Free field theories of spin-mass trajectories and quantum electrodynamics in the null plane

George R. Bart
Harry S. Truman College, Chicago, Illinois 60640

Stanley Fenster*
High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439
(Received 25 July 1977)

The ten generators of the Poincaré algebra for quantum electrodynamics and other gauge field theories are given in the null plane such that they all explicitly correspond, in the free-field case, to the Bacry-Chang group-theoretic forms. The internal oscillator content is extracted for both gauge theories and dual resonance models. In contrast to manifestly covariant or other theories, Bacry-Chang–type generators have the advantages of not referring to dependent spin components and of being rational in the canonical variables. The last property implies a simple position-space representation. Since the forms are independent of spin magnitude and allow inclusion of charge quantum numbers at will, they seem to represent an advantageous free-particle starting point for a hadron field theory with positive spin-mass trajectories (SMT) and with interaction. The interaction terms from manifestly covariant theories are considered in the null plane and found to be cubic and quartic in the fields. A straightforward extension of these interactions to SMT has not been found. The dual model, however, encompasses SMT and is known to have interactions even though the full details of the model's interaction terms are not worked out here. Consequently, the approach indicates how a realistic spectrum might be achieved without composite hadrons and incorporating full Poincaré invariance.

I. INTRODUCTION

Experiment indicates that hadron resonances may come in infinite spin-mass trajectories. Unhappily, the construction of a rigorously realistic theory of interacting trajectories seems to be an arduous task. In the years 1966–1968 infinite-component wave equations were examined and attempts were made to fit their spectra to hadrons[1] and to the hydrogen atom.[2] The latter succeeded, but the former suffered from spacelike solutions[3] and/or unrealistic spin-mass spectra. It was never proved, however, that these disadvantages could not be overcome by more complex second-order equations.[4] Contemporaneously, the infinite-momentum-frame method was discovered[5] and its relation to Dirac's "front form" realized.[6] Some theorists comprehended that the null-plane (NP) approach might lead to a relativistically covariant theory containing a positive mass spectrum.[7] Thus NP field theory was devised and applied to QED,[8-11] to a $\overline{\psi}\psi\phi$ model,[12] to fermion—massive-boson theories,[13-14] and to Yang-Mills theories.[15,16] Advantages of the NP approach were clearly perceived in this work. For example, in numerical QED calculations of the fourth-order contributions to the magnetic moment of the electron, the inherent elimination of redundant field components and the avoidance of subsidiary conditions reduced the computer time by a factor 2 to 5.[17]

Although the experimental evidence for purely linear Regge trajectories is meager, they are an appealing simplification from some theoretical viewpoints. In fact, in the dual model, which evolved during the last decade, they are an essential ingredient.[18] Several authors have analyzed the dual model as an NP field theory[19-21]—the first one to incorporate infinitely rising trajectories. Relativistic covariance was proved ingeniously.[22] The method of proof, however, did not exhibit ten quantized Poincaré group generators with interaction terms and without redundant components. No one could incorporate currents with rigor and it seemed that 26 transverse space dimensions were required.[23,24]

Also during this period, Dirac and others advanced the first quantized models having linearly rising positive spin-mass spectra.[25] An NP formulation of the Dirac model was devised and interpreted as describing a composite relativistic object possessing two constituents interacting via action-at-a-distance forces.

In this paper we illustrate how an NP prescription elegantly describes trajectories with positive mass spectra. First, however, we examine some aspects of the NP technique which seem likely to provide a theory of hadrons. In Sec. II, the universal form of the Poincaré generators appropriate to the NP technique is derived.[9,26,27] In Sec. III, all standard free field theories are reduced to a second-quantized version of the universal form. It is pointed out that the number of field components does not appear. Since fields for infinitely rising trajectories must have an infinite number of components, this is a crucial advantage in the construction of a trajectory field theory. In Sec. IV, a free-field theory of spin-mass trajectories is constructed by invoking the method of dynamical groups to replace the mass-squared and the spin symbols by products of harmonic oscillators. Several examples of internal spaces are mentioned, ranging from the simplest, the Dirac model, to the most complex, the dual model. In Sec. V, we show how the isospin-type groups can be introduced easily; we describe how the SU(6) × 0(3) model of hadrons is naturally built with the NP Hamiltonian technique. In Sec. VI, the modification of the Poincaré generators is discussed when interaction is present. Vertices and seagulls are presented for QED and the general kinematic structure is described. The zero-slope limit of the dual-model vertex is calculated and compared with those of standard field theories. The dual-model seagull term has not yet been explicitly deduced but should be extractable from a certain known diagram. Section VII is a statement of the algebraic problem posed by a hadron theory in NP form. In Appendix A we derive the form of the internal-coordinate angular momentum operators j_a in terms the components of the Pauli-Lubanski vector W_μ. The reduction of a vector-gluon field theory to NP form is sketched in Appendix B.

II. NULL-PLANE REPRESENTATION OF THE POINCARÉ GENERATORS

Denote the generators in covariant tensorial form by $P_\mu, M_{\mu\nu}$ ($\mu = 0,1,2,3$) with metric $(g_{00}, g_{11}, g_{22}, g_{33}) = (1,-1,-1,-1)$. The commutation relations are

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\rho] = ig_{\nu\rho}P_\mu - ig_{\mu\rho}P_\nu,$$
$$[M_{\mu\nu}, M_{\rho\kappa}] = ig_{\nu\rho}M_{\mu\kappa} + ig_{\mu\kappa}M_{\nu\rho} - ig_{\nu\kappa}M_{\mu\rho} - ig_{\mu\rho}M_{\nu\kappa}. \tag{2.1}$$

The Pauli-Lubanski vector is the sum of products

$$W_\mu = -\tfrac{1}{2}\epsilon_{\mu\nu\kappa\lambda}P^\nu M^{\kappa\lambda}, \tag{2.2}$$

obeying

$$[W_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, W_\lambda] = ig_{\nu\lambda}W_\mu - ig_{\mu\lambda}W_\nu,$$
$$[W_\mu, W_\nu] = i\,\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma. \tag{2.3}$$

The Casimir operators are P^2 and W^2, and it is true that $W \cdot P = 0$. It is convenient to introduce three-vectors J_a, K_a via

$$M_{ab} = \epsilon_{abc}J_c, \quad M_{a0} = K_a \quad (a = 1,2,3) \tag{2.4}$$

satisfying

$$[J_a, J_b] = i\epsilon_{abc}J_c, \quad [J_a, K_b] = i\,\epsilon_{abc}K_c,$$
$$[K_a, K_b] = -i\epsilon_{abc}J_c, \tag{2.5}$$

in terms of which

$$W_0 = P_aJ_a, \quad W_a = P_0J_a + \epsilon_{abc}P_bK_c. \tag{2.6}$$

These vectors are stepping stones to the special combinations used in null-plane theory.

Unfortunately, there are a variety of notations and normalizations used in the literature for these quantities. Our choice emerges from a unitary transformation selected so that the metric $g_{\mu\nu}$ is transformed to

$$\hat{g}_{\mu\nu} = U_\mu{}^\rho g_{\rho\sigma}U^\sigma{}_\nu = \hat{g}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.7}$$

where the real matrix U is given by

$$U^\mu{}_\nu = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 0 & -1\sqrt{2} \end{pmatrix}. \tag{2.8}$$

Under this transformation a vector A^μ is transformed to $\hat{A}^\mu = U^\mu{}_\nu A^\nu$. To avoid an excess of carets, it is convenient to use A^+, A^-, A_+, A_- instead of $\hat{A}^0, \hat{A}^3, \hat{A}_0, \hat{A}_3$, respectively. Thus

$$A^+ = A_- = \hat{A}^0 = \hat{A}_3 = (A^0 + A^3)/\sqrt{2}$$

and

$$A^- = A_+ = \hat{A}^3 = \hat{A}_0 = (A^0 - A^3)/\sqrt{2}.$$

Note $A^\mu B_\mu = A_+B_- + A_-B_+ - A_iB_i$. Tensors are treated in a similar way and we define

$$F_i = M_{i+} = (1/\sqrt{2})(K_i - \epsilon_{ij}J_j),$$
$$G_i = M_{i-} = (1/\sqrt{2})(K_i + \epsilon_{ij}J_j). \tag{2.9}$$

Then we have

$$W_i = P_-\epsilon_{ij}F_j - P_+\epsilon_{ij}G_j + K_3\epsilon_{ij}P_j,$$
$$W_+ = P_+J_3 + \epsilon_{ij}P_iF_j, \tag{2.10}$$
$$W_- = -P_-J_3 + \epsilon_{ij}G_iP_j.$$

The commutation relations of the generators are

$$[K_3,P_\pm]=\pm iP_\pm,\quad [J_3,F_i]=i\epsilon_{ij}F_j,$$
$$[K_3,F_i]=iF_i,\quad [J_3,G_i]=i\epsilon_{ij}G_j,$$
$$[K_3,G_i]=-iG_i,$$
$$[F_i,P_-]=iP_i,\quad [G_i,P_+]=iP_i, \tag{2.11}$$
$$[F_i,P_j]=i\delta_{ij}P_+,\quad [G_i,P_j]=i\delta_{ij}P_-,$$
$$[F_i,G_j]=i\delta_{ij}K_3-i\epsilon_{ij}J_3.$$

Additionally, we have

$$[K_3,W_\pm]=\pm iW_\pm,\quad [J_3,W_i]=i\epsilon_{ij}W_j,$$
$$[F_i,W_j]=i\delta_{ij}W_+,\quad [F_i,W_-]=iW_i,$$
$$[G_i,W_+]=iW_i,\quad [G_i,W_j]=i\delta_{ij}W_-,$$
$$[W_+,W_i]=i\epsilon_{ij}(P_+W_j-W_+P_j), \tag{2.12}$$
$$[W_-,W_i]=i\epsilon_{ij}(W_-P_j-P_-W_j),$$
$$[W_1,W_2]=i(W_+P_- -W_-P_+),$$
$$[W_+,W_-]=i\epsilon_{ij}P_iW_j;$$

other commutators are zero.

The spin s and mass M defining an irreducible representation are related to the Casimir operators as follows:

$$M^2=P'^2,\quad -M^2s(s+1)=W'^2,$$

where P'^2 and W'^2 are the eigenvalues of the Casimir operators at the representation in question. It is a remarkable fact that the three quantities

$$j_i=\frac{1}{M}\left(W_i-\frac{W_+}{P_+}P_i\right),\quad j_3=\frac{W_+}{P_+} \tag{2.13}$$

satisfy the commutation relations of angular momentum

$$[j_a,j_b]=i\epsilon_{abc}j_c, \tag{2.14}$$

in the timelike case $M^2>0$. This form of the j_a and their commutator follow from the rest-frame expressions by the Wigner boosting procedure. The details are given in Appendix A. After substituting for W_μ from (2.10) one can solve (2.13) to express G_1, G_2, and J_3 in terms of j_a.

Consequently, we obtain the universal NP form of the Poincaré generators for the case $P^2>0$:

$$G_i=\tfrac{1}{2}\{Q_i,P_-\}+P_iQ_++\epsilon_{ik}(Mj_k+P_kj_3)/P_+,$$
$$J_3=Q_1P_2-Q_2P_1+j_3, \tag{2.15}$$

where

$$Q_i=F_i/P_+,\quad Q_+=\tfrac{1}{2}\{K_3,1/P_+\},\quad F_i=P_+Q_i,$$
$$P_-=\frac{P_iP_i+M^2}{2P_+},\quad K_3=\tfrac{1}{2}\{Q_+,P_+\}, \tag{2.16}$$

and

$$[Q_i,P_j]=i\delta_{ij},\quad [Q_+,P_+]=i.$$

The ordering seems to be unique up to a change of variable, which can give the order in Ref. 26. The ordering is *post facto* justified by the realization of the Poincaré group commutation relations. Notice that we may regard all the generators as given in terms of the canonical pairs (Q_+P_+), (Q_iP_i), and M and j_a. The latter two are in turn to be specified in terms of canonical internal coordinate pairs as will be discussed in Sec. IV. It is important to note that M and M^2 are allowed to be operators in this formulation, contrary to the case in other formulations[25] or in standard field theories. The latter are discussed in Sec. III.

To understand the significance of the j_a, one should examine the Casimir operators of the Poincaré group $P_\mu P^\mu$ and $W_\mu W^\mu$. When the former is put equal to M^2, the latter comes out $-M^2j_aj_a$; which shows that j_aj_a is a Lorentz-invariant quantity, the rest-frame spin. Also, j_3 is the only invariant of the seven-dimensional kinematic subgroup[28] (J_3,K_3,F_i,P_i,P_+) known as $P_{2,2}$.

III. STANDARD FREE-FIELD THEORIES IN THE NULL-PLANE HAMILTONIAN FORMALISM

Beginning with the covariant Lagrangian for QED, others have illustrated how NP coordinates and NP γ matrices are defined and used to completely eliminate redundant components from the Hamiltonian.[10,11] Similarly, expressions for the NP Poincaré generators without redundant components have been given for a model with interacting (pseudo-) scalar bosons and Dirac fermions.[12] We present for the first time all ten NP Poincaré generators for the vector-gluon model, which includes QED as a special case ($m_V=0$). Only the free field expressions are given in this section; the additional interaction terms are stated in Sec. VI. An outline of our derivation is given in the Appendix B; here we have merely summarized the results.

The fermion commutation relations for $x^-=y^-$ are

$$\{\Psi_\alpha(x),\Psi_\beta^\dagger(y)\}=\delta_{\alpha\beta}\delta^3(x-y),\quad \alpha,\beta=1,2$$
$$\{\Psi_\alpha(x),\Psi_\beta(y)\}=0, \tag{3.1}$$
$$\{\Psi_\alpha^\dagger(x),\Psi_\beta^\dagger(y)\}=0,$$

where $\delta^3(x-y)=\delta(x^+-y^+)\delta^2(x-y)$. The vector field,

$$A=\begin{pmatrix}A_1\\A_2\\A_3\end{pmatrix}, \tag{3.2}$$

has the commutation relations, for $x^-=y^-$,

$$[A_\alpha(x),A_\beta(y)]=-\frac{i}{4}\delta_{\alpha\beta}\epsilon(x^+-y^+)\delta^2(x-y),$$

$$\alpha, \beta = 1, 2, 3. \tag{3.3}$$

More conventional commutation relations result from the introduction of the auxiliary field $\bar{A} = (\bar{A}_1, \bar{A}_2, \bar{A}_3)$ defined by

$$\bar{A}_\alpha(x) = -i\partial_+ A_\alpha(x). \tag{3.4}$$

Differentiation of the preceding commutator, for $x^- = y^-$, yields

$$[A_\alpha(x), \bar{A}_\beta(y)] = \tfrac{1}{2}\delta_{\alpha\beta}\delta^3(x-y), \quad \alpha, \beta = 1, 2, 3. \tag{3.5}$$

Taking $x^- = 0$ our results are

$$\begin{aligned}
P_i &= \int d^2x dx^+ \mathcal{P}_i = \int d^2x dx^+ (\Psi^\dagger i\partial_i \Psi + \bar{A} i\partial_i A),\\
P_+ &= \int d^2x dx^+ \mathcal{P}_+ = \int d^2x dx^+ (\Psi^\dagger i\partial_+ \Psi + \bar{A} i\partial_+ A),\\
P_- &= \int d^2x dx^+ \mathcal{P}_- = \frac{1}{4i}\int d^2x dx^+ dy^+ \epsilon(x^+ - y^+)[\Psi^\dagger(x)(m^2 - \vec{\nabla}^2)\Psi(y) + \bar{A}(x)(m_V{}^2 - \vec{\nabla}^2)A(y)],\\
J_3 &= \int d^2x dx^+ (x_1 \mathcal{P}_2 - x_2 \mathcal{P}_1 + \Psi^\dagger j_3^f \Psi + \bar{A} j_3^V A),\\
K_3 &= -\int d^2x dx^+ (x^+ \mathcal{P}_+ + \tfrac{1}{2} i\, \Psi^\dagger \Psi),\\
F_i &= \int d^2x dx^+ x_i \mathcal{P}_+,\\
G_i &= -\int d^2x dx^+ \Big\{x^+ \mathcal{P}_i - x_i \mathcal{P}_- - \frac{1}{2i}\int dy^+ \epsilon(x^+ - y^+)[\Psi^\dagger(x)\epsilon_{ik}(j_3^f i\partial_k + j_k^f m - \tfrac{1}{2}\epsilon_{kj}\partial_j)\Psi(y)\\
&\qquad + \bar{A}(x)\epsilon_{ik}(j_3^V i\partial_k + j_k^V m_V)A(y)]\Big\}.
\end{aligned} \tag{3.6}$$

Here $j_a^f = \sigma_a/2$ and

$$j_1^V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad j_2^V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$j_3^V = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.7}$$

which satisfy $[j_a, j_b] = i\epsilon_{abc} j_c$. The unusual j_a^V representation comes naturally from Yan's choice of fields using the "null-plane gauge."[14]

The verification of the Poincaré group commutation relations is nontrivial because A_α ($\bar{A}_\alpha$) does not commute with itself. Diagonalization is achieved by a Fourier transformation with respect to x^+, x^1, x^2:

$$\begin{aligned}
\Psi_\alpha(x) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^2p \int_0^\infty \frac{dp_+}{(2p_+)^{1/2}}[b_\alpha(p)e^{ipx} + d_\alpha^\dagger(p)e^{-ipx}],\\
\Psi_\alpha^\dagger(x) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^2p \int_0^\infty \frac{dp_+}{(2p_+)^{1/2}}[b_\alpha^\dagger(p)e^{-ipx} + d_\alpha(p)e^{ipx}],\\
A_\alpha(x) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^2p \int_0^\infty \frac{dp_+}{2p_+}[a_\alpha(p)e^{ipx} + a_\alpha^\dagger(p)e^{-ipx}],
\end{aligned} \tag{3.8}$$

with

$$\begin{aligned}
\{b_\alpha(p), b_\beta^\dagger(q)\} &= (2\pi)^3 2p_+ \delta_{\alpha\beta}\delta^3(p-q),\\
\{b_\alpha(p), b_\beta(q)\} &= \{b_\alpha^\dagger(p), b_\beta^\dagger(q)\} = 0,\\
\{d_\alpha(p), d_\beta^\dagger(q)\} &= (2\pi)^3 2p_+ \delta_{\alpha\beta}\delta^3(p-q),\\
\{d_\alpha(p), d_\beta(q)\} &= \{d_\alpha^\dagger(p), d_\beta^\dagger(q)\} = 0,\\
\{a_\alpha(p), a_\beta^\dagger(q)\} &= (2\pi)^3 2p_+ \delta_{\alpha\beta}\delta^3(p-q),\\
\{a_\alpha(p), a_\beta(q)\} &= \{a_\alpha^\dagger(p), a_\beta^\dagger(q)\} = 0.
\end{aligned} \tag{3.9}$$

Note that if $f(p_+)$ is the x^+ Fourier transform of $f(x^+)$, then the transforms of

$$-\frac{i}{2}\int dy^+ \epsilon(x^+ - y^+) f(y^+) \text{ and } -\frac{1}{2}\int dy^+ |x^+ - y^+| f(y^+) \tag{3.10}$$

are, respectively, $f(p_+)/p_+$ and $f(p_+)/p_+^2$. Hence one can see by inspection that the free-field generators are in the universal form of Sec. II. One can also observe that charge quantum numbers and other group indices may be added; they result in a mere contraction over the representation index. Additionally, there is the crucial observation that in all cases the free fields occur in a matrix contraction whose form is independent of the spin. To get different spins one adopts different representations of the j_a. The method of dynamical groups introduces harmonic-oscillator representations of the j_a and M^2 with profound consequences.

IV. FREE-FIELD THEORY OF TRAJECTORIES

When the j_a and M^2 have been specified in terms of a set of harmonic oscillators, we say that an internal space of hadrons has been selected. It is a Fock space spanned by oscillators $a_{n,\alpha}$, $a^\dagger_{n,\alpha}$, where α is a charge-type index and n is a generalized Poincaré mode number, which has some space interpretation. The index n may be discrete, continuous, unbounded, or naturally partitioned into sets. The "simplest example"[8,25,29] contains two oscillators a and b with $[a,a^\dagger]=1$, $[b,b^\dagger]=1$, and

$$j_+\equiv j_1+ij_2=a^\dagger b\,,\quad j_-\equiv j_1-ij_2=ab^\dagger\,,\qquad j_3=\tfrac{1}{2}(a^\dagger a-b^\dagger b). \tag{4.1}$$

A familiar form for these is

$$j_a=\phi^\dagger\tfrac{1}{2}\sigma_a\phi\,, \tag{4.2}$$

where

$$\phi=\begin{pmatrix}a\\ b\end{pmatrix}. \tag{4.3}$$

The operator M^2, which must commute with j_a, is

$$M^2=\tfrac{1}{2}(a^\dagger a+b^\dagger+1)+m_0^{\,2}. \tag{4.4}$$

It is true as an operator identity that

$$j^2\equiv j_a j_a=(M^2-m_0^{\,2})^2-\tfrac{1}{4}\,, \tag{4.5}$$

with

$$j^2=\tfrac{1}{4}(a^{\dagger 2}a^2+b^{\dagger 2}b^2)+\tfrac{1}{2}a^\dagger ab^\dagger b+\tfrac{3}{4}(a^\dagger a+b^\dagger b)\,, \tag{4.6}$$

and m_0 a constant. The rest-frame space-directional significance of the oscillators is determined by their commutators with j_a. In the special model just mentioned a and b constitute a two-component spinor. In general their commutator with j_a may be a product of a's. Neither j_a nor M^2 need to be quadratic. If a standard field theory of spin-$\frac{1}{2}$ or spin -1 particles is expressed in the Schwinger formalism,[29] the spin part of the single-particle state contains only a single oscillator $a^\dagger_{n,\alpha}|0\rangle$; the representation of the Poincaré group is irreducible. For example, in the gluon model of Sec. III terms like $\frac{1}{2}\sigma_a$ which act on single-particle states

$$\begin{pmatrix}\Psi^\dagger_1(x)\\ \Psi^\dagger_2(x)\end{pmatrix}|0\rangle$$

correspond to j^f_a, given by Eq. (4.2), which acts on single-particle states

$$[\Psi^\dagger_1(x)a^\dagger+\Psi^\dagger_2(x)b^\dagger]|0\rangle.$$

The method of dynamical groups[2,29,30] generalizes the standard theory to trajectories by allowing any number of multiplications $a^\dagger_{n_1,\alpha_1}a^\dagger_{n_2,\alpha_2}\cdots a^\dagger_{n_k,\alpha_k}|0\rangle$ in the specification of a single-trajectory state. A reducible representation with all spins is thus obtained.

A second example is the well-known three-oscillator representation:

$$j_a=\epsilon_{abc}r_b p_c\,,\qquad [r_a,p_b]=i\delta_{ab}. \tag{4.7}$$

A general case with N oscillators has been studied.[30] Here is a cubic one[8]:

$$j_i=\tfrac{1}{4}\{\{r_k,p_k\},p_i\}-\tfrac{1}{4}\{p_kp_k-1,r_i\}\,,\qquad j_3=r_1p_2-r_2p_1. \tag{4.8}$$

The operator M^2 must be rotationally invariant. [Note, with the j_a in Eq. (4.8) the sum r_kr_k, for example, is *not*.] A function of j_aj_a always works, but by no means exhausts the possibilities.

Noteworthy is the dual model, whose cubic construction is different. There are twelve "kinds" of oscillators[30] for each "mode". A kind is labeled by κ $(=1,2,\ldots,12)$ and a mode by m $(=\pm1,\pm2,\ldots)$. Furthermore, for each pair of these there is a "transverse" label i $(=1,2)$. These oscillators $\alpha^{m,\kappa}_i$ satisfy

$$[\alpha^{m,\kappa}_i,\alpha^{n,\lambda}_j]=n\omega\delta^{m+n,0}\delta_{ij}\delta^{\kappa\lambda}. \tag{4.9}$$

Out of them we form operators

$$Mj_k=\frac{i}{2}\epsilon_{kl}\sum_{\substack{m=-\infty\\ m\neq0}}^{\infty}\sum_{\substack{n=-\infty\\ n\neq0\\ m+n\neq0}}^{\infty}\sum_{\kappa,\lambda=1}^{12}\frac{1}{n}\alpha^{n,\lambda}_l\alpha^{m,\kappa}_r\alpha^{-n-m,\kappa}_r\,,$$

$$j_3=-\frac{i}{\omega}\sum_{n=-\infty}^{+\infty}\sum_{\kappa=1}^{12}\frac{1}{n}\alpha^{n,\kappa}_1\alpha^{-n,\kappa}_2\,, \tag{4.10}$$

$$M^2=-\omega+\sum_{n=1}^{\infty}\sum_{\kappa=1}^{12}\alpha^{n,\kappa}_i\alpha^{-n,\kappa}_i\,.$$

Here ω is an arbitrary constant. Poincaré algebra closure is enforced through the relation

$$[Mj_1,Mj_2]=iM^2j_3\,. \tag{4.11}$$

The rotational property of α is unusual. A state is formed beginning with product-type operators (omit κ)

$$\alpha^{m_1}_1\alpha^{m_2}_1\alpha^{m_3}_1\cdots\alpha^{n_1}_2\alpha^{n_2}_2\alpha^{n_3}_2\cdots.$$

A sum of such operators is required to act on $|0\rangle$ to produce a state. The relation

$$[M^2,\alpha^{m,\kappa}_j]=m\omega\alpha^{m,\kappa}_j \tag{4.12}$$

shows that products of α with an identical superscript sum have a common mass. Only such products may, therefore, constitute the terms in a sum creating an eigenmass state. Starting, for example, with a state formed with a single product of α's, one may generate the partner (j_3) states by acting with step-up operators $Mj_+ \equiv Mj_1 + iMj_2$ and step-down operators $Mj_- \equiv Mj_1 - iMj_2$ using the relation

$$[Mj_k, \alpha_l^{m,\kappa}] = -i\omega \sum_{\substack{n=-\infty \\ n\neq 0 \\ m+n\neq 0}}^{\infty} \sum_{\lambda,\mu=1}^{12} \left(\tfrac{1}{2}\epsilon_{kl}\delta_{jr}\delta_{\lambda\mu} + \frac{m}{n}\epsilon_{kr}\delta_{lj}\delta_{\lambda\kappa}\right) \alpha_j^{n+m,\lambda}\alpha_r^{-n,\mu}\,. \tag{4.13}$$

The (mass)^2 of a state is the eigenvalue of M^2; after finding it one may find the spin by operating with $(Mj_1)^2 + (Mj_2)^2 + M^2j_3^2$ to obtain $M^2j(j+1)$ and thus j itself. We remark that, strictly speaking, the j_a are quadratic since M^2 and j_3 are quadratic, and Mj_i are cubic in oscillators. A second remark is that the above is not exactly the dual model in 26 space-time dimensions that one usually finds. Using the κ variable gives a larger spectrum,[30] but in four dimensions, as we always require. Note that choosing j_a and/or M^2 to be of higher order in oscillator variables does not yet introduce interactions.

To match the oscillator representation there is a set of basis states with appropriate spin indices. The δ function in the commutation relations should be suitably generalized. We eschew discussion of T, C, P.[31,32]

V. THE SU(6) × O(3) MODEL

For a more realistic choice of M^2 and j_a, we use the quark model as a guide. The three quarks in a baryon are labeled by an index ρ $(=1,2,3)$. The ρth quark is associated with a pair of oscillators a^ρ, b^ρ. Since there are two relative space-position variables for three quarks, two additional sets of three oscillators A_a and B_a $(a=1,2,3)$ are required.[33] These oscillators all commute with each other and satisfy $[a, a^\dagger]=1$. Replace A_a and B_a by Hermitian combinations

$$R_a = \frac{1}{\sqrt{2}}(A_a + A_a^\dagger)\,, \quad P_a = -\frac{i}{\sqrt{2}}(A_a - A_a^\dagger)\,,$$
$$r_a = \frac{1}{\sqrt{2}}(B_a + B_a^\dagger)\,, \quad P_a = -\frac{i}{\sqrt{2}}(B_a - B_a^\dagger)\,. \tag{5.1}$$

Then choose

$$j_k = \phi^{\rho\dagger}\tfrac{1}{2}\sigma_k\phi^\rho + \epsilon_{kij}(R_iP_j + r_ip_j)\,, \tag{5.2}$$

where

$$\phi^\rho = \begin{pmatrix} a^\rho \\ b^\rho \end{pmatrix}. \tag{5.3}$$

A simple choice for M^2 is[34]

$$M^2 = \tfrac{1}{2}(P_aP_a + p_ap_a) + \tfrac{1}{2}\omega^2(R_aR_a + r_ar_a)\,. \tag{5.4}$$

One specifies that only the ground state of a^ρ, b^ρ (which appear in j_a but not in M^2) may be occupied. Generalizations of M^2 include spin-orbit coupling (ϕ to R) and additional potentials $V(R_aR_a, r_ar_a)$; the j_a are left unchanged. Charge-type labels may be added in the usual way to obtain a realistic hadron spectrum. Evidently, the analog of quark-antiquark mesons may be constructed with any potential function of the interquark distance (R_aR_a). This model now contains the spectrum but not the interactions of the usual models. It is relativistic as it stands; a correct incorporation of interactions keeping this property is yet to be found (see below).

VI. INTERACTIONS

It has long been recognized that the introduction of an interaction changes only P_- and G_i which acquire terms proportional to powers of a coupling constant g that are cubic and higher in the fields. The free theory is recovered by putting $g=0$. Both sets of ten generators $G(g)$ and $G(0)$ must satisfy the Poincaré algebra.

If we define the interaction terms I_- and I_i by

$$P_-(g) = P_-(0) + I_-(g)\,,$$
$$G_i(g) = G_i(0) + I_i(g)\,, \tag{6.1}$$

then the interaction terms for the vector-gluon model described in Appendix B are

$$\begin{aligned} I_-(g) &= \int d^2x\,dx^+\,\mathcal{G}_-(x) \\ &= \frac{g}{2}\int d^2x\,dx^+\,dy^+\,\epsilon(x^+ - y^+)\{m_V\Psi^\dagger(x)\Psi(x)A_3(y) + \Psi^\dagger(x)\Psi(x)\partial_kA_k(y) + \partial_k[\Psi^\dagger(x)\Psi(x)A_k(y)] \\ &\qquad - \tfrac{1}{2}\Psi^\dagger(x)A_k(x)(\partial_k + i\epsilon_{kl}\sigma_3\partial_l + \epsilon_{kl}\sigma_l m)\Psi(y) + \tfrac{1}{2}\Psi^\dagger(x)(\overleftarrow{\partial}_k - i\epsilon_{kl}\sigma_3\overleftarrow{\partial}_l + \epsilon_{kl}\sigma_l m)\Psi(y)A_k(y)\} \\ &\quad + \frac{g^2}{4i}\int d^2x\,dx^+\,dy^+[\epsilon(x^+ - y^+)\Psi^\dagger(x)A_k(x)(\delta_{kl} + i\epsilon_{kl}\sigma_3)A_l(y)\Psi(y) - i|x^+ - y^+|\Psi^\dagger(x)\Psi(x)\Psi^\dagger(y)\Psi(y)]\,. \end{aligned} \tag{6.2}$$

By means of integration by parts, the arrangement of the differentiation arrows in Eq. (6.2) has been chosen so that I_i has the particularly simple expression

$$I_i(g)=\int d^2x dx^+ x_i \mathcal{J}_-(x). \qquad (6.3)$$

It should be stressed that these are exact expressions in g; they just happen to be quadratic. To our knowledge only $P_-(g)$ has appeared previously in the literature.[10,11,35] To identify with the NP formulation of QED in Ref. 11, one puts

$$m_V=0,\quad g=e,\quad A=(A_1,A_2),\quad \Psi_{\text{new}}=U\Psi,$$
$$U=\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}. \qquad (6.4)$$

(The third component of A decouples if $m_V=0$.) Also found in Ref. 11 is the momentum-space expression for P_-, $m_V=0$, which is discussed in terms of Feynman-type diagrams. In contradistinction to the quantities of Eq. (3.6), it is unclear how to generalize the QED expressions for I_- and I_i to allow Ψ and A to have an infinite number of spin components. For example, the term $\partial_k A_k$ presumably would generalize to $\partial_k \Gamma^{kjm} A_{jm}$, where Γ^{kjm} is some Clebsch-Gordan coefficient and the indices (j,m) run over the infinite number of spin components of A; when $j=1$, $\Gamma^{klm}=\delta^{km}$. However, the value of Γ^{kjm} for other $j\neq 1$ is obscure.

We remark that, as was also the case for $\bar\psi\psi\phi$ theory, the Poincaré group commutation relations must be verified by direct tedious computation. It would be more satisfactory if one could find an operator U which carries out the transformation from the set $G(0)$ to $G(g)$:

$$G(g)=U(g)G(0)U^{-1}(g). \qquad (6.5)$$

Since similarity transformations preserve commutation relations, the Poincaré algebra would automatically be maintained. We have tried and failed to find U for QED. This is a challenging, purely mathematical problem. It is also worthwhile because the solution might indicate how to construct U's for hadron theories.

An analysis of the structure of $P_-(g)$ for the $\bar\psi\psi\phi$, gluon, and dual models shows that, in general, the interaction Hamiltonian contains a linear and a quadratic term in g. The linear term couples three particles at a vertex and contains either a mass or the universal transverse two-vector[36]

$$\vec{\mathcal{P}}=\eta_1\vec{p}_2-\eta_2\vec{p}_1=\eta_2\vec{p}_3-\eta_3\vec{p}_2=\eta_3\vec{p}_1-\eta_1\vec{p}_3, \qquad (6.6)$$

where for the ith particle

$$p_i^\mu=\left(\frac{m_i^2+\vec{p}_i^{\,2}}{2\eta_i},\ \vec{p}_i,\eta_i\right)$$

and

$$\vec{p}_1+\vec{p}_2+\vec{p}_3=0,\quad \eta_1+\eta_2+\eta_3=0. \qquad (6.7)$$

The coefficients are functions of η_1,η_2,η_3. Similarly, the g^2 term couples four particles and depends on η_i ($i=1$ to 4) with $\sum_i\eta_i=0$, but it is independent of the $\vec{p}_i$. These terms, for the rather complex Yang-Mills Lagrangian, have been recently obtained by Casher.[16] Nonetheless, such covariant Lagrangian theories embody no trajectory. The only existing interacting field theory with a trajectory is the dual model. Unfortunately, the interaction Hamiltonian for this model has not yet been extracted, but steps have been taken in the right direction. We can at present understand completely the term linear in e and cubic in the fields by virtue of the zero-slope-limit concept applied to the three-Reggeon vertex. The relevant expression[19,21,37] is

$$\exp\left[\frac{1}{2}\sum_{r,s}\sum_{n,m}N_{rs}^{nm}\vec{a}_n^{\,r}\cdot\vec{a}_m^{\,s}+\sqrt{2}\vec{\mathcal{P}}\cdot\sum_r\frac{1}{\alpha_r}\sum_m f_m\left(-\frac{\alpha_{r+1}}{\alpha_r}\right)\vec{a}_m^{\,r}\right], \qquad (6.8)$$

where

$$N_{rs}^{nm}=-\frac{\alpha_1\alpha_2\alpha_3}{\alpha_r\alpha_s}\,\frac{mn}{n\alpha_r+m\alpha_s}\times f_m(-\alpha_{r+1}/\alpha_r)f_n(-\alpha_{s+1}/\alpha_r), \qquad (6.9)$$

and

$$f_n(\gamma)=\frac{1}{n\gamma}\binom{n\gamma}{n}. \qquad (6.10)$$

Here r $(=1,2,3)$ labels a particle at the vertex, $\vec{a}_n^{\,r}$ is an nth-mode oscillator, α_r is a longitudinal momentum $(\sum_{r=1}^3\alpha_r=0)$, and $\vec{\mathcal{P}}$ is discussed above. We expand the exponential, keeping only terms containing the product of $\vec{a}_1^{\,1}$, $\vec{a}_1^{\,2}$, and $\vec{a}_1^{\,3}$ and linear in each, thus retaining only the single excitations of the trajectory. Furthermore, we keep only linear terms in $\vec{\mathcal{P}}$, noting that $f_1(\gamma)=1$, and find

$$\sum_{\text{cyclic}}(\vec{\mathcal{P}}\cdot\vec{a}^{\,1})(\vec{a}^{\,2}\cdot\vec{a}^{\,3})$$

is the coefficient of the vertex isospin factor ϵ^{abc} in Yang-Mills theory. No extra factors of η are present.

It remains to check the e^2 term of the dual-model Hamiltonian. It has not yet been explicitly computed, but it corresponds to the instantaneous four-string vertex.[19-21,37] It should provide terms[16]

$$\frac{(\eta_1-\eta_2)(\eta_3-\eta_4)}{4(\eta_1+\eta_2)(\eta_3+\eta_4)}\,\delta_{j_1j_2}\delta_{j_3j_4}$$

and

$$\delta_{j_1j_3}\delta_{j_2j_4}-\delta_{j_1j_4}\delta_{j_2j_3}.$$

At present this has the status of a conjecture.

VII. CONCLUSIONS

The NP Hamiltonian method has several advantages, in comparison with the infinite-component wave-equation technique, which point towards its usefulness for devising an interacting trajectory theory of hadrons. The masses are all real, and rising spin-mass spectra are easily accommodated, e.g., see Eq. (4.4). Also, charge and other quantum numbers are accommodated straightforwardly. The distinction between fermions and bosons arises from the difference in commutation relations, and fermion antiparticles are indicated by an interchange of creation and annihilation operators in the momentum expansion of the field.

It is important to realize the significance of the internal oscillator coordinates in NP theories as opposed to internal (relative) Minkowski coordinates in theories of composite hadrons. The hadrons appearing in NP theories might be called "pseudocomposites" to point out the absence of a relative Minkowski coordinate together with a spin-mass multiplet structure. In both cases the usual center-of-mass Minkowski four-vector appears. Note that, in general, for a model of pseudocomposites to be relativistic, the internal oscillators must have unusual (but precise) Poincaré transformation properties.

Because no three-trajectory vertex was found for the infinite-component wave-equation system, no interactions of such particles are known. At the present time this is also true for the NP Hamiltonian formulation, but we have just begun to look for a vertex. If such a vertex is found it might lead to an NP theory of elementary pseudocomposite hadron trajectories. Perhaps it will be possible to obtain a vertex by constructing the operator U, of Sec. VI, in the form $U=\exp(i\Lambda)$ where Λ will be cubic and quartic in the fields with coefficients which are exponential functions of the oscillators and the universal vector $\vec{\mathcal{P}}$, as in the dual model.

We remark that if baryons are NP pseudocomposite objects, then currently existing procedures for interactions and couplings, including decays and electromagnetic processes, will not be relativistic. Also, parton ideas may not be applied to them.

ACKNOWLEDGMENT

One of us (S. F.) thanks Dr. Tohru Eguchi for illuminating discussions and a generous interest in this work. We also wish to thank Professor Norman Fuchs for informing us that Professor Davison Soper discussed massive QED in his 1971 Stanford thesis.

APPENDIX A

Here we derive, by the Wigner boosting method, the j_a expressions used in Sec. II. It is well known that if $P_\mu|p'\rangle=p'_\mu|p'\rangle$; $P_i|0\rangle=0$, and $P_+|0\rangle=2^{-1/2}M|0\rangle$, then $|p'\rangle$ may be expressed as

$$|p'\rangle=\exp\left(ip'_i\frac{F_i}{p'_+}\right)\exp(i\alpha'K_3)|0\rangle, \qquad \text{(A1)}$$

where $e^{\alpha'}=\sqrt{2}p'_+/M$. Since we may diagonalize one W_μ as well as all four P_μ, $|p'\rangle$ may also be chosen as an eigenstate of, say, W_+; thus we may write $|p',w'_+\rangle$. The two-by-two representation of these boosts is familiar. Put

$$J_a=\tfrac{1}{2}\sigma_a,\quad K_a=\tfrac{i}{2}\sigma_a \qquad \text{(A2)}$$

and find

$$e^{i\alpha'K_3}=\begin{pmatrix}e^{\alpha'/2} & 0\\ 0 & e^{-\alpha'/2}\end{pmatrix}=\begin{pmatrix}\left(\frac{p'_+\sqrt{2}}{M}\right)^{1/2} & 0\\ 0 & \left(\frac{M}{p'_+\sqrt{2}}\right)^{1/2}\end{pmatrix}. \qquad \text{(A3)}$$

Since

$$F_1=\begin{pmatrix}0 & 0\\ \frac{-i}{\sqrt{2}} & 0\end{pmatrix},\quad F_2=\begin{pmatrix}0 & 0\\ \frac{i}{\sqrt{2}} & 0\end{pmatrix}, \qquad \text{(A4)}$$

it follows that

$$\exp\left(\frac{i}{p'_+}\,p'_iF_i\right)=\begin{pmatrix}1 & 0\\ \frac{p'_1+ip'_2}{p'_+\sqrt{2}} & 1\end{pmatrix}, \qquad \text{(A5)}$$

and that[26]

$$\exp\left(i\frac{p'_i}{p'_+}F_i\right)\exp(i\alpha'K_3)=\begin{pmatrix}\frac{p'_+\sqrt{2}}{M} & 0\\ \frac{p'_1+ip'_2}{Mp'_+\sqrt{2}} & \frac{M}{p'_+\sqrt{2}}\end{pmatrix}\equiv B_T. \qquad \text{(A6)}$$

Any four-vector X_ρ satisfies

$$[M_{\mu\nu},X_\rho]=ig_{\nu\rho}X_\mu-ig_{\mu\rho}X_\nu, \qquad \text{(A7)}$$

and has its eigenvalues represented as a two-by-two matrix

$$X' = x_0' + x_a'\sigma_a , \tag{A8}$$

upon which the boost acts in the form $B_T X B_T^\dagger$. The two examples at hand are $X = P$ and $X = W$. In an arbitrary frame P is given by

$$P = \begin{pmatrix} \sqrt{2}\,p_+' & p_1' - ip_2' \\ p_1' + ip_2' & \sqrt{2}\,p_-' \end{pmatrix}; \tag{A9}$$

in the rest frame P has the form

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} .$$

It is easy to see that $M\ B_T B_T^\dagger = P$. The equation $W_+ = P_+ J_3 + \epsilon_{ij} P_i F_j$ shows that in this same rest frame w_+' is related to j_3' by $w_+' = 2^{-1/2} M\ j_3'$, where $J_3|0\rangle = j_3'|0\rangle$. Similarly, $W_i|0\rangle$ suggests that the set

$$\frac{\sqrt{2}}{M} W_+ ,\quad \frac{1}{M} W_i , \tag{A10}$$

might satisfy the angular momentum commutation relations when they act on rest states:

$$\left[\frac{\sqrt{2}\,W_+}{M}, \frac{W_i}{M} \right] |0\rangle = i\epsilon_{ij} \frac{W_j}{M} \ |0\rangle , \tag{A11}$$

$$\left[\frac{W_1}{M}, \frac{W_2}{M} \right] |0\rangle = i\, \frac{W_+\sqrt{2}}{M} \ |0\rangle . \tag{A12}$$

We can easily find three combinations of the Poincaré generators which obey the same commutation relations when acting on boosted $|p'w_+'\rangle$ states by applying the unitary transformation

$$U(p') = \exp\left(\frac{i}{p_+'}\ p_i' F_i \right)\ \exp(i\alpha' K_3) \tag{A13}$$

to the J_a, because

$$\begin{aligned} \langle 0 | J_a | 0 \rangle &= \langle 0 | U^{-1} U J_a U^{-1} U | 0 \rangle \\ &= \langle p' | U J_a U^{-1} | p' \rangle \\ &= \langle p' | j_a | p' \rangle . \end{aligned} \tag{A14}$$

Carrying out the transformation we find Eq. (2.14).

APPENDIX B

Here we sketch the derivation of Eqs. (3.6), (6.2), (6.3); starting with the massive vector-gluon model Lagrangian:

$$\begin{aligned} \mathcal{L} = &-\tfrac{1}{2} B^{\mu\nu}(\partial_\mu B_\nu - \partial_\nu B_\mu) + \tfrac{1}{4} B^{\mu\nu} B_{\mu\nu} + \tfrac{1}{2} m_V{}^2 B^\mu B_\mu \\ &+ \bar\psi'(\gamma^\mu \tfrac{1}{2} i \overleftrightarrow{\partial}_\mu - m)\psi' - g\bar\psi'\gamma_\mu\psi' B^\mu . \end{aligned} \tag{B1}$$

Our discussion is closely related to similar work in Ref. 14. The prime anticipates a particular phase transformation for ψ. By standard methods we construct the unsymmetrical energy-momentum tensor

$$\hat T^{\mu\nu} = -\hat g^{\mu\nu}\mathcal{L} + \hat B^{\lambda\mu}\partial^\nu \hat B_\lambda + \frac{i}{2}\,\bar\psi'\,\hat\gamma^\mu\,\overleftrightarrow{\partial}^\nu\psi' . \tag{B2}$$

The meaning of the caret is given in Sec. II; hereafter we will drop the caret on all terms.

The field equations are

$$\begin{aligned} &[\gamma^\mu(i\partial_\mu - gB_\mu) - m]\psi' = 0 , \\ &\bar\psi'[\gamma^\mu(-i\overleftarrow{\partial}_\mu - gB_\mu) - m] = 0 , \\ &B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu , \\ &\partial_\nu B^{\nu\mu} + m_V{}^2 B^\mu = j^\mu , \end{aligned} \tag{B3}$$

where

$$j^\mu = g\,\bar\psi'\gamma^\mu\psi' . \tag{B4}$$

In the NP frame there are only seven independent field components which we take to be B_+, B_{+i}, and the nonzero components of $\Lambda_+\psi'$, and $\bar\psi'\Lambda_-\gamma^0 = \psi'^\dagger\Lambda_+$, where $\Lambda_\pm$ are the projection operators defined by

$$\Lambda_\pm = \Lambda^\mp = 2^{-1/2}\gamma_0\gamma_\pm = 2^{-1/2}\gamma^\pm\gamma^0 . \tag{B5}$$

Thus

$$\Lambda_+ = \Lambda^- = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \end{pmatrix},\quad \Lambda_- = \Lambda^+ = \begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{pmatrix}, \tag{B6}$$

if

$$\gamma^\mu = \begin{pmatrix} 0 & \delta_0^\mu - \delta_a^\mu\sigma^a \\ \delta_0^\mu + \delta_a^\mu\sigma^a & 0 \end{pmatrix} \text{ and } \sigma^a = -\sigma_a , \tag{B7}$$

where σ_a are the usual Pauli 2×2 matrices. The field equations split into two sets, namely the equations of motion of the independent field components,

$$\begin{aligned} &\partial_- B_+ = -B_{+-} + \partial_+ B_- , \\ &\partial_- B_{+i} = j_i + \partial_+ B_{i-} + \partial_j B_{ji} - m_V{}^2 B_i , \\ &(i\partial_- - gB_-)\Lambda_+\psi' = \gamma_0[\gamma_k(i\partial_k - gB_k) + m]\Lambda_-\psi' , \end{aligned} \tag{B8}$$

and the constraint equations for the redundant field components, e.g.,

$$(i\partial_+ - gB_+)\Lambda_-\psi' = \gamma_0\,[\gamma_k(i\partial_k - gB_k) + m]\Lambda_+\psi'. \tag{B9}$$

The constraint equations for the redundant vector-meson components may be solved to yield

$$B_i(x)=\tfrac{1}{2}\int dy^+\epsilon(x^+-y^+)[B_{+i}(y)+\partial_i B_+(y)],$$

$$B_{+-}(x)=\tfrac{1}{2}\int dy^+\epsilon(x^+-y^+)[\partial_i B_{+i}(y)+m_V{}^2B_+(y)-j_+(y)], \tag{B10}$$

$$B_{-i}(x)=\tfrac{1}{4}\int dy^+|x^+-y^+|[(\vec{\nabla}^2-m_V{}^2)B_{+i}(y)-2\partial_i\partial_j B_{+j}(y)-2m_V{}^2\partial_i B_+(y)+\partial_i j_+(y)+\partial_+ j_i(y)],$$

$$B_-(x)=\tfrac{1}{4}\int dy^+|x^+-y^+|[2\partial_i B_{+i}(y)+(\vec{\nabla}^2+m_V{}^2)B_+(y)-j_+(y)].$$

Here $\vec{\nabla}^2$ is the transverse two-dimensional Laplacian, $\epsilon(x)=\text{sign}(x)$, and $y^\mu=(y^+,x^i,x^-)$.

The presence of the B_+ term in Eq. (B9) causes a special difficulty. It is at this point that the gauge invariance of the manifestly covariant theory is required. The usual method of dealing with the problem is to put $B_+=0$. Another way to solve the constraint, Eq. (B9), for $\Lambda_-\psi'$ is to introduce

$$\psi(x)=\psi'(x)e^{ig\Lambda(x)}, \tag{B11}$$

where

$$\Lambda(x)=\tfrac{1}{2}\int dy^+\epsilon(x^+-y^+)B_+(y). \tag{B12}$$

This transforms the field equations for ψ' to similar equations for ψ in which B_μ is replaced by

$$\begin{aligned}\bar{B}_\mu(x)&=B_\mu(x)-\partial_\mu\Lambda(x)\\&=\tfrac{1}{2}\int dy^+\epsilon(x^+-y^+)B_{+\mu}(y).\end{aligned} \tag{B13}$$

Note $\bar{B}_+\equiv 0$, $\bar{B}_{\mu\nu}=B_{\mu\nu}$, and $j_\mu=g\bar{\psi}'\gamma_\mu\psi'=g\bar{\psi}\gamma_\mu\psi$. Thus, this transformation efffectively imposes the gauge condition $B_+=0$ in Eq. (B9). The solution of the transformed constraint equation for $\Lambda_-\psi$ is

$$\Lambda_-\psi(x)=\frac{1}{2i}\int dy^+\epsilon(x^+-y^+)\gamma_0\times\{\gamma_k[i\partial_k-\bar{B}_k(y)]+m\}\Lambda_+\psi(y). \tag{B14}$$

The Poincaré generators are given by Noether's theorem as

$$P_\mu=\int d^2x dx^+ T_{+\mu},$$

$$M_{\mu\nu}=\int d^2x dx^+\Big(x_\mu T_{+\nu}-x_\nu T_{+\mu}+B_{\sigma+}C^{\sigma\lambda}{}_{\mu\nu}B_\lambda+\frac{i}{2}\bar{\psi}\{\gamma_+,\Sigma_{\mu\nu}\}\psi\Big), \tag{B15}$$

where

$$d^2x=dx^1dx^2,\qquad C^{\sigma\lambda\mu\nu}=g^{\sigma\mu}g^{\lambda\nu}-g^{\lambda\mu}g^{\sigma\nu} \tag{B16}$$

and

$$\Sigma^{\mu\nu}=\tfrac{1}{4}[\gamma^\mu,\gamma^\nu]. \tag{B17}$$

The task now is to use the solutions of the constraint equations to eliminate all of the redundant components, B_i, B_{+-}, B_{-i}, $\Lambda_-\psi$, etc., and thereby obtain P_μ and $M_{\mu\nu}$ in a form where the integrands explicitly contain only the independent components. As an example consider

$$\begin{aligned}P_i&=\int d^2x dx^+\Big(B_{-+}\partial_i B_+-B_{k+}\partial_i B_k+\frac{i}{2}\bar{\psi}\gamma_+\overleftrightarrow{\partial}_i\psi+j_+\partial_i\Lambda\Big)\\&=\int d^2x dx^+\Big(\frac{i}{2}\bar{\psi}(x)\gamma_+\overleftrightarrow{\partial}_i\psi+j_+(x)\partial_i\Lambda(x)\\&\qquad-\tfrac{1}{2}\int dy^+\epsilon(x^+-y^+)\{[\partial_k B_{+k}(y)+m_V{}^2B_+(y)-j_+(y)]\partial_i B_+(x)-B_{+k}(x)\partial_i[B_{+k}(y)+\partial_k B_+(y)]\}\Big)\\&=\int d^2x dx^+\Big(i\sqrt{2}\,\psi^\dagger(x)\Lambda_+\partial_i\psi(x)+\tfrac{1}{2}\int dy^+\epsilon(x^+-y^+)[m_V{}^2B_+(x)\partial_i B_+(y)+B_{+k}(x)\partial_i B_{+k}(y)]\Big),\end{aligned} \tag{B18}$$

where the surface terms coming from an integration by parts have been put equal to zero.

It is convenient to define

$$\Psi=2^{1/4}\begin{pmatrix}\psi_1\\ \psi_4\end{pmatrix},\quad \Psi^\dagger=2^{1/4}(\psi_1^*,\psi_2^*), \tag{B19}$$

$$\bar{A}=(-iB_{+1},-iB_{+2},-im_VB_+),$$

$$A(x)=\begin{pmatrix}\bar{B}_1(x)\\ \bar{B}_2(x)\\ m_V\Lambda(x)\end{pmatrix}=\frac{i}{2}\int dy^+\epsilon(x^+-y^+)\bar{A}^T(y). \tag{B20}$$

Then P_i takes the form given in Eq. (3.6). A more instructive example is

$$J_3=\int d^2x dx^+\Big(x_1 T_{+2} - x_2 T_{+1} - B_{+1}B_2 + B_{+2}B_1 + \frac{i}{2}\bar{\psi}\gamma_+\gamma_1\gamma_2\psi\Big)\,. \quad \text{(B21)}$$

By taking account of the parts integration used in the P_i reduction and making use of the identity

$$\bar{\psi}\gamma_+\gamma_i\gamma_j\psi = -\Psi^\dagger(\delta_{ij}+i\epsilon_{ij}\sigma_3)\Psi\,, \quad \text{(B22)}$$

one obtains

$$\begin{aligned}J_3 &= \int d^2x dx^+\Big(x_1\mathcal{P}_2 - x_2\mathcal{P}_1 + \tfrac{1}{2}\Psi^\dagger\sigma_3\Psi - \tfrac{1}{2}\int dy^+\epsilon(x^+-y^+)[B_{+1}(x)B_{+2}(y) - B_{+2}(x)B_{+1}(y)]\Big)\\ &= \int d^2x dx^+(x_1\mathcal{P}_2 - x_2\mathcal{P}_1 + \Psi^\dagger j_3^f\Psi + \bar{A}j_3^V A)\,, \end{aligned} \quad \text{(B23)}$$

where $\mathcal{P}_i$ is the integrand in Eq. (3.6), $j_i^f=\frac{1}{2}\sigma_i$, and the j_i^V are given in Sec. III. The other generators are obtained by similar but more lengthy calculations.

---

*Work performed under the auspices of the United States Energy Research and Development Administration.

[1] A. O. Barut, D. Corrigan, and H. Kleinert, Phys. Rev. Lett. 20, 167 (1968).

[2] Y. Nambu, Phys. Rev. 160, 1171 (1967); C. Fronsdal and L. E. Lundberg, Phys. Rev. D 3, 524 (1971); A. O. Barut and W. Rasmussen, *ibid.* 3, 956 (1971).

[3] E. Abers, I. T. Grodsky, and R. E. Norton, Phys. Rev. 159, 1222 (1967).

[4] H. Bacry and J. Nuyts, Phys. Rev. 157, 171 (1967); F. Gürsey and M. Koca, Nuovo Cimento 1A, 429 (1971).

[5] S. Fubini and G. Furlan, Physics (N.Y.) 1, 229 (1965).

[6] P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).

[7] H. Leutwyler, J. R. Klauder, and L. Streit, Nuovo Cimento 66A, 536 (1970).

[8] K. Bardakci and M. Halpern, Phys. Rev. 176, 1686 (1968).

[9] R. A. Neville and F. Rohrlich, Nuovo Cimento 1A, 625 (1971).

[10] J. B. Kogut and D. E. Soper, Phys. Rev. D 1, 2901 (1970).

[11] J. D. Bjorken, J. B. Kogut, and D. E. Soper, Phys. Rev. D 3, 1382 (1971).

[12] D. Flory, Phys. Rev. D 1, 2795 (1970).

[13] S.-J. Chang and T.-M. Yan, Phys. Rev. D 7, 1133 (1973); S.-J. Chang, R. G. Root, and T.-M. Yan, *ibid.* 7, 1147 (1973).

[14] T.-M. Yan, Phys. Rev. D 7, 1760 (1973); 7, 1780 (1973).

[15] E. Tomboulis, Phys. Rev. D 8, 2736 (1973).

[16] A. Casher, Phys. Rev. D 14, 452 (1976).

[17] S. J. Brodsky, R. Roskies, and R. Suaya, Phys. Rev. D 8, 4574 (1973).

[18] The latest review is J. Scherk, Rev. Mod. Phys. 47, 123 (1975). Apologies for several hundred omissions.

[19] S. Mandelstam, Nucl. Phys. B64, 205 (1973); Phys. Rep. 13C, 260 (1974).

[20] M. Kaku and K. Kikkawa, Phys. Rev. D 10, 1110 (1974); 10, 1823 (1974).

[21] E. Cremmer and J. L. Gervais, Nucl. Phys. B76, 209 (1974).

[22] S. Mandelstam, Nucl. Phys. B83, 413 (1974).

[23] C. Lovelace, Phys. Lett. 34B, 500 (1971).

[24] P. Goddard, J. Goldstone, C. Rebbi, and C. B. Thorn, Nucl. Phys. B56, 109 (1973).

[25] P. A. M. Dirac, Proc. R. Soc. London A328, 1 (1972); L. C. Biedenharn, M. Y. Han, and H. Van Dam, Phys. Rev. D 8, 1735 (1973); L. P. Staunton, *ibid.* 8, 2446 (1973); 10, 1760 (1974).

[26] H. Bacry and N. P. Chang, Ann. Phys. (N.Y.) 47, 407 (1968).

[27] F. Gürsey and S. Orfanidis, Nuovo Cimento 11A, 225 (1972).

[28] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, J. Math. Phys. 17, 977 (1976), Table VII.

[29] J. Schwinger, in *Perspectives in Physics*, edited by L. C. Biedenharn and H. Van Dam (Academic, New York, 1965).

[30] V. V. Mikhailov, Teor. Mat. Fiz. 18, 342 (1974) [Theor. Math. Phys. 18, 243 (1974)]; and previous papers.

[31] P. Ramond, Nucl. Phys. B59, 412 (1973).

[32] E. P. Wigner, in *Group Theoretical Concepts and Methods in Elementary Particles*, edited by F. Gürsey (Gordon and Breach, New York, 1963), p. 37.

[33] R. Horgan and R. H. Dalitz, Nucl. Phys. B66, 1351 (1973).

[34] D. Faiman and A. W. Hendry, Phys. Rev. 173, 1720 (1968); related models are in D. B. Lichtenberg, Phys. Rev. 178, 2197 (1969); and in T. Eguchi, Phys. Lett. 59B, 457 (1975).

[35] D. Soper, Phys. Rev. D 4, 1620 (1971); also in his thesis, Stanford, 1971 (available from University Microfilms, Ann Arbor, Michigan), D. Soper derives P_- for several models and discusses the associated vertices and seagulls. His P_- for massive QED is equal to ours, up to an integral of a total derivative. He does not give expressions for G_i. However, for QED, he does give expressions for the commutators of G_i and the fields, Ψ and A. Although technically correct these expressions still contain the redundant components; also they do not show the presence of the Bacry-Chang group-theoretic form.

[36] See Eq. (2.30) of Ref. 21, Eq. (3.36) of Ref. 27, and Table I of Ref. 11.

[37] J. L. Torres-Hernandez, Phys. Rev. D 11, 3565 (1975).