terms omitted in (46d) is the nonstatic, relative-time effect imbedded in $\chi_{2b}(b,s)$ which would suggest the rather complicated nature to be expected of the missing (but small) terms of (46d).

Notice that the dominant part of the potential in the relativistic Schrödinger theory defined by Eqs. (45) is spherically symmetric and has a definite relation via Eqs. (45d) and (42) with the underlying field theory. Formal treatments[11, 12] of two-particle relativistic potential scattering show that an arbitrary nonlocal potential $U(\vec{r},\vec{p})$, where $\vec{p}$ is the c.m. momentum, is in general necessary. However, it has never been clear how the arbitrary potentials of Refs. 11, 12 were connected to the fundamental meson propagators. Our result suggests that a quite simple relativistic wave equation and potential combination duplicate the *leading terms* of a high-energy expansion of the full set of generalized ladder diagrams. The momentum dependence of the $U(\vec{r},\vec{p})$ appropriate to the relativistic wave equation (45) is apparently of order ϵ^2 and hence is expected to be small at high energy. The relativistic kinematics imbedded in Eq. (45) has been used in formulating the eikonal corrections to high-energy multiple diffraction theory.[13]

VI. CONCLUSIONS

The point of view upon which this work rests is that a well-defined high-energy, small-t, expansion of the relativistic scattering amplitude is generated by the relativistic eikonal expansion of the generalized ladder set of Feynman diagrams. This expansion has been carefully defined and evaluated in some detail to second order in corrections to the eikonal approximation in this paper.

Our results are suggested as a benchmark against which approximate descriptions of relativistic scattering should be tested. For example, one may examine the validity of various approximations to the exact kernel of either the Bethe-Salpeter or the Blankenbecler-Sugar equations by determining whether they produce the same high-energy expansion as the full set of generalized ladder diagrams. The validity of various relativistic wave equations and kinematics may be similarly tested.

Our results show that high-energy scattering has a strong resemblance to potential scattering; however, the second-order corrections to the eikonal limit at relativistic speeds show that the relative time plays a role.

APPENDIX A: FIRST-ORDER EIKONAL CORRECTION

To reduce Eq. (32) to obtain Eqs. (33) and (34), consider the four-vector

$$\begin{aligned}I^\mu&=\int_{-\infty}^{\infty}d\tau'\int_{\sigma}^{\infty}d\tau\,\partial^\mu D(b-2\kappa_1\tau+2\kappa_2\tau')\\&=(4k\sqrt{s})^{-1}\int_{-\infty}^{\infty}dy_0\int_{\gamma}^{\infty}dy_3\partial^\mu D(y)\ ,\end{aligned} \tag{A1}$$

where $\gamma=-v_2(y_0+2\sigma\sqrt{s})$, and using

$$y=(2\kappa_{20}\tau'-2\kappa_{10}\tau,\vec{b},\ -2k(\tau+\tau')) \tag{A2}$$

and

$$v_1=k/\kappa_{10},\quad v_2=k/\kappa_{20}\ . \tag{A3}$$

Perform a Fourier-transform of $D(y)$, shift y_3 to absorb v_2y_0, perform the y_0 integral to obtain a δ function, and finally perform the q_0 integral with the δ function and find

$$\begin{aligned}I^\mu=&(4k\sqrt{s})^{-1}(2\pi)^{-3}\\&\times\int_{-2v_2\sqrt{s}\sigma}^{\infty}dy_3\int d^3q(iq^\mu)e^{-i\vec{q}\cdot\vec{y}}\\&\times\tilde{D}(-(1-v_2^2)q_3^2-\vec{q}_1^{\,2})\ ,\end{aligned} \tag{A4}$$

where

$$q^{\mu}=(-v_2q_3,\vec{q})\ .$$

Scaling q_3 and y_3 by $(1-v_2^{\,2})^{1/2}$ and $1/(1-v_1^{\,2})^{1/2}$, respectively, and using Eq. (21b), we find

$$I^{\mu}=i\,\frac{m_1m_2}{k\sqrt{s}}\int_z^{\infty}dy_3\left(\frac{-v_2}{(1-v_2^{\,2})^{1/2}}\frac{\partial}{\partial y_3},\vec{\nabla}_b,\frac{1}{(1-v_2^{\,2})^{1/2}}\frac{\partial}{\partial y_3}\right)U(\vec{b},y_3)\ , \tag{A5}$$

where $z=-2v_2\sigma\sqrt{s}/(1-v_2^{\,2})^{1/2}$. Then use (34) to find

$$I^{\mu}=i\left(\frac{-v_2}{(1-v_2^{\,2})^{1/2}}\frac{\partial}{\partial z},\vec{\nabla}_b,\frac{1}{(1-v_2^{\,2})^{1/2}}\frac{\partial}{\partial z}\right)\chi_-(\vec{b},z)$$

$$\equiv i\,\Delta_2^{\mu}\,\chi_-(\vec{b},z)\ , \tag{A6}$$

which defines the differential operator Δ_2^{μ} (Δ_1^{μ} is given by replacing v_2 by $-v_1$). Similarly,

$$I'_{\mu}=\int_{-\infty}^{\sigma}d\tau'\int_{-\infty}^{\sigma}d\tau\,\partial_{\mu}D(b-2\kappa_1\tau+2\kappa_2\tau')$$

$$=i\left(\frac{-v_2}{(1-v_2^{\,2})^{1/2}}\frac{\partial}{\partial z},\ -\vec{\nabla}_b,\frac{-1}{(1-v_2^{\,2})^{1/2}}\frac{\partial}{\partial z}\right)\chi_+(\vec{b},z). \tag{A8}$$

Surprisingly, upon contracting, the Lorentz factors disappear, leaving

$$I^{\mu}I'_{\mu}=\vec{\nabla}\chi_+\cdot\vec{\nabla}\chi_-\ . \tag{A9}$$

The only remaining vestige of relativity lies in the kinematic factors, $m_1m_2/k\sqrt{s}$, used in the definition of $\chi_{\pm}$. Performing the parallel operations on the remaining term gives the same result except with $v_2\to-v_1$ in the Δ_{μ} operators. Finally, convert the σ,σ' integrations in (32) to a common z integration with multiplying factor

$$[(1-v_1^{\,2})^{1/2}/v_1+(1-v_2^{\,2})^{1/2}/v_2]/(2\sqrt{s})$$

$$=(m_1+m_2)/(2k\sqrt{s}) \tag{A10}$$

to obtain the result, Eq. (33).

APPENDIX B: FIRST-ORDER λ CORRECTION TO THE EIKONAL

Begin by forming the λ correction in the compact functional formalism:

$$-i(2\pi)^{-4}\mathcal{T}_{\lambda}^{(1)}\delta^4(k_1+k_2-k_1'-k_2')=\mathcal{K}[\langle k_1'|T_1^{(1)}|k_1\rangle\langle k_2'|T_2^{(0)}|k_2\rangle+(1\leftrightarrow2)]|_{A_1=A_2=0}\ . \tag{B1}$$

The λ correction to the single-particle external potential amplitude is found using Eq. (13):

$$\langle k_1'|T^{(1)}(A_1)|k_1\rangle=-(2\pi)^{-4}\int d^4x_1\,e^{-ik_1'\cdot x_1}\left[\exp\left(-i\int_0^{\infty}d\tau A(x_1-2\kappa_1\tau)\right)-1\right]$$
$$\times\lambda2k[p_z-k(\cos\tfrac{1}{2}\theta)]\left[\exp\left(-i\int_{-\infty}^{0}d\tau A(x_1-2\kappa_1\tau)\right)-1\right]e^{ik_1'\cdot x_1} \tag{B2}$$

with the c.m. frame used. Since $(p_z-k\cos\frac{1}{2}\theta)$ annihilates the plane wave, commute it past the bracket to give

$$\langle k_1'|T^{(1)}(A_1)|k_1\rangle=2\lambda k(2\pi)^{-4}\int d^4x_1e^{-i(k_1'-k_1)\cdot x_1}\int_{-\infty}^{0}d\tau\frac{\partial}{\partial z_1}A_1(x_1-2\kappa_1\tau)$$
$$\times\left[\exp\left(-i\int_{-\infty}^{\infty}d\tau A(x_1-2\kappa_1\tau\right)-\exp\left(-i\int_{-\infty}^{0}d\tau A(x_1-2\kappa_1\tau)\right)\right]. \tag{B3}$$

Using Eq. (4) operating with $\mathcal{K}$ yields, after the overall momentum-conserving δ function is extracted,

$$\mathcal{T}_{\lambda}^{(1)}=2\lambda k\int d^4y\,e^{iq\cdot y}\,\frac{d}{id\alpha}\int_{-\infty}^{0}d\tau\int_{\alpha}^{\infty}d\tau'\,\frac{\partial}{\partial y_3}D(y-2\kappa_1\tau+2\kappa_2\tau')$$
$$\times\left[\exp\left(-\int_{-\infty}^{\infty}d\tau\int_{\alpha}^{\infty}d\tau'D(y-2\kappa_1\tau+2\kappa_2\tau')\right)\right.$$
$$\left.-\exp\left(-\int_{-\infty}^{0}d\tau\int_{\alpha}^{\infty}d\tau'D(y-2\kappa_1\tau+2\kappa_2\tau)\right)\right]+(1\leftrightarrow2)\ . \tag{B4}$$

Letting $y = b - 2\kappa_1\sigma + 2\kappa_2\sigma'$ so $\partial/\partial y_3 = -k^{-1}(\partial/\partial\sigma + \partial/\partial\sigma')$ permits the y_0 and y_3 integrations to be performed:

$$\mathcal{T}^{(1)}_\lambda = -i\,\lambda 2k\sqrt{s}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} dt'\, D(\vec{b} - 2\kappa_1\sigma + 2\kappa_2 t')$$
$$\times\left[e^{i\chi_0(b)} - \exp\left(-\int_{-\infty}^{\sigma} dt \int_{-\infty}^{\infty} dt'\, D(b - 2\kappa_1 t + 2\kappa_2 t')\right)\right] + (1 \leftrightarrow 2)\,. \quad \text{(B5)}$$

Using $d/d\sigma$ to pull down the t' integral in the second term gives, upon integration,

$$\mathcal{T}^{(1)}_\lambda = -i2k\sqrt{s}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\lambda[-i\chi_0(b)e^{i\chi_0(b)} + e^{i\chi_0(b)} - 1] + (1 \leftrightarrow 2)\,. \quad \text{(B6)}$$

Because $\lambda \to 0$ as $\vec{q} \to 0$, the unit term in the bracket does not contribute giving Eq. (35) when added to the $1 \leftrightarrow 2$ term.

APPENDIX C: SECOND-ORDER EIKONAL CORRECTION

The second-order correction is written in the compact functional formalism as

$$-i(2\pi)^{-4}\mathcal{T}^{(2)}\delta^4(k_1 + k_2 - k_1' - k_2') = \mathcal{K}[\langle k_1'|T_1^{(2)}(A_1)|k_1\rangle\langle k_2'|T_2^{(0)}(A_2)|k_2\rangle$$
$$+ (1 \leftrightarrow 2) + \langle k_1'|T_1^{(1)}(A_1)|k_1\rangle\langle k_2'|T_2^{(1)}(A_2)|k_2\rangle]|_{A_1 = A_2 = 0}\,, \quad \text{(C1)}$$

corresponding, respectively, to the three terms

$$\mathcal{T}^{(2)} = \mathcal{T}^{(2)}_a + \mathcal{T}^{(2)}_b + \mathcal{T}^{(2)}_c\,. \quad \text{(C2)}$$

The evaluation of $\mathcal{T}^{(2)}$ will be presented in three steps. The first step is to evaluate $T^{(2)}(A)$ and then $\mathcal{T}^{(2)}_a$ ($\mathcal{T}^{(2)}_b$ follows by the replacement $v_1 \leftrightarrow -v_2$). Secondly $\mathcal{T}^{(2)}_c$ is computed, and finally the conglomerate of terms is reduced to Eq. (49). For efficiency the following shorthand will be used:

$$L_\infty(t) \equiv \int_{-\infty}^{\infty} dt, \quad L_+(\tau) \equiv \int_0^{+\infty} d\tau, \quad L_-(t') \equiv \int_{-\infty}^{0} dt', \quad L_{+\alpha}(t) \equiv \int_\alpha^\infty dt,$$
$$L_{-\alpha}(t) \equiv \int_{-\infty}^{\alpha} dt, \quad L_\tau(t) \equiv \int_0^\tau dt, \quad A_1(x,t) \equiv A_1(x - 2\kappa_1 t), \quad A_2(x,t') \equiv A_2(x - 2\kappa_2 t'), \quad \text{(C3)}$$
$$D(x,t,t') \equiv D(x - 2\kappa_1 t + 2\kappa_2 t')\,.$$

A. Evaluation of $\mathcal{T}^{(2)}_a$

To evaluate this term, the second-order correction to the external potential amplitude is needed. Using Eqs. (13) and (23) one finds

$$\langle k_1'|T^{(2)}(A)|k_1\rangle = \langle k_1'|AgNgNgA|k_1\rangle$$
$$= -i(2\pi)^{-4}\int_0^\infty d\tau \int d^4x\, e^{-ik_1'\cdot x}\exp[-iL_+(t)A(x,t)]$$
$$\times (p - k_1')\cdot(p - k_1)\exp[-iL_\tau(t)A(x, t-\tau)]$$
$$\times (p - k_1')\cdot(p - k_1)\exp[-iL_-(t)A(x, t-\tau)]e^{ik_1\cdot x}\,. \quad \text{(C4)}$$

Define $q \equiv k_1 - k_1'$ and commute the $L_-(t)$ phase to the left:

$$\langle k_1'|T^{(2)}(A)|k_1\rangle = i(2\pi)^{-4}\int_0^\infty d\tau \int d^4x\, e^{iq\cdot x}\exp[-iL_\infty(t)A(x,t)]$$
$$\times [p + q - \partial L_-(t)A(x,t)]\cdot[p - \partial L_-(t)A(x,t)]$$
$$\times [p + q - \partial L_-(t)A(x, t-\tau)]\cdot \partial L_-(t)A(x, t-\tau)\,. \quad \text{(C5)}$$

Assuming $A(x)$ vanishes on a surface $x \to \infty$, the following combination vanishes:

$$0 = \int d^4x\, e^{iq\cdot x}\exp[-iL_\infty(t)A(x,t)][p^\nu + q^\nu - \partial^\nu L_\infty(t)A(x,t)]F(x)\,.$$

Therefore, we have

$$\langle k_1'|T^{(2)}(A)|k_1\rangle = i(2\pi)^{-4}\int_0^\infty d\tau\int d^4x\, e^{iq\cdot x}\exp[-iL_\infty(t)A(x,t)]$$
$$\times\big(\partial^\nu L_+(t)A(x,t)[p_\nu-\partial_\nu L_-(t)A(x,t)][\partial^\mu L_+(t)A(x,t-\tau)][\partial_\mu L_-(t)A(x,t-\tau)]$$
$$+\{\partial^\nu L_+(t)A(x,t)[p_\nu-\partial_\nu L_-(t)A(x,t)],\, p^\mu$$
$$-\partial^\mu L_\infty(t)A(x,t-\tau)\}_-\,\partial_\mu L_-(t)A(x,t-\tau)\big). \qquad \text{(C6)}$$

Evaluation of the commutator terms gives Eq. (27). Next form the amplitude $\mathcal{T}_a^{(2)}$:

$$-i(2\pi)^{-4}\mathcal{T}_a^{(2)}(b,s)\delta^4(k_1'+k_2'-k_1-k_2)=\mathcal{K}\langle k_2'|T_2^{(0)}(A_2)|k_2\rangle\langle k_1'|T_1^{(2)}(A_1)|k\rangle\big|_{A_1=A_2=0}. \qquad \text{(C7)}$$

Let $\mathcal{K}$ first operate on $T_2^{(0)}(A_2)$ shifting its argument by

$$\int d^4y_1 D(y_1-x_2+2\kappa_2 t)\delta/\delta A_1(y_1).$$

But as $T_2^{(0)}(A_2)$ is in exponential form [analogous to Eq. (14)] the result is itself a shifting operator acting on the more difficult $T_1^{(2)}(A_1)$ term. Perform these shifting operations, transform to center-of-mass and relative coordinates, integrate over c.m. coordinates and use $y=b-2\kappa_1\sigma+2\kappa_2\sigma'$ to find

$$\mathcal{T}_a^{(2)}(b,s)=-i4k\sqrt{s}\int d^2b\, e^{i\vec q\cdot\vec b}\int_0^\infty d\tau\int_{-\infty}^\infty d\sigma\, e^{i\chi_0(b)}$$
$$\times\{-[L_\infty(t)L_{+\sigma}(t')\partial^\nu D(b,t',t)][L_\infty(t)L_{-\sigma}(t')\partial_\nu D(b,t',t)]$$
$$\times[L_\infty(t)L_{+\sigma}(t')\partial^\mu D(b,t'-\tau,t)][L_\infty(t)L_{-\sigma}(t')\partial_\mu D(b,t'-\sigma,t)]$$
$$+[L_\infty(t)L_{+\sigma}(t')\partial^\nu D(b,t',t)][L_\infty(t)L_{+\sigma}(t')\partial_\nu\partial^\mu D(b,t'-\tau,t)][L_\infty(t)L_{-\sigma}(t')\partial_\mu D(b,t'-\sigma,t)]$$
$$+[L_\infty(t)L_{+\sigma}(t')\partial^\nu D(b,t',t)][L_\infty(t)L_{-\sigma}(t')\partial_\nu D(b,t'-\tau,t)][L_\infty(t)L_{+\sigma}(t')\partial^\mu D(b,t'-\tau,t)]$$
$$-[L_\infty(t)L_{+\sigma}(t')\partial^\nu D(b,t',t)][L_\infty(t)L_\infty(t')\partial_\nu\partial^\mu D(b,t',t)][L_\infty(t)L_{-\sigma}(t')\partial_\mu D(b,t'-\tau,t)]$$
$$+[L_\infty(t)L_{-\sigma}(t')\partial^\nu D(b,t',t)][L_\infty(t)L_{+\sigma}(t')\partial_\nu\partial^\mu D(b,t',t)][L_\infty(t)L_{-\sigma}(t')\partial_\mu D(b,t'-\tau,t)]$$
$$+[L_\infty(t)L_{+\sigma}(t')\partial^\nu D(b,t',t)][L_\infty(t)L_{-\sigma}(t')\partial_\nu\partial^\mu D(b,t',t)][L_\infty(t)L_{-\sigma}(t')\partial_\mu D(b,t'-\tau,t)]$$
$$-[L_\infty(t)L_{+\sigma}(t')\partial^\mu\partial^\nu D(b,t',t)][L_\infty(t)L_{-\sigma}(t')\partial_\mu\partial_\nu D(b,t',t)]\}. \qquad \text{(C8)}$$

Using methods analogous to those used in Appendix A, transform (C8) to

$$\mathcal{T}_a^{(2)}(b,s)=-i4k\sqrt{s}\int d^2b\, e^{i\vec q\cdot\vec b}e^{i\chi_0(b)}(m_2{}^2/s)(2k)^{-2}$$
$$\times\Big\{-\int_{-\infty}^\infty dz\,\vec\nabla\chi_-(b,z)\cdot\vec\nabla\chi_+(b,z)\int_{-\infty}^z dz_1\vec\nabla\chi_-(b,z_1)\cdot\vec\nabla\chi_+(b,z_1)$$
$$+i\int_{-\infty}^\infty dz\,\vec\nabla\chi_-(b,z)\cdot\vec\nabla\Big[\int_{-\infty}^z dz_1\vec\nabla\chi_-(b,z_1)\cdot\vec\nabla\chi_+(b,z_1)\Big]$$
$$+i\int_{-\infty}^\infty dz\,\nabla_l[\nabla_m\chi_-(b,z)\nabla_m\chi_+(b,z)]\int_{-\infty}^z dz_1\nabla_l\chi_+(b,z_1)$$
$$-i\int_{-\infty}^\infty dz\,\nabla_l\chi_-(b,z)\nabla_l\nabla_m\chi_0(b)\int_{-\infty}^z dz_1\nabla_m\chi_+(b,z_1)$$
$$+\int_{-\infty}^\infty dz\,\nabla_l\nabla_m\chi_-(b,z)\int_{-\infty}^z dz_1\nabla_l\nabla_m\chi_+(b,z_1)\Big\}, \qquad \text{(C9)}$$

where l and m sum 1 to 3. $\mathcal{T}_b^{(2)}$ follows from (C9) by the replacement m_1 for m_2.

B. Computation of $\mathcal{T}_c^{(2)}$

As usual begin by forming $\mathcal{T}_c^{(2)}$ in the functional formalism

$$-i(2\pi)^{-4}\mathcal{T}_c^{(2)}(b,s)\delta^4(k_1+k_2-k_1'-k_2')=\mathcal{K}\langle k_1'|T_1^{(1)}(A_1)|k_1\rangle\langle k_2'|T_2^{(1)}(A_2)|k_2\rangle\big|_{A_1=A_2=0}. \qquad \text{(C10)}$$

Using Eq. (26) perform the $\mathcal{K}$ operation in the straightforward way setting $A_1=A_2=0$ at the end, integrate over c.m. coordinates, and set the relative coordinate $y=b-2\kappa_1\sigma+2\kappa_2\sigma'$ in the by now familiar way to obtain

$$\mathcal{T}_c^{(2)}(b,s)=-i4k\sqrt{s}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\exp[-L_\infty(t)L_\infty(t')D(b,t,t')]\int_{-\infty}^{\infty}d\sigma\int_{-\infty}^{\infty}d\sigma'$$
$$\times\{-[L_\infty(t)L_{+\sigma'}(t')\partial_\nu D(b,t,t')][L_\infty(t)L_{-\sigma'}(t')\partial^\nu D(b,t,t')]$$
$$\times[L_{-\sigma}(t)L_\infty(t')\partial^\mu D(b,t,t')][L_{+\sigma}(t)L_\infty(t')\partial_\mu D(b,t,t')]$$
$$+[L_\infty(t)L_{-\sigma'}(t')\partial^\nu D(b,t,t')][L_{-\sigma}(t)L_{+\sigma'}(t')\partial_\nu\partial^\mu D(b,t,t')][L_{+\sigma}(t)L_\infty(t')\partial_\mu D(b,t,t')]$$
$$+[L_\infty(t)L_{+\sigma'}(t')\partial^\nu D(b,t,t')][L_{-\sigma}(t)L_{-\sigma'}(t')\partial_\nu\partial^\mu D(b,t,t')][L_{+\sigma}(t)L_\infty(t')\partial_\mu D(b,t,t')]$$
$$+[L_\infty(t)L_{-\sigma'}(t')\partial^\nu D(b,t,t')][L_{+\sigma}(t)L_{+\sigma'}(t)\partial_\nu\partial^\mu D(b,t,t')][L_{-\sigma}(t)L_\infty(t')\partial_\mu D(b,t,t')]$$
$$+[L_\infty(t)L_{+\sigma'}(t')\partial^\nu D(b,t,t')][L_{+\sigma}(t)L_{-\sigma'}(t')\partial_\nu\partial^\mu D(b,t,t')][L_{-\sigma}(t)L_\infty(t')\partial_\mu D(b,t,t')]$$
$$-[L_{-\sigma}(t)L_{+\sigma'}(t')\partial_\mu\partial_\nu D(b,t,t')][L_{+\sigma}(t)L_{-\sigma'}(t')\partial^\mu\partial^\nu D(b,t,t')]$$
$$-[L_{+\sigma}(t)L_{+\sigma'}(t')\partial_\mu\partial_\nu D(b,t,t')][L_{-\sigma}(t)L_{-\sigma'}(t')\partial^\mu\partial^\nu D(b,t,t')]\}\,. \qquad \text{(C11)}$$

Further reduction of $\mathcal{T}_c^{(2)}$ is left to the next section.

C. Reduction of $\mathcal{T}_a^{(2)}+\mathcal{T}_b^{(2)}+\mathcal{T}_c^{(2)}$

Begin by considering the terms in $\mathcal{T}_a^{(2)}$, $\mathcal{T}_b^{(2)}$, and $\mathcal{T}_c^{(2)}$ of fourth degree in the causal propagator, $D(x)$. The first term in Eq. (C11) is such a term and is reduced by the methods of Appendix A to

$$-i4k\sqrt{s}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}e^{i\chi_0(b)}(m_1m_2/s)(2k)^{-2}\left[i\int_{-\infty}^{\infty}dz\,\vec{\nabla}\chi_-(b,z)\cdot\vec{\nabla}\chi_+(b,z)\right]^2. \qquad \text{(C12)}$$

This is combined with the fourth-degree terms in $\mathcal{T}_a^{(2)}$ and $\mathcal{T}_b^{(2)}$, (C9) by using the equality

$$-\int_{-\infty}^{\infty}dz\,\vec{\nabla}\chi_-(b,z)\cdot\vec{\nabla}\chi_+(b,z)\int_{-\infty}^{z}dz_1\vec{\nabla}\chi_-(b,z_1)\cdot\vec{\nabla}\chi_+(b,z_1)=\frac{1}{2}\left[i\int_{-\infty}^{\infty}dz\,\vec{\nabla}\chi_-(b,z)\cdot\vec{\nabla}\chi_+(b,z)\right]^2. \qquad \text{(C13)}$$

Using Eq. (33), the degree-four terms combine to give the first term in Eq. (37).

In considering the terms of degree three in $D(x)$ first examine the degree-three terms of Eq. (C9). Begin by writing $\chi_0(b)$ as $\chi_+(b,z)+\chi_-(b,z)$, then Fourier-transform and do all line integrals, adding or subtracting a small imaginary part where necessary to ensure convergence. Let $Y_a(b)$ denote the essence of these third-degree terms defined by

$$[\text{Degree three terms of } \mathcal{T}_a^{(2)}]=-i4k\sqrt{s}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}e^{i\chi_0(b)}Y_a(b)\,,$$

then

$$Y_a(b)=i(m_2{}^2/s)(2k)^{-2}\int\frac{d^3q'd^3q''d^3q'''}{(2\pi)^9}(\vec{q}'\cdot\vec{q}'')(\vec{q}''\cdot\vec{q}''')(m_1m_2/k\sqrt{s})^3$$
$$\times\tilde{U}(q'^2)\tilde{U}(q''^2)\tilde{U}(q'''^2)\exp[-i(\vec{q}'+\vec{q}''+\vec{q}''')\cdot\vec{b}](2\pi)\,\delta(q_z'+q_z''+q_z''')$$
$$\times\left[\frac{1}{(q_z'+i\epsilon)(q_z''-i\epsilon)(q_z'''+i\epsilon)^2}+\frac{1}{(q_z'-i\epsilon)(q_z''-i\epsilon)(q_z'''+i\epsilon)^2}\right.$$
$$\left.-\frac{1}{(q_z'-i\epsilon)(q_z''+i\epsilon)(q_z'''+i\epsilon)(q_z''+q_z'''+i\epsilon)}-\frac{1}{(q_z'-i\epsilon)(q_z''-i\epsilon)(q_z'''+i\epsilon)(q_z''+q_z'''+i\epsilon)}\right]. \qquad \text{(C14)}$$

Use the δ function to evaluate the q_z'' integral. Next let $q_z'\to -q_z'$ and $q_z'''\to -q_z'''$ on the third fraction, then rearrange the fractions to obtain

$$Y_a(b)=i(m_2{}^2/s)(2k)^{-2}(2\pi)^{-8}\int d^3q'd^2q''d^3q'''(\vec{q}'\cdot\vec{q}'')(\vec{q}''\cdot\vec{q}''')(m_1m_2/k\sqrt{s})^3$$
$$\times\tilde{U}(q'^2)\tilde{U}(q''^2)\tilde{U}(q'''^2)\frac{1}{(q_z'''+i\epsilon)^2}\left[\frac{1}{(q_z'+i\epsilon)^2}+\frac{1}{(q_z'-i\epsilon)^2}\right]. \qquad \text{(C15)}$$

Now retransform the fractions back into line integrals, and reinvert the Fourier transforms to find

$$Y_a(b)=(m_2{}^2/s)(2k)^{-2}\int_{-\infty}^{\infty} dz\,(m_1m_2/k\sqrt{s})\nabla_l\nabla_m U(b,z)$$
$$\times\left[\int_z^{\infty} dz\,\nabla_l\chi_-(\vec{b},z)+\int_{-\infty}^{z} dz\,\nabla_l\chi_+(b,z)\right]\int_{-\infty}^{z} dz'\nabla_m\chi_+(z')\,, \tag{C16}$$

where l and m are summed $1\to 3$. The corresponding term for the degree-three terms of $\mathcal{T}_b^{(2)}$ is given by replacing m_2 by m_1.

Next consider $Y_c(b)$ defined by

$$[\text{Degree-three terms of } \mathcal{T}_c^{(2)}]=-i4k\sqrt{s}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}e^{i\chi_0(b)}Y_c(b)\,.$$

After Fourier-transforming and evaluating all line integrals, find

$$Y_c(b)=(16\kappa_{10}\kappa_{20}k\sqrt{s})^{-2}(2\pi)^{-8}\int d^2q'\int d^4q''\int d^2q'''\,e^{-i(\vec{q}_\perp'+\vec{q}_\perp''+\vec{q}_\perp''')\cdot\vec{b}}$$
$$\times\tilde{D}\left[-\frac{1-v_1{}^2}{(v_1+v_2)^2}(q_0''+v_2q_z'')^2-\vec{q}_\perp'\cdot\vec{q}_\perp'\right]\tilde{D}(q''^2)\tilde{D}\left[-\frac{1-v_2{}^2}{(v_1+v_2)^2}(q_0''-v_1q_z'')^2-\vec{q}_\perp'''\cdot\vec{q}_\perp'''\right]$$
$$\times\left(\frac{-v_1}{v_1+v_2}q_0''^2+\frac{1-v_1v_2}{v_1+v_2}q_0''q_z''+\frac{v_2}{v_1+v_2}q_z''^2-\vec{q}_\perp'\cdot\vec{q}_\perp''\right)$$
$$\times\left(\frac{-v_2}{v_1+v_2}q_0''^2-\frac{1-v_1v_2}{v_1+v_2}q_0''q_z''+\frac{v_1}{v_1+v_2}q_z''^2-\vec{q}_\perp''\cdot\vec{q}_\perp'''\right)$$
$$\times\left[\frac{1}{(q_0''+v_2q_z''+i\epsilon)^2}+\frac{1}{(q_0''+v_2q_z''-i\epsilon)^2}\right]\left[\frac{1}{(q_0''-v_1q_z''+i\epsilon)^2}+\frac{1}{(q_0''-v_1q_z''-i\epsilon)^2}\right]. \tag{C17}$$

Next define

$$p'=\left(\frac{-v_1}{v_1+v_2}(q_0''+v_2q_z''),\vec{q}_\perp',\frac{-1}{v_1+v_2}(q_0''+v_2q_z'')\right)\,, \tag{C18a}$$

and

$$p'''=\left(\frac{-v_2}{v_1+v_2}(q_0''-v_1q_z''),\vec{q}_\perp''',\frac{1}{v_1+v_2}(q_0''-v_1q_z'')\right)\,, \tag{C18b}$$

and use $p_z'\leftrightarrow p_z'''$ symmetry to obtain

$$Y_c(b)=2[16\kappa_{10}\kappa_{20}k\sqrt{s}(v_1+v_2)]^{-2}(2\pi)^{-8}\int d^2p'\int d^4q''\int d^2p'''\exp[-i(\vec{p}_\perp'+\vec{q}_\perp''+\vec{p}_\perp''')\cdot\vec{b}]$$
$$\times(p'\cdot q'')(q''\cdot p''')\tilde{D}(p'^2)\tilde{D}(q''^2)\tilde{D}(p'''^2)$$
$$\times\left[\frac{1}{(p_z'+i\epsilon)^2}+\frac{1}{(p_z'-i\epsilon)^2}\right]\frac{1}{(p_z'''+i\epsilon)^2}\,. \tag{C19}$$

Now backtrack; insert p_z' and p_z''' integrals along with the Fourier integral representations (in y_1 and y_2) of the appropriate δ functions according to (C18), then reintroduce the line integrals for the transformed fractions. Next scale p_z' and p_z''' by $(1-v_1{}^2)^{-1/2}$ and $(1-v_2{}^2)^{-1/2}$, respectively, and finally let

$$t=(y_2-y_1)/(v_1+v_2)\quad\text{and}\quad z=(v_2y_1+v_1y_2)/(v_1+v_2) \tag{C20}$$

to obtain

$$Y_c(b)=i(2m_1m_2/s)(2k)^{-2}\left[\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dz\,(4k\sqrt{s})^{-1}i\partial_\mu\partial_\nu D(t,b,z)\int_{z_1}^{\infty}dz'\,\Delta_1^\mu\chi_-(\vec{b},z')+\int_{-\infty}^{z_1}dz'\Delta_1^\mu\chi_+(\vec{b},z')\right]$$
$$\times\int_{-\infty}^{z_2}dz'\,\Delta_2^\nu\chi_+(\vec{b},z')\,, \tag{C21}$$

where $z_1=y_1(1-v_1{}^2)^{-1/2}$, $z_2=y_2(1-v_2{}^2)^{-1/2}$, and where Δ_1 and Δ_2 are defined by Eq. (A6). When combined with Y_a and Y_b, (C16), and when the Δ^μ operators inside the z' integrals are converted to ∂^μ operators outside the integrals, one obtains the degree-three term, $i\bar{\chi}_2$, Eq. (37d).

The second-degree terms of $\mathcal{T}_a^{(2)}$ and $\mathcal{T}_b^{(2)}$ are already in the proper form so only the degree-two terms of $\mathcal{T}_c^{(2)}$ need be considered. Examine $H_2(b)$ defined by

$$[\text{Degree-two terms of } \mathcal{T}_c^{(2)}] = i4k\sqrt{s}\int d^2b\, e^{i\vec{q}\cdot\vec{b}} e^{i\chi_0(b,s)} H_2(b)\,.$$

As usual begin by Fourier-transforming, next perform the line integrals giving two δ functions and a fourth-degree fraction

$$H_2(b) = (2\pi)^{-8}\int d^4q \int d^4q' e^{-i(\vec{q}_\perp+\vec{q}'_\perp)\cdot\vec{b}}(q^\mu q'_\mu)^2\tilde{D}(q^2)\tilde{D}(q'^2)(2\pi)^2\delta(2\kappa_1(q+q'))\delta(2\kappa_2(q+q'))$$
$$\times\left[\frac{1}{(2\kappa_1 q+i\epsilon)(2\kappa_2 q-i\epsilon)(2\kappa_1 q'-i\epsilon)(2\kappa_2 q'+i\epsilon)}+\frac{1}{(2\kappa_1 q-i\epsilon)(2\kappa_2 q-i\epsilon)(2\kappa_1 q'+i\epsilon)(2\kappa_2 q'+i\epsilon)}\right]. \quad \text{(C22)}$$

Use the δ functions to evaluate the q_0' and q_z' integrals, then rewrite the fractions to obtain

$$H_2(b) = -(4k\sqrt{s})^{-1}(4\kappa_{10}\kappa_{20})^{-2}(2\pi)^{-6}\int d^2q_\perp \int d^2q'_\perp e^{-i(\vec{q}_\perp+\vec{q}'_\perp)\cdot\vec{b}}\int dq_0\int dq_z(-q_0^2+q_z^2-\vec{q}_\perp\cdot\vec{q}'_\perp)^2\tilde{D}(q^2)\tilde{D}(q_0^2-q_z^2-\vec{q}_\perp\cdot\vec{q}'_\perp)$$
$$\times\frac{1}{(q_z-i\epsilon)^3}\frac{\partial}{\partial v_1}\frac{\partial}{\partial v_2}\frac{1}{v_1+v_2}\left(\frac{2}{q_0-v_1q_z+i\epsilon}-\frac{1}{q_0+v_2q_z+i\epsilon}-\frac{1}{q_0+v_2q_z-i\epsilon}\right). \quad \text{(C23)}$$

Reverse the order in which the parametric differentiation and momentum integration are performed and use the relation

$$\frac{1}{x\pm i\epsilon}=\frac{P}{x}\mp i\pi\delta(x)$$

applied to the fractions within the bracket. The principal-value part vanishes if a Yukawa or superposition of Yukawas is taken for the form of $D(q^2)$, and so the δ-function part only is considered:

$$H_2(b) = i(2k_1\cdot k_2/s)(2k)^{-2}(4k\sqrt{s})^{-2}(2\pi)^{-5}\int d^2q_\perp\int d^2q'_\perp e^{-i(\vec{q}_\perp+\vec{q}'_\perp)\cdot\vec{b}}$$
$$\times\int d\eta\,\tilde{D}(-\eta^2-q_\perp^2)\tilde{D}(-\eta^2-q_\perp'^2)(\eta^2-\vec{q}_\perp\cdot\vec{q}'_\perp)^2(\eta-i\epsilon)^{-3}. \quad \text{(C24)}$$

Introduce an η' integration using a δ function to maintain the equality and find

$$H_2(b) = i(2k_1\cdot k_2/s)(2k)^{-2}(4k\sqrt{s})^{-2}(2\pi)^{-5}\int_{-\infty}^{\infty}dz\int d^2q_\perp\int d^2q'_\perp\, e^{-i(\vec{q}_\perp+\vec{q}'_\perp)\cdot\vec{b}}$$
$$\times\int_{-\infty}^{\infty}d\eta\int_{-\infty}^{\infty}d\eta'\, e^{-i(\eta+\eta')z}\tilde{D}(-\eta^2-q_\perp^2)\tilde{D}(-\eta'^2-q_\perp'^2)[-\eta\eta'-\vec{q}_\perp\cdot\vec{q}'_\perp]^2(\eta'+i\epsilon)^{-2}(\eta-i\epsilon)^{-1}. \quad \text{(C25)}$$

The fractions may now be returned to line integral form and the Fourier transforms inverted to find

$$H_2(b) = (2k_1\cdot k_2/s)(2k)^{-2}\int_{-\infty}^{\infty}dz\,\nabla_l\nabla_m\chi_-(\vec{b},z)\int_{-\infty}^{z}dz'\,\nabla_l\nabla_m\chi_-(\vec{b},z') \quad \text{(C26)}$$

using (35) and (21b), and where l and m are summed 1 → 3. Combining the degree-two terms of $T_a^{(2)}$ and $T_b^{(2)}$ with (C26) gives (37b) when the $(2k)^{-2}$ factor is extracted.

This completes the derivation of (37).

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