Gauge groups without triangular anomaly* †

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Suppose that G is a simple gauge group governing a unified gauge theory. We shall then prove that the existence or absence of the triangular anomaly is equivalent to the same question for symmetrized third-order Casimir invariants of G. Consequently, we show that the group SU(n) ($n \geq 3$) is the only simple Lie group with possible triangular anomaly. For this case, the anomaly coefficient has been explicitly computed in terms of the $n-1$ parameters specifying irreducible representations of the group SU(n). Various anomaly-free groups have been discussed, and it is argued that the best candidates for anomaly-free simple gauge groups are E_6, SO($4n+2$) ($n \geq 2$), and the vectorlike SU(n) ($n \geq 3$) theories.

One problem facing the unified non-Abelian gauge theory[1] is the possible occurrence of the triangular anomaly which may spoil[2] the renormalizability of the theory. Georgi and Glashow[3] gave a general condition for the cancellation of the triangular anomaly. Changing notations, their results are stated as follows. Suppose that G is a simple Lie group underlying the unified gauge theory, and let X_μ ($\mu = 1, 2, \ldots, p$) be infinitesimal generators of G with Lie commutation relation

$$[X_\mu, X_\nu] = C_{\mu\nu}^{\lambda} X_\lambda\,, \tag{1}$$

where the repeated index on λ implies an automatic summation over $\lambda = 1, 2, \ldots, p$. For the sake of simplicity, we shall use interchangeably the same symbol G for both the Lie group as well as its Lie algebra, since this will not in practice cause any confusion. Suppose that positive and negative chiral components of all elementary fermions entering in the theory form separately[4] the basis of two representations $\{\rho_+\}$ and $\{\rho_-\}$ of G, which are not necessarily irreducible. If $t_\mu^{(+)}$ and $t_\mu^{(-)}$ are representation matrices of X_μ in representations $\{\rho_+\}$ and $\{\rho_-\}$, respectively, then the condition for the cancellation of the triangular anomaly is given by

$$h_{\mu\nu\lambda}^{(+)} = h_{\mu\nu\lambda}^{(-)}\,, \tag{2}$$

where completely symmetric triple-linear forms $h_{\mu\nu\lambda}^{(\pm)}$ are defined by

$$h_{\mu\nu\lambda}^{(\pm)} = \mathrm{Tr}(\{t_\mu^{(\pm)}, t_\nu^{(\pm)}\}_+ t_\lambda^{(\pm)})\,. \tag{3}$$

If both representations $\{\rho_\pm\}$ are self-contragradient, i.e., if we can find nonsingular matrices $S_\pm$ satisfying

$$[t_\mu^{(\pm)}]^T = -S_\pm^{-1} t_\mu^{(\pm)} S_\pm\,, \tag{4}$$

then it is easy to see that

$$h_{\mu\nu\lambda}^{(\pm)} = 0 \tag{5}$$

identically, where the superscript T in Eq. (4) stands for the transpose matrix. It is known in the literature[5-9] that all representations of the groups

$$\mathrm{SU}(2),\ \mathrm{SO}(2l+1),\ \mathrm{SO}(4l),\ G_2,\ F_4,\ E_7,\ E_8,\ \mathrm{Sp}(2l) \tag{6}$$

are self-contragradient, so that Eq. (5) holds always for these groups. We may remark that this fact is also related to the ambivalence[10] of these groups as well as absence[11] of genuine odd-order Casimir invariants for these Lie algebras. At any rate, these groups lead to anomaly-free theories. Moreover, Georgi and Glashow[3] proved by a direct computation that all SO(n) groups with $n \neq 6$ are also anomaly-free, thus eliminating the case SO($4l+2$) ($l \geq 2$). Therefore, they concluded that simple Lie groups with possible anomaly are restricted to SU(n) ($n \geq 3$) and E_6. In this paper, we shall not separately count the exceptional case SO(6) group in this list, since SO(6) is locally isomorphic to SU(4).

The purpose of this note is to generalize those results found by Georgi and Glashow. We shall first prove that E_6 is also anomaly-free. Hence, the only simple gauge groups with a possible anomaly are the SU(n) ($n \geq 3$). As we shall show in the Appendix, these facts are intimately related to the fact of nonexistence of genuine third-order Casimir invariants for simple Lie algebras except for the algebra A_n ($n \geq 2$) corresponding to the SU($n+1$) group. Second, we can compute the explicit form of $h_{\mu\nu\lambda}^{(\pm)}$ for the SU(n) ($n \geq 3$) in terms of eigenvalues of the third-order Casimir invariant in the representations $\{\rho_+\}$ and $\{\rho_-\}$. Towards this end, we introduce the standard n^2-1 Hermitian traceless $n \times n$ matrices λ_α ($\alpha = 1, 2, \ldots, n^2-1$) which satisfy the normalization condition

$$\mathrm{Tr}(\lambda_\alpha \lambda_\beta) = 2\delta_{\alpha\beta}, \quad \mathrm{Tr}\lambda_\alpha = 0\,. \tag{7}$$

Moreover, the completely antisymmetric $f_{\alpha\beta\gamma}$ and completely symmetric $d_{\alpha\beta\gamma}$ triple-linear forms are defined as usual by[12]

$$f_{\alpha\beta\gamma}=\frac{1}{4i}\,\mathrm{Tr}([\lambda_\alpha,\lambda_\beta]\lambda_\gamma)\,,$$
$$d_{\alpha\beta\gamma}=\frac{1}{4}\,\mathrm{Tr}(\{\lambda_\alpha,\lambda_\beta\}_+\lambda_\gamma)\,, \tag{8}$$

so that the product $\lambda_\alpha\lambda_\beta$ can be expanded as

$$\lambda_\alpha\lambda_\beta=\frac{2}{n}\,\delta_{\alpha\beta}E+(d_{\alpha\beta\gamma}+if_{\alpha\beta\gamma})\lambda_\gamma\,. \tag{9}$$

Here, E is the $n\times n$ unit matrix. Accordingly, we choose the basis of the algebra A_{n-1} so that the representation matrices $t_\alpha^{(\pm)}$ satisfy

$$[t_\alpha^{(\pm)},t_\beta^{(\pm)}]=if_{\alpha\beta\gamma}t_\gamma^{(\pm)}\,. \tag{10}$$

We shall then prove the following in the Appendix. We have first

$$h_{\alpha\beta\gamma}^{(\pm)}=K^{(\pm)}d_{\alpha\beta\gamma} \tag{11}$$

for all $\alpha,\beta,\gamma=1,2,\ldots,n^2-1$, and second the constants $K^{(\pm)}$ can be evaluated in terms of eigenvalues of the third-order Casimir invariants. Since any representation of the semisimple Lie algebra is fully reducible, it really suffices to consider the case of an irreducible representation, where the value of $K^{(\pm)}$ is a sum of contributions from irreducible components contained in the reduction of $\{\rho_\pm\}$. We shall refer hereafter by $\{\rho\}$ to the generic irreducible component representation contained in this reduction of $\{\rho_\pm\}$. If t_α is the representation matrix of X_α in $\{\rho\}$ and if we set

$$\mathrm{Tr}(\{t_\alpha,t_\beta\}_+t_\gamma)=d_{\alpha\beta\gamma}K(\rho)\,, \tag{12}$$

then $K^{(\pm)}$ is a sum of $K(\rho)$'s.

It is convenient for our purpose to embed the SU(n) group into the U(n) group whose irreducible representations are specified[13] by n integers satisfying

$$f_1\geq f_2\geq\cdots\geq f_n\,. \tag{13}$$

Let e be an arbitrary integer. Then, all representations with signatures $(f_1+e,f_2+e,\ldots,f_n+e)$ of the U(n) group correspond to a single irreducible representation $\{\rho\}$ of the SU(n), specified by standard $n-1$ non-negative integers $m_1,m_2,\ldots,m_{n-1}$ with correspondence

$$m_j=f_j-f_{j+1},\quad 1\leq j\leq n-1\,. \tag{14}$$

Let us now define σ_j $(1\leq j\leq n)$ by

$$\sigma_j=f_j+\tfrac{1}{2}(n+1)-j-\frac{1}{n}\sum_{k=1}^{n}f_k \tag{15}$$

which satisfy a constraint

$$\sum_{j=1}^{n}\sigma_j=0 \tag{16}$$

as well as the ordering

$$\sigma_1>\sigma_2>\cdots>\sigma_n\,. \tag{17}$$

As we may easily check, σ_j are invariant under the substitution

$$f_k\to f_k+e,\quad k=1,2,\ldots,n \tag{18}$$

for any arbitrary constant e, so that σ_j actually depend only upon $m_1,m_2,\ldots,m_{n-1}$. The value of $K(\rho)$ is now calculated in the Appendix to be

$$K(\rho)=\frac{n}{(n^2-1)(n^2-4)}\,d(\rho)\sum_{j=1}^{n}(\sigma_j)^3\quad(n\geq3)\,, \tag{19}$$

where $d(\rho)$ is the dimension of the irreducible representation $\{\rho\}$ with

$$d(\rho)=\frac{\prod_{j<k}^{n}(\sigma_j-\sigma_k)}{1!\,2!\,3!\cdots(n-1)!}\,. \tag{20}$$

The contragradient representation $\{\rho^*\}$ of $\{\rho\}$ is characterized[14] by integers $\tilde\sigma_1,\tilde\sigma_2,\ldots,\tilde\sigma_n$ such that

$$\tilde\sigma_j=-\sigma_{n+1-j},\quad 1\leq j\leq n\,. \tag{21}$$

From Eq. (19) it is easy to check that

$$K(\rho^*)=-K(\rho),\quad d(\rho^*)=d(\rho)\,, \tag{22}$$

so that any self-contragradient representation $\rho=\rho^*$ leads to an anomaly-free theory with $K(\rho)=0$, in accordance with the statement made earlier.

If $\{\rho_+\}$ is equivalent to $\{\rho_-\}$, then we have $K^{(+)}=K^{(-)}$ so that the anomaly is automatically canceled between two representations $\{\rho_\pm\}$. As has been already remarked by Georgi and Glashow,[3] this corresponds to the case of vectorlike models. However, more interesting cases are when we have $K^{(+)}=K^{(-)}=0$ identically whether $\{\rho_+\}$ is equivalent to $\{\rho_-\}$ or not. We may refer to such cases as natural cancellation of the anomaly. Then, all representations of all simple groups except for the SU(n) ($n\geq3$) as well as self-contragradient representations of the SU(n) ($n\geq3$) are cases of natural cancellation. More generally, suppose that $\{\rho_+\}$ and/or $\{\rho_-\}$ are reducible and that their irreducible components always contain $\{\rho\}$ and its contragradient $\{\rho^*\}$ in pairs as well as any number of self-contragradient representations. Then, in view of Eq. (22), we find $K^{(\pm)}=0$ so that this case also corresponds to the natural cancellation. However, there are other cases where the cancellation mechanism is less obvious. An example is the SU(5) theory of Georgi and Glashow,[15] where the representation space is chosen to be a direct sum of $\{5^*\}$ and $\{10\}$ for $\{\rho_+\}$. We may check from Eq. (19) that contributions from $\{5^*\}$ and $\{10\}$ cancel each other in this case to give an anomaly-free theory. The deeper reason behind this cancellation may be due to the fact that the SU(5) group can be regarded[16] as a subgroup of

SO(10) and that the 16-dimensional irreducible representation of the SO(10) reduces to a direct sum $\{1\} \oplus \{5^*\} \oplus \{10\}$ representation of the SU(5) subgroup.

So far, we were assuming implicitly that G is the dynamical group governing the gauge theory which unifies all strong, weak, and electromagnetic interactions. However, if we are interested only in the cancellation of the triangular anomaly in the Weinberg-Salam theory (referred to as WS), of the weak and electromagnetic interaction, then we may regard G simply as a classification (rather than dynamical) group of all elementary fermions entering in the theory, provided that G contains the WS group $SU(2) \otimes U(1)$ as its proper subgroup and that the sum of all WS weak currents of given types can be expressed in terms of currents of the group G.

If G is the dynamical gauge group for all strong, weak, and electromagnetic interactions, then presently available experimental data on the strong interaction can be used to further restrict the group G. As we shall show elsewhere, any simple group G listed in Eq. (6) as well as any self-contragradient representation of the $SU(n)$ $(n \geq 3)$ will lead to a difficulty of explaining violations of the quark-line rule.[17] If we combine this fact with the required absence of the triangular anomaly, then the only simple groups compatible with these two requirements are E_6 and $SO(4n+2)$ $(n \geq 2)$ as well as possibly the $SU(n)$ $(n \geq 3)$ groups. For the $SU(n)$ group, the representations to be used must be non-self-contragradient with no anomaly. Although such representations can always be found as we noted, such cases are rather accidental except for the vectorlike models in which $\{\rho_+\}$ and $\{\rho_-\}$ are equivalent. Therefore, we argue that possibly the best candidates for the simple group G are E_6, $SO(4n+2)$, and $SU(n)$ $(n \geq 3)$ groups. For the $SU(n)$, only vectorlike models are preferred. We should emphasize that this conclusion may change if G is assumed to be semisimple rather than being simple.

Next, we shall investigate a problem when G is now assumed to be a classification rather than a dynamical group in a sense stated already. We have then to distinguish two cases; where (i) both leptons and quarks separately form two representations of G, or (ii) leptons and quarks together (but not separately) form a single representation of G. Although the second case is perhaps more interesting,[18] we shall consider here only case (i) since it is easier to analyze. Let $\{\rho_\pm\}$ denote now representation spaces of all positive and negative chiral components of all quarks, respectively. We may assume without a great loss of generality that at least one of $\{\rho_+\}$ and $\{\rho_-\}$ is irreducible, and we denote it by $\{\rho\}$. If all quarks have standard fractional electric charges of the form $-\frac{1}{3}$ plus integers, and if the charge operator Q is a member of a Cartan subalgebra of G, then the dimension $d = d(\rho)$ of the representation $\{\rho\}$ must be of necessity an integral multiple of 3, since the simplicity of G requires $\mathrm{Tr}\,Q = 0$ in the representation. Together with the assumption of the natural-cancellation mechanism $K(\rho) = 0$, this imposes a severe constraint for admissible representations $\{\rho\}$. Let n be the rank of G with fundamental weights $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$. Then, the highest weight Λ of the representation $\{\rho\}$ is expressed as[19]

$$\Lambda = m_1\Lambda_1 + m_2\Lambda_2 + \cdots + m_n\Lambda_n , \tag{23}$$

where m_j $(1 \leq j \leq n)$ are non-negative integers specifying the representation $\{\rho\}$. Here we shall adopt the lexiconal ordering of n simple roots of G as in Patera and Sankoff.[20] Then, we can enumerate all possible groups and their admissible irreducible representations $\{\rho\}$ satisfying the condition $K(\rho) = 0$ with dimension $d(\rho)$ less than 24 in Table I. Note that we neglected the cases of SU(2) groups, since it does not contain the WS group $SU(2) \otimes U(1)$.

Note added. After this paper was completed, Professor H. Georgi informed me that the anomaly coefficient $K(\rho)$ for the $SU(n)$ group had been computed in a somewhat different form by J. Banks and H. Georgi [Phys. Rev. D 14, 1159 (1976)]. Their notations q_j $(1 \leq j \leq n-1)$ and $A(R)$ with $N = n$, are related to ours by

$$q_j = m_j + 1 = \sigma_j - \sigma_{j+1}, \quad 1 \leq j \leq n-1$$

$$A(R) = 2K(\rho) .$$

Conversely, our σ_j is expressible as

$$\sigma_j = \frac{1}{n}\left(\sum_{k=j}^{n} (n-k)q_k - \sum_{k=1}^{j-1} kq_k\right), \quad 1 \leq j \leq n .$$

We also note that for special cases $n = 3$ and $n = 4$ the constraint

$$\sum_{j=1}^{n} \sigma_j = 0$$

leads to

$$2I_3(\rho) = \sum_{j=1}^{3} (\sigma_j)^3 = 3\sigma_1\sigma_2\sigma_3, \quad \text{for SU(3)}$$

$$2I_3(\rho) = \sum_{j=1}^{4} (\sigma_j)^3 = 3(\sigma_1+\sigma_2)(\sigma_1+\sigma_3)(\sigma_1+\sigma_4), \quad \text{for SU(4)}$$

which immediately reproduce Eqs. (11) and (12) of the paper by Banks and Georgi. The author would like to thank Professor H. Georgi for calling his attention to the above-mentioned paper.

TABLE I. List of all admissible irreducible representations $\{\rho\}$ and groups G satisfying the natural-cancellation mechanism with dimension less than 24. See the text for the symbol m_j as well as for details.

d = dimension of the representation $\{\rho\}$	Type of simple lie group G	Admissible irreducible representation $\{\rho\}$
6	(a) SU(4)	(0,1,0)
	(b) Sp(6)	(1,0,0)
9	SO(9)	(1,0,0,0)
12	(a) Sp(12)	$m_1=1,\ m_j=0\ (2\leq j\leq 6)$
	(b) SO(12)	
15	(a) SU(4)	(1,0,1)
	(b) SO(15)	$m_1=1,\ m_j=0\ (2\leq j\leq 7)$
18	(a) SO(18)	$m_1=1,\ m_j=0\ (2\leq j\leq 9)$
	(b) Sp(18)	
21	(a) SO(7)	(0,1,0)
	(b) Sp(6)	(2,0,0)
	(c) SO(21)	$m_1=1,\ m_j=0\ (2\leq j\leq 10)$
24	(a) SU(5)	(1,0,0,1)
	(b) SO(24)	$m_1=1,\ m_j=0\ (2\leq j\leq 12)$
	(c) Sp(24)	

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APPENDIX

First, we shall make a brief mathematical preparation in order to prove various statements made in the text. Let G be any (not necessarily semisimple) Lie algebra with Lie equation (over complex number field)

$$[X_\mu, X_\nu] = C^\lambda_{\mu\nu} X_\lambda \,. \tag{A1}$$

Then, a collection of p elements T_μ ($\mu = 1, 2, \ldots, p$) belonging to members of the universal enveloping algebra of G is called a vector operator of G, if we have

$$[X_\mu, T_\nu] = C^\lambda_{\mu\nu} T_\lambda \,. \tag{A2}$$

Obviously, X_μ is a vector operator, but this is not in general only one. We can also define vector operators T_μ in a given representation $\{\rho\}$ of G, if X_μ and T_μ are now $d\times d$ matrices satisfying Eqs. (A1) and (A2). Here, $d=d(\rho)$ is the dimension of the representation $\{\rho\}$.

Hereafter, we restrict ourselves to the case that G is a simple Lie algebra of rank n, and that $\{\rho\}$ is irreducible with highest weight Λ as in Eq. (23) of the text. Let $n_V(\rho)$ be the number of all linearly independent vector operators in the representation $\{\rho\}$. Then, it has been proved elsewhere[21] that we have

$$n_V(\rho) = n - n_0(\rho) \,, \tag{A3}$$

where $n_0(\rho)$ is the number of m_j's which are zero. In other words, $n_V(\rho)$ is equal to the number of m_j's which are positive. In particular, we are interested in the case in which $\{\rho\}$ is the adjoint representation $\{\rho_0\}$ of G. Note that the adjoint representation must be irreducible because of the simplicity of G. With the lexiconal orderings of simple roots as in Ref. 20, the highest weight Λ_0 of the adjoint representation $\{\rho_0\}$ of a simple Lie algebra G is given by[22]

$$\Lambda_0 = \begin{cases} \Lambda_1 + \Lambda_n, & A_n \ (n\geq 1)\,, \\ \Lambda_2, & B_n \ (n\geq 3) \text{ and } D_n \ (n\geq 4)\,, \\ 2\Lambda_1, & C_n \ (n\geq 2)\,, \\ \Lambda_1, & G_2,\ F_4,\ E_7, \text{ and } E_8\,, \\ \Lambda_6, & E_6\,. \end{cases} \tag{A4}$$

Therefore, the theorem quoted above implies $n_V(\rho_0)=2$ for A_n ($n\geq 2$), but $n_V(\rho_0)=1$ for all other algebras. As we shall see shortly, this is equivalent to the fact that the algebra A_n ($n\geq 2$) alone has one symmetrized third-order Casimir invariant, while all other simple algebras possess none.

Let us set

$$f_\mu = \mathrm{ad} X_\mu \,, \tag{A5}$$

so that its (α,β) matrix element $(\alpha,\beta=1,2,\ldots,p)$ is given by

$$(f_\mu)_{\alpha\beta} = C^\alpha_{\mu\beta}\,. \tag{A6}$$

Then, $p\times p$ matrices f_μ satisfy, of course,

$$[f_\mu, f_\nu] = C^\lambda_{\mu\nu} f_\lambda\,. \tag{A7}$$

Also, Cartan's criteria[19] of semisimplicity implies that the bilinear form $g_{\mu\nu}$ defined by

$$g_{\mu\nu} = \mathrm{Tr}(f_\mu f_\nu) \tag{A8}$$

is nonsingular with its inverse $g^{\mu\nu}$. Let $p\times p$ matrices h_μ be a vector operator in the adjoint representation $\{\rho_0\}$, so that

$$[f_\mu, h_\nu] = C^\lambda_{\mu\nu} h_\lambda\,. \tag{A9}$$

We introduce $h^\alpha_{\lambda\beta}$ and $h_{\lambda\mu\nu}$ by

$$\begin{aligned} h^\alpha_{\lambda\beta} &= (h_\lambda)_{\alpha\beta}\,,\\ h_{\lambda\mu\nu} &= g_{\lambda\tau} h^\tau_{\mu\nu}\,. \end{aligned} \tag{A10}$$

In particular, if we choose h_μ to be equal to f_μ, then the corresponding triple-linear form

$$f_{\mu\nu\lambda} = g_{\mu\tau} C^\tau_{\nu\lambda} \tag{A11}$$

is completely antisymmetric. Also, in terms of $h_{\lambda\mu\nu}$, Eq. (A9) is shown to be equivalent to[23]

$$C^\lambda_{\alpha\beta} h_{\lambda\mu\nu} + C^\lambda_{\alpha\mu} h_{\beta\lambda\nu} + C^\lambda_{\alpha\nu} h_{\beta\mu\lambda} = 0\,. \tag{A12}$$

We can prove[23] that $h_{\lambda\mu\nu}$ satisfying (A12) can be chosen to be either completely symmetric or completely antisymmetric and, moreover, that the completely antisymmetric $h_{\mu\nu\lambda}$ must be proportional to $f_{\mu\nu\lambda}$. The result quoted just after Eq. (A4) implies then that only the triple-linear form $h_{\mu\nu\lambda}$ satisfying Eq. (A12) is precisely $f_{\mu\nu\lambda}$ (apart from a multiplicative constant) for all simple Lie algebras except for A_n $(n\geq 2)$ and that we have[24] an additional completely symmetric form $d_{\mu\nu\lambda}$ for A_n $(n\geq 2)$. Hereafter, we define and normalize $f_{\mu\nu\lambda}$ and $d_{\mu\nu\lambda}$ as in Eq. (8) of the text for the case of the SU(n) group, corresponding to the algebra A_{n-1}.

Next, for any $h_{\mu\nu\lambda}$ satisfying Eq. (A12) we set

$$h^{\lambda\mu\nu} = g^{\lambda\alpha} g^{\mu\beta} g^{\nu\gamma} h_{\alpha\beta\gamma}\,. \tag{A13}$$

Then, $h^{\lambda\mu\nu}$ is also either completely symmetric or antisymmetric. If we set

$$I_3 = h^{\lambda\mu\nu} X_\lambda X_\mu X_\nu\,, \tag{A14}$$

then it is not difficult to prove[23] that I_3 is a Casimir invariant of G. However, for the antisymmetric case $h_{\lambda\mu\nu} = f_{\lambda\mu\nu}$, we can always reduce I_3 into a constant multiple of the second-order Casimir invariant

$$I_2 = g^{\mu\nu} X_\mu X_\nu\,. \tag{A15}$$

Therefore, only the case corresponding to completely symmetric $h_{\lambda\mu\nu} = d_{\lambda\mu\nu}$ defines a genuine third-order Casimir invariant. Conversely, if I_3 is a symmetrized third-order Casimir invariant of G, we can always rewrite I_3 in the form (A14), where the completely symmetric $h_{\lambda\mu\nu}$ obtained in this way via Eq. (A13) satisfy Eq. (A12). This fact will be reported elsewhere. Summarizing these facts, we have shown that the algebra A_n $(n\geq 2)$ has only one symmetrized third-order Casimir invariant, while all other simple Lie algebras have none. This can also be seen[25] from known orders of all algebraically independent Casimir invariants of simple Lie algebras in the mathematical literature.

After these preparations, we will now prove our main results stated in the text. Let t_μ $(\mu=1,2,\ldots,p)$ be the representation matrix of X_μ in a generic irreducible representation $\{\rho\}$ of G, and set

$$h_{\mu\nu\lambda} = \mathrm{Tr}(\{t_\mu, t_\nu\}_+ t_\lambda)\,. \tag{A16}$$

Obviously, $h_{\mu\nu\lambda}$ is completely symmetric. Moreover, it satisfies Eq. (A12), if we note a trivial identity

$$\mathrm{Tr}([t_\alpha, Q]) = 0, \quad Q = \{t_\mu, t_\nu\}_+ t_\beta\,.$$

Therefore, by the result stated above, we find

$$h_{\mu\nu\lambda} = 0$$

identically for all simple Lie algebras except for A_n $(n\geq 2)$. Second, since the algebra A_n $(n\geq 2)$ can have only one symmetric form $d_{\lambda\nu\mu}$, we conclude that

$$h_{\mu\nu\lambda} = d_{\mu\nu\lambda} K(\rho)\,, \tag{A17}$$

where the proportionality constant $K(\rho)$ depends in general upon the representation $\{\rho\}$. The numerical value of $K(\rho)$ can be determined from

$$d_{\mu\nu\lambda} h_{\mu\nu\lambda} = d_{\mu\nu\lambda} d_{\mu\nu\lambda} K(\rho)\,. \tag{A18}$$

Define now the second- and third-order Casimir invariants of the algebra A_{n-1} [or SU(n) group] by

$$\begin{aligned} I_2 &= X_\lambda X_\lambda\,,\\ I_3 &= d_{\mu\nu\lambda} X_\mu X_\nu X_\lambda\,, \end{aligned} \tag{A19}$$

where the X_μ's are normalized by

$$[X_\mu, X_\nu] = i f_{\mu\nu\lambda} X_\lambda\,, \tag{A20}$$

and where $f_{\lambda\mu\nu}$ and $d_{\lambda\mu\nu}$ are now given by Eq. (8). In this case, we need not distinguish the upper indices from the lower ones, since $g_{\mu\nu}$ is proportional to $\delta_{\mu\nu}$ for our choice of the basis.

If $I_2(\rho)$ and $I_3(\rho)$ are eigenvalues of these Casimir invariants in the irreducible representation $\{\rho\}$, then we obviously have

$$d_{\mu\nu\lambda}h_{\mu\nu\lambda}=2d(\rho)I_3(\rho),\qquad d_{\mu\nu\lambda}d_{\mu\nu\lambda}=4d(\rho_1)I_3(\rho_1), \tag{A21}$$

where $\{\rho_1\}$ refers to the n-dimensional representation specified by the signature $(1,0,0,\ldots,0)$ with generator $t_\alpha=\frac{1}{2}\lambda_\alpha$. The eigenvalue $I_3(\rho)$ can be computed immediately as in Ref. 21. Or we may reduce it to a form studied by many authors[26-28] as follows: Let the lower-case latin indices a,b,c,d run from 1 to n, and define

$$B^a_b=\sum_{\alpha=1}^{n^2-1}(\lambda_\alpha)_{ab}X_\alpha,\qquad X_\alpha=\frac{1}{2}\sum_{a,b=1}^{n}(\lambda_\alpha)_{ba}B^a_b. \tag{A22}$$

When we note an identity

$$\sum_{\alpha=1}^{n^2-1}(\lambda_\alpha)_{ab}(\lambda_\alpha)_{cd}=2\left(\delta_{ad}\delta_{bc}-\frac{1}{n}\delta_{ab}\delta_{cd}\right), \tag{A23}$$

then (A20) is rewritten in the standard form

$$[B^a_b,B^c_d]=\delta^a_dB^c_b-\delta^c_bB^a_d,\qquad \sum_{a=1}^{n}B^a_a=0. \tag{A24}$$

After some computations, we find now

$$I_2(\rho)=\frac{1}{2}\left[\sum_{j=1}^{n}(\sigma_j)^2-\frac{1}{12}n(n^2-1)\right], \tag{A25}$$

$$I_3(\rho)=\frac{1}{2}\sum_{j=1}^{n}(\sigma_j)^3, \tag{A26}$$

where σ_j $(j=1,2,\ldots,n)$ are defined by Eq. (15) of the text. Especially for the fundamental representation $\{\rho_1\}$, we compute

$$I_2(\rho_1)=\frac{1}{2n}(n^2-1),\qquad d_{\mu\nu\lambda}d_{\mu\nu\lambda}=4nI_3(\rho_1)=\frac{(n^2-1)(n^2-4)}{n}, \tag{A27}$$

so that Eqs. (A21) and (A18) reproduce Eq. (19) of the text.

For the contragradient representation $\{\rho^*\}$ of $\{\rho\}$, we have[29]

$$I_2(\rho^*)=I_2(\rho),\qquad I_3(\rho^*)=-I_3(\rho). \tag{A28}$$

As the result, we have $I_3(\rho)=0$ for any self-contragradient representation $\{\rho\}$. Let us consider the converse problem. We can prove easily that $I_3(\rho)=0$ implies $\{\rho\}=\{\rho^*\}$ for the cases of SU(3) and SU(4) if we use Eqs. (16) and (17). However, for the case[30] SU(n) ($n\geq 5$), the same is not true. For example, SU(5) has a non-self-contragradient representation $f_1=8$, $f_2=3$, $f_3=2$, $f_4=-6$, and $f_5=-7$ with $2I_3(\rho)=10^3+4^3+2^3+(-7)^3+(-9)^3=0$. For SU($n$) ($n\geq5$), we have algebraically independent odd-order Casimir invariants I_5, I_7, etc. so that we could have $I_3(\rho)=0$, but $I_5(\rho)\neq0$, so that $\rho\neq\rho^*$.

Concluding this paper, we simply remark that any calculation of an N-fermion closed-loop diagram is related to a study of the Nth order Casimir invariant I_N of G. The present triangular problem is a special case of $N=3$.

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[1]See, e.g., H. Fritzsch and P. Minkowski, Ann. Phys. (N.Y.) 93, 193 (1975), and references quoted therein. In the present paper, we consider only the case of simple Lie groups, since any semisimple Lie algebra is a direct sum of simple algebras.

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[3]H. Georgi and S. L. Glashow, Phys. Rev. D 6, 429 (1972).

[4]If we wish, we could treat the case in which all positive and negative chiral components together (but not separately) form a representation of G. In this case, the result is mathematically equivalent to the case where we consider only one of the $\{\rho_+\}$ and $\{\rho_-\}$ representations in what follows.

[5]A. I. Mal'cev, Izv. Akad. Nauk, SSSR, Ser. Mat. 8, 143 (1944) [in Amer. Math. Soc. Transl. No. 33 (1950)].

[6]J. Tits, *Lecture Notes in Mathematics*, No. 40 (Springer, New York, 1967).

[7]L. C. Biedenharn, J. Nuyts, and H. Ruegg, Commun. Math. Phys. 2, 231 (1966).

[8]M. L. Mehta, J. Math. Phys. 7, 1824 (1966); M. L. Mehta and P. K. Srivastava, *ibid.* 7, 1833 (1966).

[9]A. K. Bose and J. Patera, J. Math. Phys. 11, 2231 (1970).

[10]The author would like to express his gratitude to Professor L. C. Biedenharn for illuminating this point as well as its connection with the Frobenius-Schur invariant of the group. See also, W. T. Sharp, L. C. Biedenharn, E. DeVries, and A. J. Van Zanten, Can. J. Math. 27, 246 (1975). Note that the ambivalence leads to real characters.

[11]If a Lie algebra G does not have any symmetrized

odd-order Casimir invariants, then eigenvalues of all Casimir invariants have the same values for any $\{\rho\}$ and its contragradient one $\{\rho^*\}$ so that $\{\rho\}$ must be self-contragradient. The orders of all algebraically independent Casimir invariants of simple Lie algebras can be found in A. Borel and C. Chevalley, Am. Math. Soc. Mem. 14, 1 (1955); G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups (I)* (Springer, New York, 1972), p. 144. Only algebras E_6, D_{2l+1}, and A_n ($n \geq 2$) have odd-order Casimir invariants.

[12]Numerical values of $f_{\alpha\beta\gamma}$ and $d_{\alpha\beta\gamma}$ for the SU(n) ($n=3,4,5,6$) groups are tabulated in M. Gell-Mann, Phys. Rev. 125, 1067 (1962); D. A. Dicus and V. S. Mathur, Phys. Rev. D 9, 1003 (1974); H. Hayashi, I. Ishiwata, S. Iwao, M. Shako, and S. Takeshita, Ann. Phys. (N.Y.) 101, 394 (1976) for $n=3$, 4, and for $n=5,6$, respectively.

[13]H. Weyl, *The Classical Groups* (Princeton Univ. Press, Princeton, N.J., 1939).

[14]The contragradient representation $\{\rho^*\}$ of $\{\rho\}$ for the SU(n) group has signature $\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_{n-1}$ such that $\tilde{m}_j = m_{n-j}$ $(1 \leq j \leq n-1)$. In the U(n) notation, this is equivalent to having $\tilde{f}_j = e - f_{n+1-j}$ $(1 \leq j \leq n)$ for some fixed integer e. Then, $\tilde{\sigma}_j = -\sigma_{n-j+1}$ immediately follows. σ_j is related to the usual l_j by $\sigma_j = l_j - \frac{1}{2}(n-1)$.

[15]H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974).

[16]H. Fritzsch and P. Minkowski, Ref. 1. The extra presence of the trivial singlet representation does not affect the discussion of the triangular anomaly.

[17]S. Okubo, Phys. Lett. 5, 165 (1963); G. Zweig, CERN Report No. TH-412, 1964 (unpublished); J. Iizuka, Prog. Theor. Phys. Suppl. 37-38, 21 (1966); J. Iizuka, K. Okada, and O. Shito, Prog. Theor. Phys. 35, 1061 (1966).

[18]Some interesting examples of the case (ii) have been discussed by J. C. Pati and A. Salam, Phys. Rev. D 8, 1240 (1973); and 10, 275 (1974), for SU(4)⊗SU(4), and by F. Gürsey and P. Sikivie, Phys. Rev. Lett. 36, 775 (1976) for E_7, while the SU(6) group has been proposed by K. Inoue, A. Kakuto, and Y. Nakano, Prog. Theor. Phys. 58, 630 (1977); Kyushu Univ. Report No. 77-HE-5, 1977 (unpublished); K. Fujikawa, INS Report No. INS-REP-291, 1977 (unpublished); M. Yoshimura, Tsukuba Univ. Report No. TU/77/62, 1977 (unpublished).

[19]E.g., see N. Jacobson, *Lie Algebras* (Interscience, New York, 1962).

[20]J. Patera and D. Sankoff, *Tables of Branching Rules for Representations of Simple Lie Algebras* (Les Presses de L'Université de Montréal, Montréal, Canada, 1973) have tabulated all irreducible representations of dimension less than 1000 for any simple Lie algebra of rank less than 8.

[21]S. Okubo, J. Math. Phys. (to be published).

[22]We need not consider here the algebras B_2 and D_3, since they are essentially identical to C_2 and A_3, respectively. With the present ordering convention of simple roots, the highest weight Λ_0 of the adjoint representations of these algebras are $\Lambda_0 = 2\Lambda_2$ for B_2 and $\Lambda_0 = \Lambda_2 + \Lambda_3$ for D_3, which are indeed equivalent to those of C_2 and A_3 given in Eq. (A4) after suitable re-ordering of simple roots, i.e., the inversion of the Dynkin diagram.

[23]S. Okubo, Univ. of Rochester Report No. UR-545, 1975 (unpublished).

[24]This fact for the algebra A_n ($n \geq 2$) has been observed before by L. C. Biedenharn, J. Math. Phys. 4, 436 (1963).

[25]A. Borel and C. Chevalley, Ref. 11. Note that only odd-order Casimir invariants of E_6 and D_n ($n \geq 3$) are of orders 5 and 9 for E_6 and n for D_n only if n is odd. The algebra A_n ($n \geq 2$) has odd-order Casimir invariants I_{2l+1} for all l with $1 \leq l \leq \frac{1}{2}n$. All others have none.

[26]A. M. Perelomov and V. S. Popov, Sov. J. Nucl. Phys. 3, 676, 819 (1966).

[27]J. D. Louck and L. C. Biedenharn, J. Math. Phys. 11, 2368 (1970).

[28]M. C. K. Aguilera-Navarro and V. C. Aguilera-Navarro, J. Math. Phys. 17, 1173 (1976).

[29]Actually, this is the reason why the eigenvalue $I_3(\rho)$ has a very simple form (A26). It is known from Refs. 21 and 27 that $I_N(\rho)$ must be a completely symmetric polynomial of σ_j's of the order N. But because of Eqs. (A28) and (16), $I_3(\rho)$ must be automatically proportional to $\sum_{j=1}^{n}(\sigma_j)^3$.

[30]When we note that $x_j = 2n\,\sigma_j$ are all integers, then the condition $I_3(\rho) = 0$ is equivalent to finding the integer solutions of the Diophantine equation

$$\sum_{j=1}^{n}(x_j)^3 = 0$$

such that all x_j are distinct from each other with conditions

$$\sum_{j=1}^{n} x_j = 0$$

and $x_j - x_k$ being integer multiples of $2n$. There always exists a solution satisfying $x_{n+1-j} = -x_j$, which corresponds to the self-contragradient representation.