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WKB wave function for systems with many degrees of freedom: A unified view of solitons and pseudoparticles

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The WKB method for systems with many degrees of freedom is developed. Using a given imaginary-time (Euclidean) classical solution of the equations of motion, we explicitly construct the WKB wave function in the classically forbidden region of configuration space. Similarly, we construct the wave function for the allowed region using a real-time (Minkowski) solution. For this purpose we use the collective-coordinate method previously developed for solitons in quantum field theory. The present WKB method is an extention of that by Banks, Bender, and Wu to systems with many degrees of freedom and field theories. This paper is intended to present ideas and the general formalism: two applications are briefly discussed: the quantization condition for periodic solutions and vacuum tunneling in field theories.

I. INTRODUCTION

The WKB approximation is one of the basic methods for studying quantum systems. It is, however, simple only for one degree of freedom. In the general case, one difficulty has been that one does not know the WKB wave functions explicitly. Keller,[1] Gutzwiller,[2] and Maslov[3] developed a method based on functional integrals in order to avoid the explicit introduction of WKB eigenstates. The method has been applied to field theory by Dashen, Hasslacher, and Neveu[4] and has led to very interesting developments in field theory based on semiclassical approximations.[5]

It is clear, however, that one cannot always avoid the use of wave functions. For instance, in the semiclassical treatment of soliton scattering by path integrals,[6] eigenstates of the Hamiltonian had to be built so as to establish the scattering formalism, although the method used there is only formal because the eigenstates so obtained involve the momentum of the field. However, the use of wave functions is obviously unavoidable if one wants to discuss matching at turning points. This problem arises especially in connection with recent studies of vacuum tunneling in field theory.[5]

In two remarkable papers[7] Banks, Bender, and Wu studied this question in particular systems with two degrees of freedom. Their basic idea was that barrier penetration occurs mostly in small tubes in configuration space around certain classical solutions so that the WKB approximation is essentially one dimensional and they could determine the ground-state wave function. These classical paths correspond to classical solutions with pure imaginary time.

Up to now, vacuum tunneling in field theory has been studied mostly by path integrals in Euclidean space-time. In a recent paper[8] we proposed an interpretation of Euclidean classical solutions in Minkowski space-time which, as we later realized, is the generalization to field theory of the ideas of Banks, Bender, and Wu. In Euclidean field theory the problem of matching is avoided through the so-called dilute-gas approximation.[9] As we shall argue later on, this is not satisfactory for theories with no mass scale, and one seems to be forced to really handle the matching at turning points in field theory.

In studying this question we found that, contrary to the common belief, general WKB eigenfunctions, to the first two orders in $\hbar$ and for a given classical trajectory, are rather simple objects which can be systematically written down once one has solved the classical problem of small fluctuations around the classical trajectory considered. This is the subject of the present paper. We should remark here that if one applies this WKB method to the soliton problems the basic formalism is similar to that of Christ and Lee.[10]

The general ideas are developed in Sec. II. Two examples of applications are given in Sec. III with special emphasis on vacuum tunneling. This paper is intended to be self-contained and to be understandable to readers who are not familiar with field theory. Hence we use the language of quantum mechanics though we also have in mind field-theory implications.

II. DETERMINATION OF WKB WAVE FUNCTION

Let us consider a system with N degrees of freedom. We denote the generalized coordinates by $\vec{R}$

and the potential by $V(\vec{R})$. The Lagrangian of the system is assumed to be $\mathcal{L}=\frac{1}{2}\dot{\vec{R}}^2-V(\vec{R})$. Field theory can be regarded formally as a system of infinite degrees of freedom, $N\to\infty$. The simplest way to deal with the $\hbar$ expansion is to introduce a parameter g such that V can be written as

$$V(\vec{R})=(1/g^2)\mathcal{V}(g\vec{R}), \tag{2.1}$$

where $\mathcal{V}$ does not depend on g. This means that in V, the nth power of $\vec{R}$ has a coefficient proportional to g^{n-2}. From (2.1) one sees that any classical solution and classical action is, respectively, proportional to g^{-1} and g^{-2}. Letting $\hbar=1$, we thus see that g^2 plays the role of $\hbar$ and the semiclassical approximation will mean expansion in g.

As is well known, in the leading order in the WKB approximation, the Schrödinger equation reduces to the Hamilton-Jacobi equation. Namely, if we let

$$H\psi=E\psi,\quad \psi=e^{i\epsilon W},$$

we get, to leading order,

$$\frac{\epsilon^2}{2}\left(\frac{\partial W}{\partial R_i}\right)^2+V(\vec{R})=E. \tag{2.2}$$

Obviously, we can choose $\epsilon^2=1$ if $E>V$ (classically allowed region) and $\epsilon^2=-1$ if $E<V$ (classically forbidden region). Equation (2.2) is the Hamilton-Jacobi equation with potential ϵ^2V and energy ϵ^2E. By the standard method, solutions of (2.2) are obtained as follows:

$$W(\vec{R})=\int^{s_1}ds\,[2\,|E-V(\vec{r}(s))|\,]^{1/2},$$
$$\left(\frac{d\vec{r}}{ds}\right)^2=1,\quad \vec{r}(s_1)=\vec{R}, \tag{2.3}$$

where the curve $\vec{r}$ is such that the integral is stationary. The classical meaning of $\vec{r}$ is best shown by introducing another parametrization denoted by τ such that ($\vec{r}_\tau\equiv d\vec{r}/d\tau$)

$$\frac{1}{2\epsilon^2}\vec{r}_\tau^{\,2}+V(\vec{r})=E. \tag{2.4}$$

Then $\vec{r}$ should satisfy

$$\frac{1}{\epsilon^2}\vec{r}_{\tau\tau}=-\vec{\nabla}V(\vec{r}). \tag{2.5}$$

Hence, $\epsilon\tau$ plays the role of time for a classical trajectory with energy E and potential energy V. In the forbidden region $\epsilon\tau$ is purely imaginary. Note that ψ is a stationary state, and it is thus clear that τ has nothing to do with the true time.

In order to obtain the functional form of $W(\vec{R})$ by (2.3) we must know the general solution of (2.4) and (2.5), i.e., the trajectory passing through a point $\vec{R}_0$ and an arbitrary point $\vec{R}$ in configuration space. In practice, however, especially in field theories one knows explicitly only a limited class of classical trajectories. So we assume that only a classical solution $\vec{r}(\tau)$ with energy E_0 is given, and consider the wave function in the vicinity of the classical trajectory in configuration space.

Then (2.3) is simply the WKB exponent for one degree of freedom which is the position along the trajectory. Hence, the dominant effect due to the existence of a classical trajectory $\vec{r}(\tau)$ is contained in the quantum mechanics of this degree of freedom. Following our general method,[11] we introduce it as a collective coordinate; that is, we extract it out of $\vec{R}$ through the change of variable:

$$\vec{R}=\vec{r}(f(q))+\sum_{a=2}^{N}\vec{n}_a(f(q))\eta^a, \tag{2.6}$$

where f is an arbitrary given function which fixes the parametrization of the curve, q is the new coordinate which indicates the position on the curve, and $\vec{n}_a(\tau)$ together with $\vec{r}_\tau(\tau)$ form a moving local reference frame at the point $\vec{r}(\tau)$. We choose it such that

$$\vec{n}_a\cdot\vec{n}_b=\delta_{ab},\quad \vec{n}_a\cdot\vec{r}_\tau=0.$$

Equation (2.6) can actually represent only a small neighborhood of the classical curve in configuration space because the vectors $\vec{r}_\tau$, $\vec{n}_a$ form only a local reference frame. Consistency will be achieved at the end when we will obtain the wave function which decreases away from the classical trajectory with an exponential decrease of order g^0. Indeed, if this is verified, the relevant values of η^a are such that $|\eta^a|\ll|\vec{r}(\tau)|$ because $\vec{r}(\tau)$, a classical solution, is of order g^{-1}. We shall come back to this point later on. For the reader who is more familiar with the $\hbar$ expansion we note that $\vec{r}(\tau)$ and $\vec{\eta}$ are of order $\hbar^0$ and $\hbar^{1/2}$, respectively, so that the same picture also emerges.

A straightforward computation shows that

$$\vec{p}\equiv-i\vec{\nabla}=\frac{\vec{r}_\tau(f(q))}{f'(\vec{r}_\tau^{\,2}-\vec{r}_{\tau\tau}\cdot\vec{\eta})}\left(p-f'\sum_{a,b}\eta^a\Gamma_a^b\zeta^b\right)+\sum_a\vec{n}_a\zeta^a, \tag{2.8}$$

$$p=-i\frac{\partial}{\partial q},\quad \zeta^a=-i\frac{\partial}{\partial\eta^a},\quad \Gamma_a^b=\vec{n}_b\cdot\vec{n}_{a\tau}$$
$$\vec{\eta}=\sum_a\eta^a\vec{n}_a,\quad f'=\frac{df}{dq}. \tag{2.9}$$

In the above expressions $\vec{r}$ and $\vec{n}_a$ are to be considered for $\tau=f(q)$, τ indices mean taking derivatives with respect to τ before replacing τ by $f(q)$. We use the same conventions hereafter.

We insert (2.6) and (2.8) into the Hamiltonian and expand in powers of g. To the order we are working, i.e., g^0 order, the ordering of operators is irrelevant. One gets

$$H \simeq \frac{p^2}{2f'^2\vec{r}_\tau^{\,2}} + V^{(0)}(q) + \frac{p^2}{f'^2(\vec{r}_\tau^{\,2})^2}\,\vec{r}_{\tau\tau}\cdot\vec{\eta} + V_a^{(1)}\eta^a + \tfrac{1}{2}(\zeta^a)^2 + \tfrac{1}{2}V_{ab}^{(2)}\eta^a\eta^b - \frac{p\Gamma_a^b\eta^a\zeta^b}{f'\vec{r}_\tau^{\,2}} + \frac{3}{2}\frac{p^2}{f'^2}\frac{(\vec{r}_{\tau\tau}\cdot\vec{\eta})^2}{(\vec{r}_\tau^{\,2})^3}\,. \tag{2.10}$$

We have expanded the potential:

$$V(\vec{R}) = V^0(q) + \eta^a V_a^{(1)}(q) + \tfrac{1}{2}\eta^a\eta^b V_{ab}^{(2)}(q) + \cdots\,. \tag{2.11}$$

It is easy to see that $V^{(n)}(q) = O(g^{n-2})$.

Let us now solve the Schrödinger equation to the first two leading orders by letting

$$H\psi = (E_0 + E_1)\psi\,,\quad E_0 = O(g^{-2})\,,\quad E_1 = O(g^0)\,, \tag{2.12}$$

$$\psi(q,\vec{\eta}) = e^{i\epsilon S_0}\tilde{\chi}(q,\vec{\eta})\,,\quad S_0 = O(g^{-2})\,,\quad \tilde{\chi} = O(g^0)\,. \tag{2.13}$$

The Schrödinger equation to orders g^{-n}, $n=4,2,1,0$, lead, respectively, to the equations

$$\frac{\partial S_0}{\partial\eta^a} = 0\,, \tag{2.14a}$$

$$\left(\frac{\partial S_0}{\partial q}\right)^2 = 2f'^2\vec{r}_\tau^{\,2}\,|E_0 - V^{(0)}|\,, \tag{2.14b}$$

$$(\epsilon^2\vec{r}_{\tau\tau}\cdot\vec{n}_a + V_a^{(1)})\eta^a = 0\,, \tag{2.14c}$$

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\eta^{a2}}\, i\epsilon\left(\frac{1}{f'}\frac{\partial}{\partial q} - \Gamma_a^b\eta^a\frac{\partial}{\partial\eta^b}\right) + \tfrac{1}{2}W_{ab}\eta^a\eta^b - \frac{i\epsilon}{2f'}\frac{\partial}{\partial q}\ln(f'\vec{r}_\tau^{\,2}) - E_1\right]\tilde{\chi} = 0\,, \tag{2.14d}$$

$$W_{ab} = V_{ab}^{(2)} + \frac{3\epsilon^2}{\vec{r}_\tau^{\,2}}\, r_{\tau\tau}^a r_{\tau\tau}^b\,. \tag{2.15}$$

Equation (2.14b) is, as expected, the leading WKB equation for q degrees of freedom, and we get

$$S_0(q) = \int^q dq'\,(2f'^2\,\vec{r}_\tau^{\,2}\,|E_0 - V^{(0)}|)^{1/2}\,. \tag{2.16}$$

It is readily checked to be of order g^{-2} if f is of order g^0.

Next, projecting equation (2.5) onto the vectors $\vec{n}_a(\tau)$, $a=2,\ldots,N$, one sees that (2.14c) is indeed satisfied since $\vec{r}(\tau)$ is a classical solution.

Our task is now to solve equation (2.14d). For this we first remark that as one could have expected, we only have the combination $(1/f')\partial/\partial q$, so that it is simpler to re-express $\tilde{\chi}$ as a function of τ redefined by $\tau = f(q)$ in any region where f is single valued. Note that f should be chosen such that f' is always nonvanishing and we assume f' to be positive. The next-to-last term in (2.14d) corresponds to the standard WKB factor of order zero in g for q quantum mechanics. It disappears if we redefine $\tilde{\chi}$ as

$$\tilde{\chi} = \frac{\chi}{(f'\vec{r}_\tau^{\,2})^{1/2}} = \frac{\chi}{(S_0')^{1/2}}\,, \tag{2.17}$$

and we have to solve the equations

$$\mathcal{H}\chi = E_1\chi\,; \tag{2.18}$$

$$\mathcal{H} \equiv -i\epsilon\left(\frac{\partial}{\partial\tau} - \Gamma_a^b\eta^a\frac{\partial}{\partial\eta^b}\right) - \frac{1}{2}\frac{\partial^2}{\partial\eta^{a2}} + \tfrac{1}{2}W_{ab}\eta^a\eta^b\,. \tag{2.19}$$

This is a nontrivial problem since both Γ and W are functions of τ. The crucial point of our method is that (2.18) can be solved if one knows a complete set of solutions for the equation of small fluctuations around $\vec{r}(\tau)$. Denote such a solution by $\vec{v}$. From (2.5) it satisfies

$$\frac{1}{\epsilon^2}v^i_{\tau\tau} = -\left.\frac{\partial^2 V}{\partial R^i\partial R^j}\right|_{\vec{R}=\vec{r}(\tau)} v^j\,. \tag{2.20}$$

We shall assume that the matrix $\partial^2V/\partial R^i\partial R^j|_{\vec{r}}$ is positive definite. Hence, (2.20) will have solutions with real exponential (oscillating exponential) behavior for $\epsilon^2 = -1$ ($\epsilon^2 = +1$). Expand $\vec{v}$ in the moving frame by

$$\vec{v} = (\vec{v}\cdot\vec{r}_\tau)\vec{r}_\tau/\vec{r}_\tau^{\,2} + u_a\vec{n}_a\,. \tag{2.21}$$

Taking the derivative of (2.5) with respect to τ, one sees that $\vec{r}_\tau$ is also a solution of (2.20). From the Wronskian argument one gets

$$\frac{d}{d\tau}(\vec{r}_\tau\cdot\vec{v}_\tau - \vec{r}_{\tau\tau}\cdot\vec{v}) = 0\,.$$

So we can choose $\vec{v}$ such that

$$\vec{r}_\tau\vec{v}_\tau = \vec{r}_{\tau\tau}\vec{v}\,. \tag{2.22}$$

From this one can check that (2.20) implies for u^a the equations

$$D_{ab}D_{bc}u_c + \epsilon^2 W_{ab}u_b = 0\,, \tag{2.23}$$

$$D_{ab} \equiv (\partial/\partial\tau)\delta_{ab} + \Gamma^b_a\,. \tag{2.24}$$

The method of solving (2.18) is based on the remark that if u satisfies (2.23), the operator

$$A \equiv e^{-i\nu\tau/\epsilon}\left[u_a\frac{\partial}{\partial\eta^a} - i\epsilon(Du)_a\eta^a\right] \tag{2.25}$$

is such that

$$[\mathcal{H}, A] = -\nu A\,. \tag{2.26}$$

Hence, if ν is positive (negative), A acts as a destruction (creation) operator on the eigenfunctions of $\mathcal{H}$. If ν is zero, A is not interpretable in terms of the creation-annihilation operator as it commutes with $\mathcal{H}$. This will be related to the well-known zeroth-mode phenomenon, which is linked to symmetry properties of V.

The set of ν and operators A which can appear will be specified by the boundary conditions of the region of configuration space considered. We shall illustrate this point with two specific examples: periodic orbit in allowed region, and the penetration problem for quantum fluctuations around a local minimum of V. The later example is obviously relevant for vacuum tunneling. We shall not discuss a true WKB matching at a turning point since we have not yet studied this problem in detail. In order to simplify the discussion we shall further assume that none of the ν's encountered vanishes. Some comments on the general case are given at the end of this section.

In the first case we will have

$$\vec{r}(\tau)=\vec{r}(\tau+T), \tag{2.27}$$

and A should be periodic of period T so that both functions (here $\epsilon=\pm 1$)

$$(u)e^{-i\nu\tau/\epsilon}\equiv g\,,\quad (Du)e^{-i\nu\tau/\epsilon}\equiv f \tag{2.28}$$

must be periodic with period T. At this point the discussion proceeds along lines similar to Ref. 6. From (2.23) and (2.28) we see that ν is such that

$$\begin{aligned}&\mathcal{B}\begin{pmatrix}g\\ f\end{pmatrix}=\nu\begin{pmatrix}g\\ f\end{pmatrix},\\ &\mathcal{B}\equiv i\begin{pmatrix}\epsilon D_{ab} & -\delta_{ab}\\ W_{ab} & \epsilon D_{ab}\end{pmatrix}.\end{aligned} \tag{2.29}$$

The periodicity condition makes $\mathcal{B}$ Hermitian with an inner product

$$(2,1)\equiv\int_0^T d\tau(g_2^* \, f_2^*)\sigma_2\begin{pmatrix}g_1\\ f_1\end{pmatrix}, \tag{2.30}$$

and the ν's are the set of eigenvalues of the operator $\mathcal{B}$. They are necessarily real since $\mathcal{B}$ is Hermitian. Since $\mathcal{B}$ is purely imaginary, we see that if $\binom{g}{f}$ is an eigenvector of $\mathcal{B}$ with eigenvalue ν, $\binom{g^*}{f^*}$ is also an eigenvector but with eigenvalue $-\nu$. Let ν_m be the set of all positive ν's and u^m be the set of corresponding small fluctuations. We define

$$A_m=e^{-i\nu_m\tau/\epsilon}\left[u_a^m\frac{\partial}{\partial\eta^a}-i\epsilon(Du^m)_a\eta^a\right]. \tag{2.31}$$

From the Hermiticity of $\mathcal{B}$ it is straightforward to check that if we normalize u^m by

$$i\epsilon\left[(Du^m)_a^* u_a^m-u_a^{m*}(Du^m)_a\right]=1\,, \tag{2.32}$$

we have

$$[A_m,A_n]=[A_m^\dagger,A_n^\dagger]=0\,,\quad [A_m,A_n^\dagger]=\delta_{nm}\,. \tag{2.33}$$

Equation (2.23) has $2N-2$ independent solutions so we get $N-1$ creation-annihilation operators.

Next we discuss the penetration problem ($\epsilon=\pm i$) for quantum fluctuations around a local minimum of V. We choose E_0 to be equal to the value of V at the minimum. Since $\epsilon^2=-1$, the classical trajectory corresponds to a maximum of potential energy. If the potential is harmonic near its minimum, it takes an infinite τ interval to reach the stability point. For definiteness we choose the corresponding limit to be $\tau\to-\infty$. S_0 is an integral with a fixed lower bound so it tends to $-\infty$ in the limit. Near the minimum the term involving e^{-S_0} is a decreasing function of the distance to the stability point. It must be of order 1 as it will be matched to the oscillator wave function of small oscillation near the minimum, which has the same behavior. On the contrary, the term involving e^{S_0} is an increasing function of the distance to the stability point. It will be matched to an exponentially small component of the wave function of small oscillations, which has similiar behavior and which appears in solving the Schrödinger equation near the minimum because, owing to tunneling, the energies differ from exact harmonic-oscillator energies.[12] We shall only discuss the matching of the e^{-S_0} term in the present paper.

We introduce

$$\Lambda_{ab}=\lim_{\tau\to-\infty}V_{ab}^{(2)}\,.$$

We can then obtain a set of solutions of (2.23) such that

$$\begin{aligned}&u_a^{m(\pm)}\to f_a^m\,e^{\pm\omega_m\tau}\,,\\ &\Lambda_{ab}f_b^m=\omega_m^2 f_a^m\,,\quad f_a^m f_a^n=\frac{\delta_{nm}}{2\omega_n}\,.\end{aligned} \tag{2.34}$$

The creation-annihilation operators will be defined by

$$\begin{aligned}&A_m=e^{-\omega_m\tau}\left[u_a^{m(+)}\frac{\partial}{\partial\eta^a}+[Du^{m(+)}]_a\eta^a\right],\\ &\tilde{A}_m=-e^{\omega_m\tau}\left[u_a^{m(-)}\frac{\partial}{\partial\eta^a}+[Du^{m(-)}]_a\eta^a\right].\end{aligned} \tag{2.35}$$

Indeed, using the Wronskian together with (2.34) one can show that

$$[A_m,A_n]=[\tilde{A}_m,\tilde{A}_n]=0\,,\quad [A_n,\tilde{A}_m]=\delta_{nm}\,. \tag{2.36}$$

From (2.34) it follows that

$$\begin{aligned}&A_m\xrightarrow[\tau\to-\infty]{} f_a^m\left(\frac{\partial}{\partial\eta^a}+\omega_m\eta^a\right),\\ &\tilde{A}_m\xrightarrow[\tau\to-\infty]{} f_a^m\left(-\frac{\partial}{\partial\eta^a}+\omega_m\eta^a\right).\end{aligned} \tag{2.37}$$

Hence A_m, $\tilde{A}_m$ tend to the creation-annihilation operators of the quantum fluctuations around the local minimum of V. The WKB eigenstates will thus match to the eigenstates of the allowed region.

Finally, going back to the general discussion ($\epsilon^2=\pm 1$), we determine the ground-state wave function. For both signs of ϵ^2 we have $N-1$ annihilation operators of the form

$$A_m = e^{-i\nu_m \tau/\epsilon}\left[u_a^m \frac{\partial}{\partial\eta^a} - i\epsilon(Du^m)_a\eta^a \right]. \tag{2.38}$$

The ground state χ_0 is the solution of

$$A_m\chi_0 = 0\,, \quad m = 2, \ldots, N \tag{2.39}$$

which gives[13]

$$\chi_0 = d(\tau)\exp[-\tfrac{1}{2}\Omega_{ab}(\tau)\eta^a\eta^b]\,, \qquad \Omega_{ab} = -i\epsilon u_a^{-1m}(Du^m)_b\,, \tag{2.40}$$

where u^{-1} is such that

$$u_a^{-1m}u_b^m = \delta_{ab}\,. \tag{2.41}$$

This matrix exists because the u_a^m, $m=2, \ldots, N$, are $N-1$ linearly independent vectors in the $(N-1)$-dimensional space orthogonal to $\vec{r}_\tau$. It is furthermore easy to check that Ω is a symmetric matrix.

Equation (2.39) determines χ_0 up to an arbitrary function $d(\tau)$ which we compute by inserting (2.40) into the equation $\mathcal{H}\chi_0 = E_1^0\chi_0$. Combining everything we finally obtain the ground-state wave function

$$\psi_0 = \frac{e^{i\epsilon S_0}}{(f'\,\vec{r}_\tau^{\,2})^{1/2}}\frac{e^{iE_1^0\tau/\epsilon}}{(\det u)^{1/2}}\exp(-\tfrac{1}{2}\Omega_{ab}\eta^a\eta^b)\,. \tag{2.42}$$

This is the generalization of the formula obtained by Banks, Bender, and Wu.[7] The excited-state wave functions are obtained by applying the creation operators to ψ_0. They will involve, in addition, the standard Hermite polynomials.

As we explained earlier, our method makes sense if ψ vanishes rapidly away from the classical path. This will be the case if Re Ω is a positive-definite matrix. In the allowed region this condition is readily checked to hold using the orthogonality relations due to the Hermiticity of $\mathcal{B}$ in our previous example. The normalization condition (2.32) gives

$$2\,\mathrm{Re}\Omega_{ab} = u^{-1n*}_{\;a}u^{-1n}_{\;b}\,, \tag{2.43}$$

which is indeed positive definite. In the case of the forbidden region we have no such proof. However if V varies very slowly and if the classical path is very close to a straight line, one has

$$\Omega_{ab} \simeq u^{-1m}_{\;a}\nu_m\, u_a^m\,, \tag{2.44}$$

which is indeed positive definite since $\nu_m > 0$. The condition of positive definiteness will hold whenever Ω does not differ drastically from (2.44).

Finally, we comment on the case where some of the ν's are zero. This will always be the case if the potential has a symmetry so that in configuration space we have a continuous set of classical solutions with the same classical action. Then the Hamitlonian commutes with the corresponding infinitesimal generators; this is reflected in the fact that the operators with $\nu=0$ commute with $\mathcal{H}$. This is the usual zeroth-mode problem of semiclassical methods, which can be solved by introducing collective coordinates following our general method.[11] In this context it is simply equivalent to performing the standard separation of variables in the Schrödinger equation for a symmetric potential before applying the WKB method; we shall not elaborate upon it here.

III. APPLICATIONS

Our result obviously has many applications in various potential and field-theory problems. We briefly discuss two of them which are related to our two examples in Sec. II and postpone more detailed discussions to forthcoming papers.

A. Quantization condition for periodic solutions

As we discussed above, a state is characterized by the occupation numbers n_i, $i=2, \ldots, N$. (Again we assume that none of the ν's vanishes.) The wave function must be periodic with period T. The creation-annihilation operators were chosen to be periodic in such a way that the quantization condition does not depend on n_i but only on the particular classical trajectory considered. As we have seen before, $u^m e^{-i\nu_m\tau/\epsilon}$ is periodic of period T. From this we conclude that

$$\det[u(\tau+T)] = \det[u(\tau)]\exp\left(+i\sum_n \nu_n T/\epsilon\right). \tag{3.1}$$

Formula (2.40) leads to

$$\psi_0(\tau+T) = \psi_0(\tau) \times\exp\left[i\left(\epsilon W(E_0) + \frac{E_1^0 T}{\epsilon} - \sum_n \nu_n\frac{T}{2\epsilon}\right)\right],$$

where $W(E)$ is the action integral over a period

$$W(E) = \oint ds[2(E-V)]^{1/2}\,. \tag{3.2}$$

The quantization condition reads (here $\epsilon = \pm 1$)

$$W(E_0) + T\left(E_1^0 - \sum_n \frac{\nu_n}{2}\right) = 2m\pi\,, \quad m \text{ integer}\,.$$

This can be put into the same form as in Ref. 4 since for any level

$$E_1 = E_1^0 + \sum_i n_i\nu_i\,,$$
$$W(E) \simeq W(E_0) + E_1 T\,,$$

and we get

$$W(E) = 2m\pi + \sum_i (n_i + \tfrac{1}{2})\nu_i T\,, \tag{3.3}$$

which agrees with the result of Ref. 4.

B. Vacuum tunneling in field theories

It will be associated with solutions with $\epsilon^2=-1$ (forbidden region) such that E_0 is the energy of the classical vacuum. In Lorentz-invariant field theories we are thus led to consider classical solutions of Euclidean field equations according to Eq. (2.5), i.e., the so-called pseudoparticle solutions first considered by Polyakov.[14] We shall briefly discuss two typical examples.

Example I. Two-dimensional Higgs model. The Lagrangian is given by

$$\mathcal{L}=-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2 -|(\partial_\mu-ieA_\mu)\phi|^2+\frac{1}{2g^2}(1-g|\phi|^2)^2, \quad (3.4)$$

where ϕ is a complex scalar field. The Euclidean solutions are the vortex solutions $\hat{A}_{\rm cl}$, $\hat{\phi}_{\rm cl}$. (From now on we use the convention of putting a caret over any quantity which is relative to Euclidean space-time, so as to distinguish it from Minkowski quantities since we shall handle both at the same time.) The vortex solutions are classified by the topological index (magnetic flux number)

$$\hat{P}=\frac{e}{2\pi}\int \hat{d}^2x\,(\partial_4\hat{A}_1-\partial_1\hat{A}_4). \quad (3.5)$$

Example II. SU(2) Yang-Mills theory in four dimensions. The Lagrangian is given by

$$\mathcal{L}=-\tfrac{1}{4}(G^a_{\mu\nu})^2,\qquad G^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g\epsilon_{abc}A^b_\mu A^c_\nu, \quad (3.6)$$

and the Pontryagin index is defined by

$$\hat{P}=\frac{g^2}{16\pi^2}\int \hat{d}^4x\,\epsilon_{ijk}\hat{G}_{4i}\hat{G}_{jk}. \quad (3.7)$$

It takes only integer values for field configurations with finite action. Throughout the discussion we choose the $A_0=0$ gauge so that the classical solution will be considered in the $\hat{A}_4=0$ gauge.

As an example we shall look only at the quantum meaning of one-pseudoparticle (one vortex) solutions. In Example II one can check in the $\hat{A}_4=0$ gauge, that the solution of Ref. 15 takes the form (λ is the size of the pseudoparticle)

$$\hat{A}_{{\rm cl}\,j}(\vec{x},x_4)\equiv\hat{A}^a_{{\rm cl}\,j}\frac{\tau^a}{2}=O\hat{A}^p_jO^{-1}-\frac{i}{g}(\partial_jO)O^{-1},\qquad j=1,2,3$$

$$O=\exp\left\{\frac{i\vec{x}\cdot\vec{\tau}}{(r^2+\lambda^2)^{1/2}}\left[\tan^{-1}\left(\frac{x_4}{(r^2+\lambda^2)^{1/2}}\right)-\frac{\pi}{2}\right]\right\},$$

$$\hat{A}^p_j=\frac{i}{g}\,\frac{x_4^2+r^2}{x_4^2+r^2+\lambda^2}\,\omega^{-1}\partial_j\omega, \quad (3.8)$$

$$\omega=\frac{x_4-i\vec{x}\cdot\vec{\tau}}{r^2+x_4^2},\quad r^2=\vec{x}^2.$$

The gauge condition $\hat{A}_4=0$ does not break gauge invariance by a time-independent gauge transformation. As a result, in (3.8), we could have replaced the term $-\pi/2$ by an arbitrary function of x, and there is an arbitrariness in the definition of $\tan^{-1}$. We choose it such that $\tan^{-1}(-\infty)=(n+\frac{1}{2})\pi$ and call $\hat{A}^n_{\rm cl}$ the function so obtained. One can see that

$$\hat{A}^n_{{\rm cl}\,k}\underset{\tau\to-\infty}{\simeq}\frac{i}{g}K^n\nabla_k(K^{-n})\equiv A^n_{vk},$$
$$\hat{A}^n_{{\rm cl}\,k}\underset{\tau\to+\infty}{\simeq}A^{n+1}_{vk},\quad K=\exp\left[\frac{i\pi\vec{x}\cdot\vec{\tau}}{(r^2+\lambda^2)^{1/2}}\right]. \quad (3.9)$$

As expected, for $\tau\to\pm\infty$ we go to the allowed region and $\hat{A}_{\rm cl}$ tends to a pure gauge term, that is, to a classical ground state of the theory. This is the situation in our second example of Sec. II. These ground states are related by the gauge transformation K successively. The same discussion can be carried out in Example I.

From our general discussion we know that the field-theory eigenstate of the Hamiltonian will be nonzero only in a neighborhood of the classical path in configuration space parametrized by the new quantum variable q through $x_4=f(g)$.

In the present examples, one gets a better insight by relating q to the topological properties of the classical solution through an appropriate choice of f, which we now discuss. In the $\hat{A}_4=0$ gauge, one has

$$\hat{P}=\int_{-\infty}^{+\infty}dx_4\frac{\partial}{\partial x_4}\hat{Q}=Q(+\infty)-Q(-\infty), \quad (3.10)$$

$$\hat{Q}[\hat{A}]=\frac{e}{2\pi}\int dx_1\hat{A}_1\quad\text{(example I)},$$

$$\hat{Q}[\hat{A}]=\frac{g^2}{16\pi^2}\int d^3x\left(\hat{A}^a_i\partial_j\hat{A}^a_k+\frac{g}{3}\epsilon_{abc}\hat{A}^a_i\hat{A}^b_j\hat{A}^c_k\right)\epsilon_{ijk}\quad\text{(example II)}. \quad (3.11)$$

We can choose f from the equation

$$q=\hat{Q}[\hat{A}^n_{\rm cl}(x,f(q))] \quad (3.12)$$

in the interval of q where one has a solution. Differentiating (3.12) we get

$$1=\frac{df}{dq}h(q),$$

$$h(q)=\frac{e}{2\pi}\int dx_1\partial_4\hat{A}^n_{{\rm cl}\,1}\quad\text{(example I)}, \quad (3.13)$$

$$h(q)=\frac{g^2}{16\pi^2}\int d^3x\,\epsilon_{ijk}(\partial_4\hat{A}^{an}_{{\rm cl}\,i})\hat{G}^{an}_{{\rm cl}\,jk}\quad\text{(example II)}.$$

One can check that h does not depend on n as it is invariant by time-independent gauge transformations. Equation (3.13) can be rewritten as

$$\frac{1}{2}\left(\frac{df}{dq}\right)^2-\frac{1}{2h^2}=0,\quad f'h>0. \quad (3.14)$$

We have a mechanical analog to a "point" with "position" f, "time" q, zero "energy," and "potential" $u=-1/2h^2$. Since $u<0$, its "velocity" never vanishes, and it always moves toward the right or left depending on the sign of h. For a pseudoparticle $h>0$, so $f'>0$; we look at this case as a specific example. From equation (3.10) one finds

$$\int_{-\infty}^{+\infty} \frac{df}{(-2u)^{1/2}} = \hat{P}\,. \tag{3.15}$$

The Pontryagin index corresponds to the "time" Δq required by the "point" to move from $f=-\infty$ to $f=+\infty$. Thus $\Delta q=1$ is the interval where (3.12) can be used for given n. Patching together the results obtained from (3.12) for all values of n one defines f for all values of q. It is found to be periodic of period 1 owing to K gauge transformations and to be such that, for $\sigma\to 0$,

$$f(n-\sigma)\to -\infty\,,\quad f(n+\sigma)\to +\infty\,.$$

The classical path is finally given by $A_{c1}(x,f(q))$, which is defined for arbitrary q by

$$A_{c1}(x,f(q))=\hat{A}^n_{c1}(x,f(q))\,,\quad n\le q\le n+1$$

so that $q\to q+1$ is equivalent to a K gauge transformation on $\hat{A}_{c1}$.

Example I is similar, and we end up with trajectories in configuration space which are periodic up to a gauge transformation K. Because our theory must be gauge invariant, the state described by ψ must satisfy this property. Since a fixed phase factor in a wave function is unobservable, we can have in general

$$\psi(q+1)=e^{i\theta}\psi(q)\,,$$

where θ is an arbitrary angle. In this way one finds very naturally the degeneracy of the vacuum.[9,16] Moreover, since we have the excited-state wave function, we can study the spectrum of excitations of the theory, which is the physically relevant problem.

Because $q\to q+1$ is equivalent to a gauge transformation, the q quantum mechanics is equivalent to that of a periodic potential. Hence θ arises as in Bloch waves of a one-dimensional crystal.

The matching problem and the determination of the wave function are possible to handle in example II since, owing to O(5) invariance of the classical solution,[17] the equation for small fluctuations is entirely solvable.

Finally, we note a crucial difference between example I (mass scale) and example II (no mass scale). In example I, f cannot be computed explicitly but since $h(q)$ is the integral of the magnetic field, one has

$$h(f)\underset{f\to\infty}{\sim} Ce^{-\mu f}\,,$$

where C is a constant and μ is the mass of vector field. This leads to the following behavior for the inverse function $q(f)$:

$$q\underset{f\to\pm\infty}{\sim} n+\frac{C}{\mu}\,e^{-\mu|f|}\,. \tag{3.16}$$

In example II, $q(f)$ can be computed as follows:

$$q(f)=\frac{1}{4}\left[\frac{3\lambda^2 f+2f^3}{(\lambda^2+f^2)^{3/2}}\right]+\text{const}\,, \tag{3.17}$$

which leads to

$$q\underset{|f|\to\infty}{\sim} n+\frac{15}{16}\left|\frac{f}{\lambda}\right|^{-4}\,. \tag{3.18}$$

Thus in example I we have an exponential behavior, while in example II we have a power behavior.

$|f|\to\infty$ corresponds to approaching the minima of the potential for q quantum mechanics. The two different behaviors (3.16), (3.18) show that these potentials behave in a very different way in these two cases. In fact, it is harmonic near the minimum in example I while the potential is much flatter in example II. As a result WKB matching will lead to rather different results. In example I one would obtain a result equivalent to the dilute-gas approximation[9] of Euclidean field theory.[18] In example II a different result will come out. This question is important for the problem of quark confinement, since in the dilute-gas approximation the Yang-Mills theory does not seem[19] to confine quarks, contrary to the initial hopes of Polyakov.[14] This problem is currently under investigation. According to Sec. II one performs the canonical transformation[20]

$$A'_i=\hat{A}_{c1i}(\vec{x},f(q))+\tilde{A}_i(\vec{x})\,,$$

$$\int d\vec{x}\,\tilde{A}_i(\vec{x})G_{4ic1}(\vec{x},f(q))=0\,,$$

which is the method we proposed earlier.[8]

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[1]J. Keller, Ann. Phys. (N.Y.) 4, 180 (1958).
[2]M. Gutzwiller, J. Math. Phys. 12, 343 (1971); 11, 1791 (1970); 10, 1004 (1969); 8, 1979 (1967).
[3]V. Maslov, Teor. Mat. Fiz. 2, 30 (1970) [Theor.

Math. Phys. 2, 21 (1970)]; V. Maslov, *Theory of Disturbances and Asymptotic Methods* (Moscow Univ. Press, Moscow, 1965); *Théorie de Perturbations et Methods Asymptotiques* (Dumod, Paris, 1972).

[4]R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974); 10, 4130 (1974); 10, 4138 (1974); 11, 3424 (1975).

[5]For recent reviews see the following: A. Neveu, Rep. Prog. Phys. 40, 599 (1977); R. Jackiw, Rev. Mod. Phys. 49, 681 (1977); J.-L. Gervais, Schladming lecture notes, 1977 (unpublished); A. Jevicki, in *Deeper Pathways in High Energy Physics*, proceedings of Orbis Scientiae, Univ. of Miami, Coral Gables, Florida, 1977, edited by A. Perlmutter and L. F. Scott (Plenum, New York, 1977).

[6]J.-L. Gervais and A. Jevicki, Nucl. Phys. B110, 113 (1976); See also A. Jevicki, Ref. 5.

[7]T. Banks, C. M. Bender, and T. T. Wu, Phys. Rev. D 8, 3346 (1973); T. Banks and C. M. Bender, *ibid.* 8, 3366 (1973).

[8]J.-L. Gervais and B. Sakita, CCNY Report No. HEP 76/11, 1976 (unpublished); the present article supersedes this paper.

[9]C. Callan, R. Dashen, and D. Gross, Phys. Lett. 63B, 334 (1976).

[10]N. Christ and T. D. Lee, Phys. Rev. D 12, 1606 (1975). We note that the scattering problem of solitons has not been completed by this method. We shall discuss it in future publications.

[11]J.-L. Gervais, A. Jevicki, and B. Sakita, Phys. Rep. 23C, 237 (1975). See also J.-L. Gervais, Ref. 5.

[12]One of us (J.-L. G.) is grateful to C. Callan for explaining this sort of mechanism.

[13]Hereafter all sums over u^m's or ν_m's only sum over the ones with $\nu_m > 0$.

[14]A. M. Polyakov, Phys. Lett. 59B, 82 (1975).

[15]A. A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin, Phys. Lett. 59B, 85 (1975).

[16]R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976).

[17]R. Jackiw and C. Rebbi, Phys. Rev. D 14, 517 (1976).

[18]This has been checked for one degree of freedom by C. Callan (unpublished).

[19]C. Callan, R. Dashen, and D. Gross, Phys. Lett. 66B, 375 (1977).

[20]Note that we are in the Schrödinger representation. Hence A and $\tilde{A}$ do not involve the time.