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Gauge-invariant formulation of the self-dual sector*

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The field-strength formulation of non-Abelian gauge theories opens the gate to a description in terms of local gauge-invariant variables. As a first step in this direction, I work out a gauge-invariant formulation of the self-dual sector. A simple extension, using the ideas of Corrigan, Fairlie, and Yates, provides a gauge-invariant characterization of broader regions within the saddle point.

I. INTRODUCTION

In a previous paper,[1] I proposed a local formulation of non-Abelian gauge theories in terms of field strengths $G^a_{\mu\nu}$. The formulation focuses attention on the inverse field strength $(\mathcal{G}^{-1})^{ab}_{\mu\nu}$, $\mathcal{G}^{ab}_{\mu\nu} \equiv f^{abc}G^c_{\mu\nu}$, and configurations with $\det\mathcal{G}=0$ (all x_μ) are singular.[2] To go beyond semiclassical expansions about nonsingular configurations, one must prescribe G contours (or regulators) near the singular configurations — such that the G-functional integral equals the original integral over potentials. I will discuss this question elsewhere. The present work is addressed to another aspect of the field-strength formulation, and I will limit myself here to nonsingular configurations.

Having a formulation in terms of variables which "rotate" only (under gauge transformation) — instead of rotating and translating, as do the potentials — points the way to a local *gauge-invariant description*. In this paper I am going to treat only the very simple case of a self- (or anti-self) dual sector — plus a simple extension, using the ideas of Corrigan, Fairlie, and Yates,[3] to broader regions of the saddle point. Hopefully some of the strategy I employ in the self-dual sector will be of aid in the more general case.

II. CHOICE OF VARIABLES AND STRATEGY

In this paper, I treat only the gauge group O(3) in four space-time dimensions.[4] The field strength $G^a_{\mu\nu}$ has $6\times 3=18$ components. Three of these can be removed by gauge transformation, and so one might expect to specify the gauge-invariant content of the theory in terms of $18-3=15$ gauge invariants.

I will not pursue the full problem here, but rather go to the self-dual sector (Euclidean metric)

$$G=\tilde{G}\ , \quad \tilde{G}^a_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^a_{\rho\sigma}\ , \quad \epsilon_{0123}=+1\ . \tag{2.1}$$

Here we expect $9-3=6$ independent gauge-invariant variables. I express self-duality by writing

$$G^a_{\mu\nu}=E^a_i\eta'_{i\mu\nu}\ , \quad E^a_i=\tfrac{1}{4}\eta'_{i\mu\nu}G^a_{\mu\nu}\ , \qquad \eta'_{i\mu\nu}=\epsilon_{0i\mu\nu}-\delta_{\mu i}\delta_{\nu 0}+\delta_{\nu i}\delta_{\mu 0}\ . \tag{2.2}$$

η' is the self-dual 't Hooft tensor.[5] Of the nine components of the electric field E^a_i (the upper index is color, the lower is spin), three can be removed by gauge transformation (and two signs fixed).

Consider the six gauge-invariant quantities

$$E^a_iE^a_j\equiv g_{ij}\ . \tag{2.3}$$

In matrix notation $E^a_i\equiv(E)_{ai}$,

$$E^TE=g \tag{2.4}$$

and $E\to OE$, O orthogonal, is the gauge transformation. For reasons of provocation, I will refer to the symmetric 3×3 gauge-invariant matrix g as the "metric tensor." From this point of view, the electric field is the "dreibein" field for the metric.

The metric tensor is adequate to describe those regions of the self-dual sector in which (det is determinant)

$$\xi\equiv 2\det E \tag{2.5}$$

has fixed sign, for then $\xi=2\,(\det g)^{1/2}$ up to that fixed sign. For the general case of a boundary (sign change of ξ) I will include ξ itself as a variable to measure the remaining sign. I found it convenient to use this overcomplete set $\{g,\xi\}$ in the algebraic details of the reformulation. It is a conceptual advantage, however, to note that a complete set of (just) six gauge-invariant variables is easily found. Such a set is

$$\{\xi,\bar{g}\equiv(\det g)^{-1/3}g=(|\xi|/2)^{-2/3}g\}\ , \tag{2.6}$$

that is, a unimodular metric $\bar{g}$ and ξ.

The reader may find it instructive to work out the preceding paragraph in a particular gauge. A convenient choice is the following: E (the matrix) upper-triangular, and $E_{11}\geqslant 0$, $E_{22}\geqslant 0$. It is not hard to solve Eq. (2.4) explicitly for $E(\bar{g},\xi)$,

$$\begin{aligned} E_{1i} &= (g_{11})^{-1/2} g_{1i}\,, \\ E_{22} &= (g_{11})^{-1/2}(g_{11}g_{22} - g_{12}{}^2)^{1/2}\,, \\ E_{23} &= (g_{11})^{-1/2}(g_{11}g_{12} - g_{12}{}^2)^{-1/2}(g_{23}g_{11} - g_{12}g_{23})\,, \\ E_{33} &= \tfrac{1}{2}\xi(g_{11}g_{22} - g_{12}{}^2)^{-1/2}\,. \end{aligned} \tag{2.7}$$

I should say before proceeding that my choice of variables was just to get off the ground in the first place; many other hopefully superior, more elegant choices can be made. My variables are not even covariant, so I can expect no more from the reformulation. It may be helpful, e.g., to work in terms of the redundant but covariant variables $I_{\mu\nu;\rho\sigma}$ and $K_{\mu\nu}$,

$$\begin{aligned} I_{\mu\nu;\rho\sigma} &\equiv G^a_{\mu\nu} G^a_{\rho\sigma} = g_{ij}\eta'_{i\mu\nu}\eta'_{j\rho\sigma}\,, \\ g_{ij} &= \tfrac{1}{16}\eta'_{i\mu\nu}\eta'_{j\rho\sigma} I_{\mu\nu;\rho\sigma}\,, \\ K_{\mu\nu} &\equiv \tfrac{1}{3}\epsilon^{ijk}(G^i\tilde{G}^j G^k)_{\mu\nu} = \xi\delta_{\mu\nu}\,. \end{aligned} \tag{2.8}$$

My first attempt to solve the problem was along the following lines: I know the field-strength saddle-point equations[1] (e is the gauge coupling),

$$\mathfrak{F}^a_{\mu\nu}(\mathcal{J}[G]) + eG^a_{\mu\nu} = 0\,, \tag{2.9a}$$

$$\mathfrak{F}^a_{\mu\nu} = \partial_\mu \mathcal{J}^a_\nu - \partial_\nu \mathcal{J}^a_\mu + \epsilon^{abc}\mathcal{J}^b_\mu \mathcal{J}^c_\nu\,, \tag{2.9b}$$

$$\mathcal{J}^a_\mu[G] = (\mathcal{G}^{-1})^{ab}_{\mu\nu}\partial_\lambda G^b_{\lambda\nu}\,. \tag{2.9c}$$

In the self-dual sector, these are equations for the electric field. If I go to a particular gauge, say the upper-triangular gauge, I have the form $E = E(\bar{g}, \xi)$ explicitly Eq. (2.7). This form is of course gauge dependent, but if I substitute into Eq. (2.9), I obtain equations for $\bar{g}$, ξ which must be *independent* of gauge choice. Unfortunately this gauge is not rotationally invariant, and so the resulting equations for $\bar{g}$, ξ are in a terrible scramble. Other gauges (that I studied) suffer from compensating drawbacks. I report this "brute-force" approach for two reasons. First, anyone who tries to will believe it can be done in principle (and hence for full Yang-Mills?). Second, it provides a grim background against which to appreciate the relatively simple results I will present below.

The basic strategy of the next section is as follows. By using Eqs. (2.2) and (2.9b), (2.9c) we can construct $\mathfrak{F}^a_{\mu\nu}(E)$ ($\mathfrak{F}$ as a function of E and its derivatives). At first sight, this is a horrible expression; with a moment's thought, however, $\mathfrak{F}$ must be a function of E such that it rotates like E under gauge transformation. It is therefore reasonable to hope[6] that it can be put in the form

$$\mathfrak{F}^a_{\mu\nu} = \theta_{\mu\nu;m}(\bar{g}, \xi)E^a_m\,, \tag{2.10}$$

where θ is a function only of the gauge invariants. I shall show that this is precisely what happens. The gauge-invariant description follows immediately. Away from the singular configurations, E has an inverse. The field equations

$$(\theta_{\mu\nu;m} + e\eta'_{m\mu\nu})E^i_m = 0 \tag{2.11}$$

are then equivalent to the gauge-invariant description

$$\theta_{\mu\nu;m} + e\eta'_{m\mu\nu} = 0\,. \tag{2.12}$$

III. FIELD-STRENGTH STRUCTURE AND GAUGE-INVARIANT FORMULATION

As described in Ref. 1, we have for O(3) in four dimensions

$$\begin{aligned} (\mathcal{G}^{-1})^{ij} &= \tilde{G}^j G^i K^{-1}\,, \\ K &= \tfrac{1}{3}\epsilon^{ijk}G^i\tilde{G}^j G^k\,. \end{aligned} \tag{3.1}$$

In the self-dual sector K is proportional to the unit matrix, and we have the "gluon" $\mathcal{J}$,

$$\begin{aligned} \mathcal{J}^i_\mu &= \xi^{-1}(G^j G^i)_{\mu\nu}\partial_\lambda G^j_{\lambda\nu}\,, \\ K &= \xi = 2\det E\,. \end{aligned} \tag{3.2}$$

Note that $\det\mathcal{G}$ not identically zero for all x_μ implies the same for ξ and $\det g$.

Our first real task in this paper is then to sort out the numerator of the gluon $\mathcal{J}^i_\mu$. I would like to do the sorting in such a manner (consistent with our strategy) as to show the gauge-transformation properties of the gluon. Each term in the gluon numerator is cubic in E, with two color indices contracted. When this pair is separated by a derivative, I separate out the gauge-invariant part with the identity

$$\begin{aligned} E^i_l\partial_\mu E^i_m &= \tfrac{1}{2}(\partial_\mu g_{lm} + \epsilon_{lmp}\mathcal{A}^\mu_p)\,, \\ \mathcal{A}^\mu_p &\equiv \epsilon_{prs}E^a_r\partial_\mu E^a_s\,. \end{aligned} \tag{3.3}$$

$\mathcal{A}^\mu_p$ is a (global) color-singlet "spin density." In this way, after some algebra, I can express the gluon as[7]

$$\begin{aligned} \mathcal{J}^a_\mu &= \xi^{-1}(\mathcal{A}^\mu_m + \eta'_{j\mu\nu}\partial_\nu\tilde{g}_{jm})E^a_m\,, \\ \tilde{g}_{jm} &\equiv g_{jm} - \tfrac{1}{2}\delta_{jm}\mathrm{Tr}(g)\,, \end{aligned} \tag{3.4}$$

where Tr means trace.

This form shows the gluon gauge-transformation property nicely. Under the infinitesimal rotation

$$\delta E^a_i = \epsilon^{abc}\chi^b E^c_i\,, \tag{3.5}$$

χ^b arbitrary, the spin density changes by

$$\begin{aligned} \delta\mathcal{A}^\mu_p &= -\xi\partial_\mu\chi^b\bar{E}^b_p\,, \\ \bar{E}^a_i &\equiv \xi^{-1}\epsilon^{abc}\epsilon_{ijk}E^b_j E^c_k\,, \\ \bar{E}^b_p E^b_q &= \delta_{pq}\,, \quad \bar{E}^b_p E^a_p = \delta^{ab}\,. \end{aligned} \tag{3.6}$$

Thus, I verify the gluon gauge transformation,

$$\delta\mathcal{J}^a_\mu = -\partial_\mu\chi^a + \epsilon^{abc}\chi^b\mathcal{J}^c_\mu\,, \tag{3.7}$$

with the spin density providing the translation term. In fact, the first term in $\mathcal{J}$.

$$\tilde{\mathcal{J}}^a_\mu \equiv \xi^{-1}\mathcal{A}^\mu_m E^a_m = -\tfrac{1}{2}\epsilon^{abc}\bar{E}^b_l \partial_\mu E^c_l , \tag{3.8}$$

has, by *itself*, the correct transformation property [Eq. (3.7)]. This gives us a simple laboratory to test our strategy.

I have computed the simpler object $\mathcal{F}^a_{\mu\nu}(\tilde{\mathcal{J}})$. Helpful identities are

$$\begin{aligned}
\epsilon^{ijk}\epsilon^{jlm}\epsilon^{kpq} &= -(\delta_{jp}\delta_{lq} - \delta_{jq}\delta_{lp})\epsilon^{jim} \\
&\quad + (\delta_{jp}\delta_{mq} - \delta_{mp}\delta_{jq})\epsilon^{jil} , \\
\bar{E}^a_i \partial_\mu E^b_i &= -E^b_i \partial_\mu \bar{E}^a_i , \quad \bar{E}^a_i \partial_\mu E^a_j = -E^a_j \partial_\mu \bar{E}^a_i , \\
\bar{E}^a_i \bar{E}^a_j &= (g^{-1})_{ij} , \quad \bar{E}^k_a = E^k_b (g^{-1})_{ba} .
\end{aligned} \tag{3.9}$$

With the help of the first four identities and Eqs. (2.3), (3.6), I can manage a contraction to 1, g or g^{-1} in each of the terms in $\epsilon^{abc}\tilde{\mathcal{J}}^b_\mu \tilde{\mathcal{J}}^c_\nu$. Then I eliminate all $\bar{E}$'s in favor of E via the last identity. After some algebra, I find that all terms with derivatives on E cancel, leaving the simple result

$$\mathcal{F}^i_{\mu\nu}(\tilde{\mathcal{J}}) = \tfrac{1}{4}\partial_\mu (g^{-1})_{ca} g_{ab} \partial_\nu (g^{-1})_{bd} \epsilon^{ijk} E^j_c E^k_d . \tag{3.10}$$

This certainly shows the correct gauge-transformation property. I can also put it in the desired (linear-in-E) form by the identity

$$\epsilon^{ijk} E^j_c E^k_d = \tfrac{1}{2}\xi E^i_a (g^{-1})_{ab} \epsilon_{bcd} . \tag{3.11}$$

Thus encouraged, I have computed the remaining terms in $\mathcal{F}^a_{\mu\nu}(\mathcal{J})$. After some similar algebra, I obtain the desired result,

$$\mathcal{F}^i_{\mu\nu} = \theta_{\mu\nu;m} E^i_m , \tag{3.12a}$$

$$\begin{aligned}
\theta_{\mu\nu;m} &= \tfrac{1}{2}\xi\left[\tfrac{1}{4}\partial_\mu (g^{-1})_{ca} g_{ab} \partial_\nu (g^{-1})_{bd} + Z_{\mu c} Z_{\nu d}\right] \\
&\quad \times (g^{-1})_{mr}\epsilon_{rcd} \\
&\quad + \partial_\mu Z_{\nu m} - \partial_\nu Z_{\mu m} \\
&\quad + \tfrac{1}{2}(g^{-1})_{ml}(\partial_\mu g_{ln} Z_{\nu n} - \partial_\nu g_{ln} Z_{\mu n}) ,
\end{aligned} \tag{3.12b}$$

$$Z_{\mu i} \equiv \xi^{-1}\eta'_{a\mu\sigma}\partial_\sigma \tilde{g}_{ai} . \tag{3.12c}$$

The first term in θ corresponds to Eq. (3.10). A nice check on the form of θ is to remember that $\mathcal{F}_{\mu\nu}$ is scale invariant under $G \to \kappa^{1/2} G$. Now, under $g \to \kappa g$, $\xi \to \kappa^{3/2}\xi$, θ is homogeneous of degree $\kappa^{-1/2}$. The form Eq. (3.12) for $\mathcal{F}^i_{\mu\nu}$ is much simplified over previous work, and will be an aid in the field-strength formulation itself.

The field-strength equations of motion Eq. (2.9) are then

$$(\theta_{\mu\nu;m} + e\eta'_{m\mu\nu})E^i_m = 0 . \tag{3.13}$$

And, as discussed in Sec. II, the gauge-invariant formulation is simply

$$\theta_{\mu\nu;m} + e\eta'_{m\mu\nu} = 0 . \tag{3.14}$$

Multiplying our result by g_{mk} puts the equations in a form resembling general relativity (say in the gauge with $g_{00} = 1$, $g_{0i} = 0$) with a cosmological constant.

Other interesting quantities are easily evaluated in terms of our variables. For example, ignoring the "Faddeev-Popov" terms,[1] I find for the field-strength action, using Eq. (3.12a),

$$\begin{aligned}
S &= \int d^4x \left(-\frac{1}{2e} G^a_{\mu\nu}\mathcal{F}^a_{\mu\nu} - \tfrac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} \right) \\
&= \int d^4x \left[-\frac{1}{e}(\theta_{0i;m} + \tilde{\theta}_{0i;m}) g_{mi} - \mathrm{Tr}(g) \right] ,
\end{aligned}$$

where $\tilde{\theta}$ is the usual dual of θ.

IV. SIMPLEST ANSATZ

The gauge-invariant equations of motion, Eq. (3.14), are conveniently grouped, following Ref. 1, as self-dual and anti-self-dual parts,

$$\theta^+_{\mu\nu;m} + e\eta'_{m\mu\nu} = 0 , \quad \theta^-_{\mu\nu;m} = 0 , \tag{4.1a}$$

$$\theta = \theta^+ + \theta^- , \quad \theta^\pm = \tfrac{1}{2}(\theta \pm \tilde{\theta}) . \tag{4.1b}$$

I will seek a solution to these in the form of the *simplest conceivable* ansatz

$$\begin{aligned}
\bar{g} &= 1 , \\
\xi &= 2\lambda^3 .
\end{aligned} \tag{4.2}$$

From Eq. (2.6), this is equivalent to $g = \lambda^2$, $\xi = 2\lambda^3$. I have used λ instead of just ξ itself only to make contact with Ref. 1. The equations of motion (4.1a), (4.1b) become, respectively,

$$-\frac{\partial_\mu\lambda\partial_\mu\lambda}{4\lambda^3} + \frac{1}{2}\frac{\Box^2\lambda}{\lambda^2} + 2e = 0 \tag{4.3a}$$

and

$$\begin{aligned}
&\delta_{mi}\left[\frac{3}{4}\left(\frac{\partial_j\lambda\partial_j\lambda}{\lambda^3} - \frac{\partial_0\lambda\partial_0\lambda}{\lambda^3}\right) + \frac{1}{2}\left(\frac{\partial_0{}^2\lambda}{\lambda^2} - \frac{1}{2}\frac{\partial_j\partial_j\lambda}{\lambda^2}\right)\right] \\
&\quad - \frac{3}{2}\frac{\partial_i\lambda\partial_m\lambda}{\lambda^3} + \frac{\partial_m\partial_i\lambda}{\lambda^2} + \epsilon_{mij}\left(\frac{3}{2}\frac{\partial_j\lambda\partial_0\lambda}{\lambda^3} - \frac{\partial_0\partial_j\lambda}{\lambda^2}\right) = 0 .
\end{aligned} \tag{4.3b}$$

These equations are precisely those implied (before the $\lambda[R]$ assumption) in Ref. 1, now divided by λ. (The ansatz there was equivalent, $E^a_i = \delta^a_i\lambda$, $\mathcal{J}^i_\mu = -\eta'_{i\mu\nu}\frac{1}{2}\partial_\nu \ln\lambda$.) Note that the simple variable change $\lambda = \phi^2$ ($\xi = 2\phi^6$) brings the self-dual part of the equations of motion [Eq. (4.3a)] to the form[8]

$$\frac{\Box^2\phi}{\phi^3} + 2e = 0 . \tag{4.4}$$

Nevertheless, as stated in Ref. 1, when taken together Eqs. (4.3a), (4.3b) have as their only common solution the translated pseudoparticle[9]

$$\lambda = \frac{4b}{e} \frac{1}{[(x-x_0)^2+b]^2} \tag{4.5}$$

with $b, x_{0\mu}$ arbitrary. In the next section, I will discuss the circumstance under which Eq. (4.4) survives without interference from the anti-self-dual Eq. (4.3b).

There are also known[5] self-dual solutions of the form $V^a_\mu = -(1/g)\mathcal{J}^a_\mu = (1/g)\eta_{a\mu\nu}\partial_\nu \ln\phi$, $\Box^2\phi = 0$ (η is the *anti*-self-dual tensor in our notation). Except for the pseudoparticle itself, however, these are not included in our simplest ansatz Eq. (4.2). (However, see Sec. V.)

V. CORRIGAN-FAIRLIE-YATES AND A BROADER GAUGE-INVARIANT CHARACTERIZATION

Corrigan, Fairlie, and Yates (CFY)[3] have recently distributed a very interesting paper that independently observes the simplification of $\mathcal{G}^{-1}$ in the self-dual sector.[1] They also observed that the *same* simplification describes broader regions of the saddle point: one need not set the anti-self-dual part to zero.

They begin with the saddle-point equations (2.9) in the form

$$\mathcal{F}^a_{\pm\mu\nu}(\mathcal{J}) + eG^a_{\pm\mu\nu} = 0\,, \tag{5.1a}$$

$$\partial_\mu G^a_{\mu\nu} + \epsilon^{abc}\mathcal{J}^b_\mu G^c_{\mu\nu} = 0\,, \tag{5.1b}$$

$$G = G_+ + G_-\,, \quad G^a_{+\mu\nu} = S^a_{+i}\eta'_{i\mu\nu}\,, \quad G^a_{-\mu\nu} = S^a_{-1}\eta_{i\mu\nu}\,, \tag{5.1c}$$

$$\partial_\mu \tilde{G}^a_{\mu\nu} + \epsilon^{abc}\mathcal{J}^b_\mu \tilde{G}^c_{\mu\nu} = 0\,. \tag{5.1d}$$

Equation (5.1d) follows from Eq. (5.1a). Instead of setting $S_- = 0$ (as I did), they combine (5.1b) and (5.1d) to obtain

$$\partial_\mu G^a_{+\mu\nu} + \epsilon^{abc}\mathcal{J}^b_\mu G^c_{+\mu\nu} = 0\,, \tag{5.2a}$$

$$\partial_\mu G^a_{-\mu\nu} + \epsilon^{abc}\mathcal{J}^b_\mu G^c_{-\mu\nu} = 0\,. \tag{5.2b}$$

Equations (5.1a) and (5.2a), (5.2b) are equivalent to the original set. If (say) G_+ has an inverse $\mathcal{G}_+^{-1}$ then

$$\mathcal{J}^a_\mu = (\mathcal{G}_+^{-1})^{ab}_{\mu\nu}\partial_\lambda G^b_{+\lambda\nu}\,. \tag{5.3}$$

On inspection of Eqs. (2.2) and (5.1c), it is evident then that $\mathcal{G}_+^{-1}$ and $\mathcal{J}^a_\mu(S_+)$ are precisely *our* forms in the self-dual sector with the *identification*

$$E^a_i \to S^a_{+i}\,. \tag{5.4}$$

Equations for $S_\pm$

$$\mathcal{F}^a_{+\mu\nu}(\mathcal{J}[S_+]) + eS^a_{+i}\eta'_{i\mu\nu} = 0\,, \tag{5.5a}$$

$$\mathcal{F}^a_{-\mu\nu}(\mathcal{J}[S_+]) + eS^a_{-i}\eta_{i\mu\nu} = 0 \tag{5.5b}$$

reduce to ours if $S^a_{-1} = 0$. In general, CFY point out that, given S_+, S_- may be computed from (5.5b). The resulting G_- will solve the original equations: We have already satisfied (5.1a) *in toto* [and hence (5.1d)], and (5.2a). Subtracting (5.2a) from (5.1d), we obtain (5.2b).

It is easy then to link up the advances of the present paper with the observation of CFY: *All results of this paper for E should be read in terms of the map Eq. (5.4)*, and the anti-self-dual parts of the equations should be used to compute S_-. One is then reading, e.g.,

$$g_{ij} \equiv S^a_{+i}S^a_{+j}\,, \quad \xi \equiv 2\det S_+\,, \tag{5.6}$$

etc., in (say) Eqs. (3.4) and (3.12). The field-strength equations read

$$(\theta^+_{\mu\nu;m} + e\eta'_{m\mu\nu})S^i_{+m} = 0\,, \tag{5.7a}$$

$$\theta^-_{\mu\nu;m}S^a_{+m} + S^a_{-m}\eta_{m\mu\nu} = 0 \tag{5.7b}$$

[and (5.7b) determines S_-]. Finally, the gauge-invariant equations for $\bar{g}$, ξ are (just the self-dual part of our previous equations)

$$\theta^+_{\mu\nu;m} + e\eta'_{m\mu\nu} = 0\,. \tag{5.8}$$

Thus, under the "simplest" ansatz (4.2), the result (4.4) stands without interference. It deserves emphasis then that the Wilczek-Corrigan-Fairlie ansatz is, within the gauge-invariant formulation, the simplest conceivable ansatz.

Note that *all gauge invariants can be constructed from* $\bar{g}$, ξ using, say,

$$\begin{aligned} G^a_{\mu\nu} &= -\frac{1}{e}\,\theta_{\mu\nu;i}S^a_{+i}\,, \\ \tilde{G}^a_{\mu\nu} &= -\frac{1}{e}\,\tilde{\theta}_{\mu\nu;i}S^a_{+i}\,. \end{aligned} \tag{5.9}$$

For example, one easily computes

$$\begin{aligned} K_{\mu\nu} &= \tfrac{1}{3}\epsilon^{abc}(G\tilde{G}G)_{\mu\nu} \\ &= -\frac{\xi}{6e^3}\epsilon_{ijk}\theta_{\mu\rho;i}\tilde{\theta}_{\rho\sigma;j}\theta_{\sigma\nu;k}\,. \end{aligned} \tag{5.10}$$

The CFY observation works as long as not both $\det S_\pm = 0$. This is not as broad as the "full" saddle point ($\det\mathcal{G} \neq 0$), which presumably needs 15 gauge-invariant variables.

Following CFY, I can also find the $\Box^2\phi = 0$ solutions[5] as a limit. As mentioned in Sec. III, under $S_+ \to \kappa^{1/2}S_+$, we have $\bar{g} \to \bar{g}$, $\xi \to \kappa^{3/2}\xi$, $\theta \to \kappa^{-1/2}\theta$. Thus with $S_+ = \kappa^{1/2}S'_+$, $\xi = \kappa^{3/2}\xi'$ we have, in the limit $\kappa \to 0$,

$$\theta^+_{\mu\nu;m}(\bar{g}, \xi') = 0\,, \tag{5.11a}$$

$$\theta^-_{\mu\nu;m}(\bar{g}, \xi')S^{a\prime}_{+m} + S^a_{-m}\eta_{m\mu\nu} = 0\,. \tag{5.11b}$$

The ansatz (4.2) for ξ' then comes to $\phi^{-3}\Box^2\phi = 0$ and (5.11b). Thus (still) $\mathcal{J}^i_\mu = -\eta'_{i\mu\nu}\partial_\nu \ln\phi$ but the solutions are *anti*-self-dual.

Note added in proof. After submission of this manuscript, I received a report from H. R. Pagels [Aspen report (unpublished)] which overlaps the work of Ref. 3.

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[3]E. Corrigan, D. B. Fairlie, and R. G. Yates, Durham report, 1977 (unpublished).

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[5]G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976); Phys. Rev. D 14, 3432 (1976).

[6]I believe a proof can be found along the lines of the construction of Sec. III. Such an understanding, useless in the light of Sec. III, might be crucial for the more general problem.

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