Killing tensor quantum numbers and conserved currents in curved space

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The relationship between relativistic quantum current conservation laws in a curved-space background and the corresponding "good quantum numbers," i.e., operators that commute with the fundamental wave operator in a first-quantized field theory, is considered. It is shown that under favorable circumstances (such as vanishing Ricci curvature) the existence of such an operator for scalar fields is automatically implied by the existence of the corresponding constant for particle trajectories in the classical limit, that is to say, by the existence of a Killing vector or a "Killing tensor" in the first- and second-order cases, respectively. Thus the fourth constant of the motion for a scalar quantum field in the Kerr metric background arises automatically from the Killing tensor defining the fourth constant of the classical motion. Another application is to the Runge-Lenz constants in the nonrelativistic hydrogen atom problem. The "Schiff conjecture" concerning the relationship between classical mechanics and first-quantized field theory in connection with the equivalence principle is discussed in passing.

I. INTRODUCTION

A key part in the early development of quantum-mechanical theories in general, and of the original nonrelativistic Schrödinger theory in particular, was played by the *"hydrogen atom problem"* by consequence of its having a well-defined and simple analytic solution. In relativistic gravitation theory an analogously important role is played by the *black-hole problem* which has been effectively solved in the particular case of the vacuum Einstein theory by the work of Israel,[1] Carter,[2,3] and Robinson[4] which has established that the only topologically spherical axially symmetric black-hole equilibrium states belong to the family of solutions of Einstein's equations discovered by Kerr in 1963,[5] (the work by Hawking[6,7] having virtually excluded the possibility that there can exist other—e.g. toroidal or nonaxisymmetric—kinds of equilibrium state). The problem has also been solved in the more general framework of the source-free Einstein-Maxwell equations, since further work by Israel,[8] Carter,[3] and Robinson[9] has also virtually established that the only well-behaved topologically spherical axisymmetric black-hole equilibrium solutions of these equations belong to the generalized Kerr family discovered by Newman *et al.*[10]

Owing to the special significance that is thus conferred on them, the Kerr and Kerr-Newman solutions have been the subject of intense study and detailed investigations including notably the stability analyses of Teukolsky and Press[11] and Stewart.[12] Much of this work would not have been feasible in practice had it not been for the remarkable and surprising fact that all the relevant wave equations have turned out to be soluble in the Kerr and Kerr-Newman background by separation-of-variables methods. The first and simplest example to come to light was that of the Hamilton-Jacobi equation for the classical geodesics and charged-particle orbits.[13] In addition to the three obvious constants of the motion (interpretable as conservation of axial angular momentum, energy, and proper mass), the classical orbits turned out to admit a more mysterious fourth separation constant, which reduces to the square of the total angular momentum in the spherical (Schwarzschild-Reissner-Nordström) limit, but which is not related to any manifest symmetry in the more general rotating solutions. The next development was the discovery[14] that this fourth separation constant had a quantum analog, in so much as the D'alembert equation—and more generally the charged Klein-Gordon equation—for a scalar field can be separated straightforwardly into four ordinary differential equations involving four corresponding separation constants. After some unsuccessful attempts by several workers, the next breakthrough was made by Teukolsky,[15,16] who discovered that the separation procedure can be extended to wave equations for massless neutral particles with nonzero spin (neutrino, photon, graviton). A further breakthrough was made by Chandrasekhar,[17] who showed how the procedure can be adapted to deal with spinning particles with nonzero mass, and most recently this line of generalization has been completed by Toop[18] and Page,[19] who have shown (independently) that Chandrasekhar's procedure can cope with the full Dirac equation for a realistic (i.e., charged as well as massive) electron field.

While many practical-minded workers have been content to know that these miraculously convenient properties of the Kerr and Kerr-Newman solutions *exist,* others have wondered whether their occurrence might be in some sense *explained* as straightforward consequences of something more fundamental. An analogous situation arises in the nonrelativistic hydrogen atom problem where, in addition to the constants of the motion arising from the obvious SO(3) rotation group symmetry, there are also the Runge-Lenz constants whose existence gives rise to the degeneracy of energy levels (Pauli[20]). These constants may be regarded as arising from a hidden SO(4) invariance acting on the bound energy eigenstates (Malkin and Mal'ko[21]). The constants in the Kerr-Newman solutions could likewise be described in terms of a hidden three-parameter *Abelian* invariance group acting on the relevant Hilbert space and containing as a subgroup the two-parameter invariance group arising from the manifest stationarity and axis-symmetry. However, the unearthing of a hidden invariance group can hardly be claimed to provide a satisfactory explanation in itself: It merely amounts to replacing one mystery by another.

The same thing could be said about the quite different approach to the question that was initiated by Penrose and Walker[22] and followed up by Hughston *et al.*[23,24] These workers have sought to derive the existence of the fourth constant of motion from another special property of the Kerr-Newman solutions, namely the fact that their Weyl conformal tensor is algebraically degenerate—type D. However, even if such a demonstration could be made complete, there would still remain the mystery of why the black-hole equilibrium states should have this degeneracy property. Indeed it would seem no less logical to use a directly opposite approach and seek to explain the type-D property as being a consequence of the existence of the constants of motion. The deduction that for vacuum Einstein or source-free Einstein-Maxwell solutions the type-D property is indeed a direct consequence of the separability properties of the wave equation has been carried out completely by Carter.[14] There remains, however, the question of whether the separability properties can themselves be derived merely from the existence of the corresponding constants of the motion. In so far as the classical orbits governed by the Hamilton-Jacobi equation are concerned, it is well known (see Eisenhart[25]) that any second-order constant of motion of the kind under consideration is associated with a second-rank symmetric tensor field satisfying an equation analogous to the familiar Killing equation for an isometry-generating vector field (see Sec. VI).

The derivation of the corresponding Hamilton-Jacobi separability properties under suitable conditions from such a tensor field has been carried out by Woodhouse,[26] thereby providing the missing link needed to complete the derivation of the type-D property from the existence of the fourth constant of motion. The objective of the converse (Penrose-Walker) program, i.e., the derivation of the existence of the "Killing tensor" field from the type-D property, has not yet been fully attained, but it has at least been possible to derive the existence of a "Killing spinor" which automatically gives rise to a constant of motion in the special case of *null* geodesics. (A related antisymmetric tensor, originally brought to light by Floyd, has been described by Penrose.[27])

Even if one could satisfactorily explain the constants of classical motion and the separability properties of the Hamilton-Jacobi equation, there would remain the problem of accounting for the corresponding properties of the quantum wave equations. The present work is intended as a first step in a program aimed at deriving the quantum constants and separability properties as consequences of the corresponding classical results. A principal result will be the demonstration (Sec. VI) that the existence of a second-order constant of motion for the scalar (charged Klein-Gordon) wave equation is an automatic consequence of the existence of a corresponding classical constant *provided* that the background space satisfies the vacuum Einstein or source-free Einstein-Maxwell equations—though not in general otherwise. (An alternative demonstration of this result in the pure vacuum case has been given by Bonanos[28] using a more specialized method). The converse must of course hold automatically, regardless of any background field equations, by Bohr's correspondence principle.

Sections II and III of this work discuss the relationship between the existence of a quantum constant of motion, as defined in terms of an operator that commutes with the fundamental wave operator (so that an eigenstate may also be a solution), and the existence of a corresponding conserved current. Although it is a slight digression from the original program, another topic that arises naturally apropos of the relationship between the first-quantized field theory and the classical limit in curved space is the "Schiff conjecture" concerning the interdependence of the strong and weak versions of the equivalence principle, which is discussed in Sec. IV.

We shall use the following notation conventions. The *composite* of two nonassociative operators will be denoted by a small circle, so that e.g. for the operator

$$\partial_a:\ \varphi \to \partial_a\varphi \equiv \frac{\partial\varphi}{\partial x^a}$$

of partial differentiation with respect to local coordinates x^a, we may express the Leibnitz rule as

$$\partial_a \circ \varphi = \partial_a\varphi + \varphi\partial_a$$

treating the scalar field as a multiplicative operator acting (associatively) on other scalar fields. The *commutator* of two operators is the difference between their *composites* (as distinct from their *products*, except in the associative case wherein the product and the composite are the same) in the direct and the reversed orders. We shall denote the commutator by square brackets and a comma as usual, which enables us to express the Leibnitz rule by

$$[\partial_a, \varphi] \equiv \partial_a \circ \varphi - \varphi \circ \partial_a = \partial_a \varphi \equiv \varphi_{,a},$$

where we have introduced the use of a comma subscript as an abbreviation to denote a partial derivative component.

When working with a vector operator ∇ of covariant differentiation with respect to some given connection we shall use a semicolon subscript to indicate the resulting components; e.g. for a contravariant vector $\vec{v}$ we shall set

$$(\nabla \otimes \vec{v})_a{}^b \equiv (\nabla_a v^b) \equiv \nabla_a v^b \equiv v^b{}_{;a}.$$

We shall use parentheses and square brackets respectively for symmetrization and antisymmetrization of indices. Our convention for the Riemann tensor may be specified by

$$[\nabla_a, \nabla_b] = \mathcal{R}_{ab}$$

in so far as actions on a vector field $\vec{v}$ are concerned, where $\mathcal{R}_{ab}$ is a matrix operator with components $\mathcal{R}_{ab}{}^c{}_d$, so that in more explicit form

$$2v^c{}_{,[b;a]} = (\mathcal{R}_{ab}\vec{v})^c \equiv R_{ab}{}^c{}_d v^d.$$

Our sign convention for the Ricci tensor is given by

$$R_{ab} = R_{ca}{}^c{}_b.$$

The operator of Lie differentiation with respect to $\vec{v}$ will be denoted by $\vec{v}\mathcal{L}$; it satisfies

$$[\vec{u}\mathcal{L}, \vec{v}\mathcal{L}] \equiv (\vec{u}\mathcal{L}\vec{v})\mathcal{L}.$$

II. GENERAL THEORY OF A PASSIVE FIELD IN A GIVEN BACKGROUND

In the passive-test-particle limit (wherein self-interactions, radiation reaction, etc. are ignored) the motions of isolated particles (or compound bodies) in a given background field are considered as being described in terms of *currents* (which are *vector densities* on the space-time manifold) that are *homogeneous bilinear* partial differential functions of fields Φ with components Φ^A ($A = 1, \ldots, S$, say) that satisfy *homogeneous linear* partial differential equations of motion of the form

$$(\mathcal{K}\Phi) = 0, \tag{2.1}$$

where $\mathcal{K}$ maps the space of fields Φ onto its *dual* or rather—when the field is complex—onto the *complex conjugate* of its dual.

Thus—using a dot as usual to distinguish complex-conjugate component indices—the field $(\mathcal{K}\Phi)$ will have components $(\mathcal{K}\Phi)_{\dot{A}} \equiv (\mathcal{K}_{\dot{A}B}\Phi^B)$. In the complex case there will in general be a particularly fundamental current $\vec{\mathcal{J}}$ with space-time components $\mathcal{J}^a$ ($a = 1, \ldots, n$, with $n = 4$ in conventional theories) that is interpretable as a *probability flux density of charge*. This flux will have the form

$$\vec{\mathcal{J}} = \vec{\mathcal{J}}(\Phi, \bar{\Phi}) \tag{2.2}$$

using a bar to denote complex conjugation, where $\vec{\mathcal{J}}(\Phi, \bar{\Psi})$ is a bilinear partial differential function, defined in terms of $\mathcal{K}$ in such a way as to satisfy the identity

$$(\bar{\Psi}\mathcal{K}\Phi - \Phi\bar{\mathcal{K}}^\dagger\bar{\Psi}) \equiv -i\partial_a \mathcal{J}^a(\Phi, \bar{\Psi}) \tag{2.3}$$

for arbitrary fields Φ and Ψ, where $\mathcal{K}^\dagger$ (with components $\mathcal{K}^\dagger_{\dot{A}B}$ is the operator *adjoint* to $\mathcal{K}$. The relation (2.3) is in fact the standard defining equation for the adjoint operator, which is thereby specified uniquely. The adjoint may be given more explicitly in terms of an Euler differentiation operation by the useful formula

$$(\bar{\mathcal{K}}^\dagger\bar{\Psi}) \equiv \frac{\delta(\bar{\Psi}\mathcal{K}\Phi)}{\delta\Phi}. \tag{2.4}$$

The relation (2.3) is not sufficient to determine $\vec{\mathcal{J}}(\Phi, \bar{\Psi})$ uniquely, since one could always add to it any identically conserved bilinear partial differential function of Φ and $\bar{\Psi}$. Nevertheless there is a particularly natural construction whereby a current with the required property is given by an expression of the form

$$2i\vec{\mathcal{J}}(\Phi, \bar{\Psi}) \equiv \left(\frac{\vec{\Delta}(\Phi\bar{\mathcal{K}}^\dagger\bar{\Psi})}{\Delta\bar{\Psi}}\bar{\Psi} - \frac{\vec{\Delta}(\bar{\Psi}\mathcal{K}\Phi)}{\Delta\Phi}\Phi\right). \tag{2.5}$$

In the two preceding equations, we have used the standard notation $\delta/\delta\Phi$ for the Euler derivative, and we have introduced the notation $\vec{\Delta}/\Delta\Phi$ for what may appropriately be called the *Noether derivative*. For an arbitrary partial differential function $\mathfrak{F}$ these derivatives are defined as follows.

A sequence of quasi-Euler derivatives (with respect to the partial derivative compounds $\Phi^A{}_{,a}$ etc.) is first introduced inductively (working *down* from the highest-order partial derivative of Φ on which $\mathfrak{F}$ depends) by

$$\ldots,$$

$$\frac{\delta\mathfrak{F}}{\delta\Phi_{,a,b}} \equiv \frac{\partial\mathfrak{F}}{\partial\Phi_{,a,b}} - \partial_c\left(\frac{\delta\mathfrak{F}}{\delta\Phi_{,a,b,c}}\right),$$

$$\frac{\delta\mathfrak{F}}{\delta\Phi_{,a}} \equiv \frac{\partial\mathfrak{F}}{\partial\Phi_{,a}} - \partial_b\left(\frac{\delta\mathfrak{F}}{\delta\Phi_{,a,b}}\right). \tag{2.6}$$

This sequence is terminated by the definition of the true Euler derivative

$$\frac{\delta\mathcal{F}}{\delta\Phi}\equiv\frac{\partial\mathcal{F}}{\partial\Phi}-\partial_a\left(\frac{\delta\mathcal{F}}{\delta\Phi_{,a}}\right). \tag{2.7}$$

The Noether derivative $\vec{\Delta}\mathcal{F}/\Delta\Phi$ may now be introduced as the partial differential *operator* with space components given by

$$\frac{\Delta^a\mathcal{F}}{\Delta\Phi}\equiv\frac{\delta\mathcal{F}}{\delta\Phi_{,a}}+\frac{\delta\mathcal{F}}{\delta\Phi_{,a,b}}\partial_b+\frac{\delta\mathcal{F}}{\delta\Phi_{,a,b,c}}\partial_b\partial_c+\cdots \tag{2.8}$$

(where the field component index, A say, is left implicit). This notation is motivated by the fact that it allows one to express the differential change $d\mathcal{F}$ in $\mathcal{F}$, due to differential changes $d\Phi$ in the various fields on which it may depend, by a concise but explicit expression of the form

$$d\mathcal{F}\equiv\sum\frac{\delta\mathcal{F}}{\delta\Phi}d\Phi+\partial_a\left(\sum\frac{\Delta^a\mathcal{F}}{\Delta\Phi}d\Phi\right), \tag{2.9}$$

where the summation is taken over the various fields involved (as well as over the component indices).

Using the general differential identity (2.9) it is easy to verify that formulas (2.4) and (2.5) do indeed lead to the divergence identity (2.3). To start with we note that if $\mathcal{F}$ has a *homogeneous linear* dependence on some particular field Φ then one can immediately deduce from (2.9) (by substituting $d\Phi=\Phi d\lambda$, where λ is a variable scale factor) that the relation

$$\mathcal{F}\equiv\Phi\frac{\delta\mathcal{F}}{\delta\Phi}+\partial_a\left(\frac{\Delta^a\mathcal{F}}{\Delta\Phi}\Phi\right) \tag{2.10}$$

will also hold as an identity. [In this last equation there is still an implicit summation over the components of the particular field Φ under consideration, but no longer over the *other* fields on which $\mathcal{F}$ may depend as in (2.9).] To see that (2.4) is indeed a a valid expression for the adjoint of $\mathcal{H}$ it now suffices to substitute $(\overline{\Psi}\mathcal{H}\Phi)$ in place of $\mathcal{F}$ in (2.10). If we now take the *difference* between the results of substituting $(\Phi\overline{\mathcal{H}}^\dagger\overline{\Psi})$ and $(\overline{\Psi}\mathcal{H}\Phi)$ in place of $\mathcal{F}$ in (2.10) we obtain the required divergence identity (2.3). It is to be noted that if instead of taking the difference we had taken the sum, we would have obtained the even simpler divergence identity

$$\partial_a\left(\frac{\Delta^a(\overline{\Psi}\mathcal{H}\Phi)}{\Delta\Phi}\Phi+\frac{\Delta^a(\Phi\overline{\mathcal{H}}^\dagger\overline{\Psi})}{\Delta\overline{\Psi}}\overline{\Psi}\right)\equiv 0, \tag{2.11}$$

which might at first sight appear to provide us with a generally valid conservation law holding *independent* of the field equation. However, on closer examination it transpires that the identity (2.4) is merely a trivial consequence of the more fundamental identity

$$\frac{\vec{\Delta}(\Phi\overline{\mathcal{H}}^\dagger\overline{\Psi})}{\Delta\overline{\Psi}}\overline{\Psi}\equiv-\frac{\vec{\Delta}(\overline{\Psi}\mathcal{H}\Phi)}{\Delta\Phi}\Phi, \tag{2.12}$$

which is satisfied by the adjoint of *any* homogeneous linear partial differential operator $\mathcal{H}$ regardless of its differential order or of the number of components involved. The proof of the integrated form (2.12) of this very useful identity will be left as an exercise for the reader—for the usual reason, namely that the author is (at present) unable to provide an elegant demonstration. (I found myself obliged to have recourse to a rather unwieldy double induction procedure.)

We shall obtain a genuine nontrivial conservation law from our original divergence identity (2.3) provided Φ lies in the null space of $\mathcal{H}$ [i.e., provided the basic field equation (2.1) is satisfied and provided *also* that Ψ lies in the null space of the adjoint operator $\mathcal{H}^\dagger$]. In order for the latter condition to be satisfied automatically when Ψ is set equal to Φ, as in expression (2.2), we are thus led to restrict ourselves to operators that are self-adjoint (or can be made so by an appropriate premultiplication), i.e., for which

$$\mathcal{H}^\dagger=\mathcal{H}. \tag{2.13}$$

When this additional axiom is satisfied, the fundamental current (2.2), which [with the aid of (2.12)] may be expressed compactly by

$$\vec{\mathcal{J}}(\Phi,\overline{\Phi})\equiv i\frac{\vec{\Delta}(\overline{\Phi}\mathcal{H}\Phi)}{\Delta\Phi}\Phi, \tag{2.14}$$

will automatically be *real*, i.e., we shall have

$$\mathcal{J}^a(\Phi,\overline{\Phi})\equiv\overline{\mathcal{J}^a(\Phi,\overline{\Phi})} \tag{2.15}$$

as may be seen by substituting (2.13) in the original expression (2.4). By substituting (2.13) in (2.3) we verify at once that the conservation law

$$\partial_a\mathcal{J}^a(\Phi,\overline{\Phi})=0 \tag{2.16}$$

will hold whenever the field equations (2.1) are satisfied.

Now it is to be noticed that on the assumption that the components Φ^A transform as scalars or as tensorial components under changes of the space coordinate system x^a the components $(\mathcal{H}_{\bar{A}B}\Phi^B)$ of $(\mathcal{H}\Phi)$ must transform as scalar *densities* or tensor *density* components, since if $\mathcal{H}$ is to be self-adjoint the quantity $(\overline{\Psi}\mathcal{H}\Phi)$ must necessarily be a *scalar density*. However, it is usually convenient to work as far as possible with strictly scalar or tensorial operators mapping the space of fields Φ onto *itself* rather than onto the complex conjugate of the dual space, and thus in particular it is desirable to express the field equations in the form

$$(H\Phi)=0 \tag{2.17}$$

for some such strictly scalar or tensorial endomorphism operator with components $H^A{}_B$. In typical theories there is a fundamental self-adjoint operator β, whose components transform as tensorial or scalar densities, that is used for transforming between the two kinds of representation, and more generally for lowering of field component indices. (It may also be used globally to determine a Hilbert structure on the space of fields.) In the simplest cases the operator β is nondifferential, which means that the self-adjointness condition

$$\beta^\dagger = \beta \tag{2.18}$$

is simply equivalent to the statement that the component matrix $\beta_{\dot{A}B}$ is Hermitian.

The fundamental operator $\mathcal{H}$ is thus given by

$$\mathcal{H} = \beta H \tag{2.19}$$

(or in terms of components $\mathcal{H}_{\dot{A}B} = \beta_{\dot{A}C} H^c{}_B$), and in terms of H the self-adjointness condition takes the form

$$H^\dagger \beta - \beta H = 0 . \tag{2.20}$$

The bilinear scalar density that plays such an important role in the foregoing discussion may be expressed directly in terms of H by

$$(\overline{\Psi}\mathcal{H}\Phi) \equiv (\tilde{\Psi} H \Phi) , \tag{2.21}$$

where $\tilde{\Psi}$ (with components $\tilde{\Psi}_A$) is the *Dirac adjoint* of Ψ (with components Ψ^A) and is defined by

$$\tilde{\Psi} \equiv \overline{\Psi}\beta . \tag{2.22}$$

The classic example of the application of the foregoing principles is of course the simple Dirac equation (see e.g. Lichnerowicz[29]) for the electron for which the operators $\mathcal{H}$ and H (acting on a field Φ with four complex spin components Φ^A) are given by

$$H = i\hbar\gamma^a\nabla_a + mI$$

(where $\hbar$ is Planck's constant and m is the electron mass) and

$$\mathcal{H} = i\hbar\alpha^a\nabla_a + m\beta$$

(the latter is the form originally used by Dirac) with

$$\alpha^a = \beta\gamma^a ,$$

where ∇_a is an operator of covariant differentiation, with components $\nabla_a{}^A{}_B \equiv \delta^A_B\partial_a + \Gamma_a{}^A{}_B$ such that α^a and β, and hence also γ^a, are covariantly constant. The self-adjointness condition is ensured by the condition that the matrices α^a (with components $\alpha^a{}_{\dot{A}b}$) be Hermitian. The matrices γ^a (with mixed undotted components $\gamma^{aA}{}_B$) can be used to construct the inverse, with components g^{ab}, of the space-time metric tensor, according to the familiar rule $\gamma^a\gamma^b + \gamma^b\gamma^a = g^{ab}I$, where I (with components $I^A{}_B = \delta^A_B$) is the unit matrix. In view of the fact that a local reference frame may be chosen for which the time-direction γ matrix, γ^0 say, has the same components as the β matrix, many standard textbooks use the symbol γ^0 in place of β to denote the latter—thus regrettably obscuring the fact that they have entirely different transformation properties—the former is the component of a *vector* with mixed undotted spinor indices $\gamma^{CA}{}_B$ while the latter is a *scalar density* with dotted and undotted subscript spinor indices $\beta_{\dot{A}B}$.

III. THE LAGRANGIAN FORMULATION AND THE NOETHER INVARIANCE CONDITION

The only fundamental axioms invoked in the above presentation of the principles of theories of passive fields in a given background were the existence of a linear field equation (2.1), the self-adjointness condition (2.13), and the expression (2.14) for the fundamental conserved current.

Although the existence of a variational formulation was not invoked, it is nevertheless evident from the foregoing analysis that the field equations will necessarily be of Lagrangian form as a direct consequence of the self-adjointness. We shall in fact have

$$(\mathcal{H}\overline{\Phi}) \equiv -\frac{\delta\mathcal{L}}{\delta\overline{\Phi}} , \tag{3.1}$$

where $\mathcal{L}$ is the real partial differential function defined by

$$\mathcal{L}(\Phi, \overline{\Phi}) \equiv -\tfrac{1}{2}(\overline{\Phi}\mathcal{H}\Phi + \Phi\mathcal{H}\overline{\Phi}) \tag{3.2}$$

[as may easily be verified using the formula (2.4)].

The field equations (2.1) are thus equivalent to

$$\frac{\delta\mathcal{L}}{\delta\overline{\Phi}} = 0 , \tag{3.3}$$

and it is to be noted that when they are satisfied we shall also have

$$\mathcal{L}(\Phi, \overline{\Phi}) = 0 . \tag{3.4}$$

The basic conserved current (2.14) may be expressed directly in terms of the Lagrangian by

$$i\vec{\mathcal{J}}(\Phi, \overline{\Phi}) \equiv 2\frac{\vec{\Delta}\mathcal{L}(\Phi, \overline{\Phi})}{\Delta\Phi}\Phi . \tag{3.5}$$

Having thus obtained a Lagrangian formulation, we may now apply the standard Noether theory (see e.g Trautman[30]) to construct additional conserved currents associated with any symmetries that the Lagrangian or its space integral may possess. Substituting $\mathcal{L}$ in the general variation formula (2.9) we obtain as a starting point the basic Noether identity

$$\partial_a\left(\frac{\Delta^a\mathcal{L}}{\Delta\Phi}d\Phi+\frac{\Delta^a\mathcal{L}}{\Delta\overline{\Phi}}d\Phi\right)-d\mathcal{L}\equiv\frac{\delta\mathcal{L}}{\delta\Phi}d\Phi+\frac{\delta\mathcal{L}}{\delta\overline{\Phi}}d\overline{\Phi}\,. \qquad (3.6)$$

From any one-parameter transformation group with generator $\dot{\Phi}$, where a dot denotes differentiation with respect to the group parameter λ say, i.e,

$$d\Phi=\dot{\Phi}d\lambda\,, \qquad (3.7)$$

we shall obtain a corresponding *Noether conservation law* if the corresponding derivative, $\dot{\mathcal{L}}\equiv d\mathcal{L}/d\lambda$ is equal to zero or to a pure divergence, i.e., *if the group leaves the space integral of* $\mathcal{L}$ *invariant* (subject to suitable zero-boundary conditions), since the right-hand side of (3.6) will drop out when the field equations are satisfied, leaving

$$\frac{\Delta^a\mathcal{L}}{\Delta\Phi}\dot{\Phi}+\frac{\Delta^a\mathcal{L}}{\Delta\overline{\Phi}}\dot{\overline{\Phi}}\ -\dot{\mathcal{L}}=0\,. \qquad (3.8)$$

In the present application $\mathcal{L}$ depends bilinearly on Φ and $\overline{\Phi}$ and we shall correspondingly restrict our attention to variations generated by a *linear* partial differential operator iK say acting on the field, i.e., we shall take

$$\dot{\Phi}=i(K\Phi) \qquad (3.9)$$

(the factor i is introduced for convenience later on). To find the condition for iK to generate an invariance group of the Lagrangian integral we start by noting that by (2.3) and (3.2) the Lagrangian may be expressed in the form

$$\mathcal{L}=-(\overline{\Phi}\mathcal{H}\Phi)-\tfrac{1}{2}i\partial_a\mathcal{g}^a\,, \qquad (3.10)$$

whence we obtain

$$i\dot{\mathcal{L}}=\overline{\Phi}(\mathcal{H}K\Phi)-(\overline{K\Phi})(\mathcal{H}\Phi)-\tfrac{1}{2}\partial_a\dot{\mathcal{g}}^a\,, \qquad (3.11)$$

which will have the form of a divergence only if

$$(\overline{\Phi}[K^{\dagger}\mathcal{H}-\mathcal{H}K]\Phi)\equiv0\,. \qquad (3.12)$$

(This quantity cannot be equal to the divergence of any nonzero bilinear partial differential function of Φ and $\overline{\Phi}$ since it contains no derivatives of $\overline{\Phi}$.) Thus we see that in order for iK to generate an invariance of the Lagrangian integral (in the sense that $\mathcal{L}$ should be equal to zero or a divergence for an *arbitrary* unperturbed field Φ) it is *necessary* as well as obviously sufficient that the operator K should satisfy

$$\mathcal{H}K=K^{\dagger}\mathcal{H}\,. \qquad (3.13)$$

If, as is usually the case, we are concerned only with operators K having the same self-adjointness property as the endomorphism H corresponding to $\mathcal{H}$, i.e., such that

$$(\beta K)^{\dagger}=\beta K\,, \qquad (3.14)$$

and if β has no vanishing eigenvalues (i.e., $\beta\Phi=0\rightarrow\Phi=0$), which will certainly be true in the standard case when β is just an operator of multiplication by a nonsingular Hermitian matrix), then (3.13) will simply be equivalent to the condition that the commutator

$$[H,K]\equiv HK-KH \qquad (3.15)$$

should vanish, i.e.,

$$[H,K]=0\,. \qquad (3.16)$$

It is evident from (3.9) that if the operator K satisfies the condition (3.13) [or more particularly the combination of (3.14) and (3.16)] then we shall have

$$(\mathcal{H}\dot{\Phi})=0 \qquad (3.17)$$

whenever Φ satisfies the basic field equation (2.1). Moreover, when this last equation (3.17) holds we can immediately derive not just one *real* conservation law—which is all that we could expect from the general Noether argument—but a *complex* conservation law of the form

$$\partial_a\mathcal{g}^a(\dot{\Phi},\overline{\Phi})=0\,, \qquad (3.18)$$

obtained by substituting $\dot{\Phi}$ and Φ respectively in place of Φ and Ψ in the basic divergence identity (2.3). We could also obtain a conservation law by interchanging Φ and $\dot{\Phi}$, but it need not be independent, due to the basic identity

$$\mathcal{g}^a(\Phi,\dot{\overline{\Phi}})\equiv\overline{\mathcal{g}^a(\dot{\Phi},\overline{\Phi})} \qquad (3.19)$$

[of which the reality condition (2.15) is a special case] that holds as a consequence of the self-adjointness of $\mathcal{H}$. It is an immediate consequence that the *total* variation of the basic current $\vec{\mathcal{g}}$ defined by (2.1), namely

$$\dot{\vec{\mathcal{g}}}\equiv\vec{\mathcal{g}}(\dot{\Phi},\overline{\Phi})+\vec{\mathcal{g}}(\Phi,\dot{\overline{\Phi}})\,, \qquad (3.20)$$

will also be conserved, i.e.,

$$\partial_a\dot{\mathcal{g}}^a=0\,; \qquad (3.21)$$

this last equation is merely equivalent to the *real part* of (3.18). It follows immediately from (3.11) that we shall also have

$$\dot{\mathcal{L}}=0 \qquad (3.22)$$

when the field equations are satisfied. Hence the basic Noether conservation law (3.8) reduces [using (2.12) and (2.14)] simply to

$$\partial_a\mathcal{g}^a(K)=0\,, \qquad (3.23)$$

where we define

$$\vec{\mathcal{g}}(K)\equiv\tfrac{1}{2}(\vec{\mathcal{g}}(\dot{\Phi},\overline{\Phi})-\vec{\mathcal{g}}(\Phi,\dot{\overline{\Phi}}))\,, \qquad (3.24)$$

or equivalently

$$\vec{\mathcal{g}}(K)\equiv\tfrac{1}{2}(\vec{\mathcal{g}}(K\Phi,\overline{\Phi})+\vec{\mathcal{g}}(\Phi,\overline{K\Phi}))\,. \qquad (3.25)$$

Thus the Noether conservation law is not independent but merely equivalent to the *imaginary* part of our original complex conservation law (3.18).

It is to be mentioned that in practice the additional non-Noether conservation law (3.21) [i.e., the real part of (3.18)] will not necessarily be genuinely independent. Thus for example in the trivial special case when K is taken simply to be the *unit operator* I—which always gives rise to an invariance—then the real part $\vec{\mathfrak{g}}^{+}$ of the complex conserved current will vanish identically, while the imaginary part reduces to the basic conserved current that was introduced at the outset, i.e., we have

$$\vec{\mathfrak{g}}(I) \equiv \vec{\mathfrak{g}} \; . \tag{3.26}$$

A nontrivial example is that of an energy-momentum conservation law resulting from a spacetime translation invariance: In Sec. V it will be shown explicitly in the particular case of a second-order scalar wave equation that the real part $\vec{\mathfrak{g}}^{+}$ of the complex conserved energy-momentum current will consist merely of a divergence, so that although it may be *locally* nonzero, its integral over a spacelike hypersurface, subject to appropriate boundary conditions, will still vanish identically.

IV. THE SCALAR WAVE EQUATION AND THE EQUIVALENCE PRINCIPLE

We shall illustrate the foregoing principles by applying them in the simplest possible case, namely that where the field Φ consists of just one *complex scalar* component obeying a *second-order* wave equation. This type of model theory has always been used as a first (exploratory or pedagogical) step in the process of refinement from a crude classical point-particle theory toward a more precise quantum theory. The scalar model has often made it easier to perceive effects (such as the Hawking particle emission process) that might have eluded discovery longer if a more sophisticated theory had been used at the outset. Its use implies that we are not only ignoring effects such as radiation reaction, as we have done throughout, but also effects associated with spin, magnetic moments, etc.

We shall at various stages be concerned with the transition from this first-quantized passive scalar field model to the corresponding classical limit in which the allowed motions are solutions of a set of ordinary differential equations whose coefficients are explicit functions of the external fields that are supposed to be acting on the particle. The classical particle may be said to be neutral (or "freely falling") if it is possible to choose a local ("freely falling") coordinate reference system in the neighborhood of any given point in such a way that the acceleration at that point is zero, that is to say, if the possible motions are describable as geodesics of a certain *projective connection* (see Ehlers and Schild[31]). The *weak* version of Einstein's equivalence principle may be expressed as the statement that this projective affine connection is *universal* in the sense of being the same for all kinds of neutral particles.

(It is this principle that the Eötvös-Dicke-Brazinsky experiments are generally accepted to have tested with very high accuracy.) The *strong* version of Einstein's equivalence principle includes a further statement to the effect that this universal (projective) connection derives from the Riemannian affine connection associated with a certain universal pseudo-Riemannian *metric* tensor. Since the requirement of compatibility with a metric imposes significant restrictions (integrability conditions) on the possible form of the connection, it is often maintained that the strong equivalence principle needs independent experimental verification (in addition to that given by the Eötvös-type experiments) of the kind provided (so far to much lower accuracy) by gravitational red-shift experiments such as those of Pound and Rebka[32] for photons, or Colella, Overhauser, and Werner[33] for neutrons. However, according to what is widely referred to in the literature (see e.g. Will[34]) as Schiff's "conjecture," such independent verification should be superfluous *provided* one accepts the standard principles of quantum mechanics—in terms of which the strong principle should be derivable as a logical consequence of the weak version of the equivalence principle.

Now we shall argue at the end of this section that far from being a difficult theorem, "Schiff's conjecture" is virtually a tautology, *provided* that the "standard principles" of quantum mechanics are interpreted as including the principle of *local Lorentz covariance*, meaning that on scales small enough for the metric to be considered as uniform the laws of physics may be considered as isotropic with respect to transformations leaving invariant the Minkowski structure defined thereby. Lorentz invariance in this sense may be considered to have been experimentally verified—though not of course with unlimited accuracy—by many types of experiment; a classic example is the Lamb-shift measurements (see e.g. Lundeen and Pipkin[35]) on the hydrogen atom, which confirm the predictions of Lorentz-covariant quantum-electrodynamic theory to at least one part in 10^8. The isotropy of Minkowski space that is *observed* in such experiments would of course be violated by the effects of externally imposed electric or magnetic fields (through the Stark and Zeeman effects), but it is obviously possible to restore effective Lorentz invariance in the *theory* of the phenomenon by treating the electrodynamic field as part of the dynamics and not as merely part of the external background. However, we would not know in advance how to do the same for a hypothetical new and as yet unknown field that might conceivably be responsible for deviations from the predictions of the strong equivalence principle. Thus, although such a field might (as with electromagnetism) be incorporated in a Lorentz-covariant theory later on, its effect could be expected to be first discovered experimentally as an apparent breaking of the isotropy of space-time, showing up *locally* through level splittings (analogous to those due to the Stark and Zeeman effects) which would presumably be of the *same order* of magnitude as the *global* frequency shifts of the type sought by experiments of the Pound-Rebka type. The chances of discovering such an effect are of course increased each time the possible accuracy of frequency comparisons is improved by a new technical advance such as was achieved by the use of the Mossbauer effect (on which the Pound-Rebka–type experiments are based). However, so long as such advances do not reveal *local* deviations from Lorentz invariance (as manifested by anomalous line splittings detectable in the laboratory) Schiff's argument would seem to imply that there is little point in trying to detect deviations from the predictions of the strong equivalence principle by *long-distance* comparisons, as was done in the rocket-launched hydrogen maser experiments recently described by Vessot.[36] (This is not to deny the potential interest of using highly accurate—e.g. hydrogen maser—clocks for testing higher-order general-relativistic effects in the neighborhood of the sun.)

We start from the fact that the *most general* second-order linear partial differential operator acting on a complex scalar field Φ may be written in the form

$$\mathcal{K}=\partial_a\mathcal{A}^{ab}\partial_b+\mathcal{B}^a\partial_a+\mathcal{C}\ , \tag{4.1}$$

where we may without loss of generality impose the symmetry condition

$$\mathcal{A}^{ab}=\mathcal{A}^{(ab)}\ . \tag{4.2}$$

In order for the form (4.1) to remain covariant under a general coordinate transformation, the complex quantities $\mathcal{A}^{ab}$, $\mathcal{B}^a$, $\mathcal{C}$ must transform respectively as components of contravariant tensor, vector, and scalar *densities*. The condition (2.13) that $\mathcal{K}$ should be self-adjoint can be expressed very simply as the requirement that the quantities $\mathcal{A}^{ab}$ and $\mathcal{C}$ be real and that $\mathcal{B}^a$ be pure imaginary:

$$\mathcal{A}^{ab}=\overline{\mathcal{A}}^{ab},\quad \mathcal{B}^a=-\overline{\mathcal{B}}^a,\quad \mathcal{C}=\overline{\mathcal{C}}\ . \tag{4.3}$$

Using the formula (2.14) we see that the fundamental current associated with (4.1) subject to (4.3) will have the form

$$\mathcal{J}^a=i\mathcal{A}^{ab}(\Phi\overline{\Phi}_{,b}-\overline{\Phi}\Phi_{,b})-2i\mathcal{B}^a\Phi\overline{\Phi}\ . \tag{4.4}$$

As the operator $\mathcal{K}$ is of second order, the corresponding Lagrangian density given by the automatic prescription (3.2) will also contain second derivatives. Since many standard procedures in theoretical physics are based on the supposition that the Lagrangian contains only first derivatives, it is useful to introduce a modified Lagrangian $\mathcal{L}_I$ from which the second derivatives have been removed by the addition of an appropriate divergence to $\mathcal{L}$. Therefore we write the Lagrangian obtained by applying (3.2) in the form

$$\mathcal{L}\equiv\mathcal{L}_\mathrm{I}-\{\tfrac{1}{2}\mathcal{A}^{ab}(\Phi\overline{\Phi})_{,b}\}_{,a}\ , \tag{4.5}$$

where

$$\mathcal{L}_\mathrm{I}\equiv\mathcal{A}^{ab}\Phi_{,a}\overline{\Phi}_{,b}+\mathcal{B}^a(\Phi\overline{\Phi}_{,a}-\overline{\Phi}\Phi_{,a})-\mathcal{C}\Phi\overline{\Phi}\ . \tag{4.6}$$

It is to be noted that as the system stands the field Φ is "unphysical" in the sense that the field equations are covariant and the current (4.4) is actually *invariant* under any transformation, whereby Φ is multiplied by a freely variable complex scalar factor, provided appropriate gauge transformations are applied to the fields $\mathcal{A}^{ab}$, $\mathcal{B}^a$, $\mathcal{C}$. When we set

$$\Phi\rightarrow\hat{\Phi}=S^{-1}\Phi \tag{4.7}$$

[which may be regarded as a local action of the group $GL(1,C)$] with

$$S=|S|\exp(i\chi)\ , \tag{4.8}$$

for some real phase angle χ, then we shall have

$$\mathcal{K}\rightarrow\hat{\mathcal{K}}=\overline{S}\mathcal{K}S\ , \tag{4.9}$$

where explicitly

$$\hat{\mathcal{K}}=\partial_a\hat{\mathcal{A}}^{ab}\partial_b+\hat{\mathcal{B}}^a\partial_a+\hat{\mathcal{C}} \tag{4.10}$$

with

$$\hat{\mathcal{A}}^{ab}=|S|^2\mathcal{A}^{ab}\ ,$$

$$\hat{\mathcal{B}}^a=|S|^2(i\mathcal{A}^{ab}\chi_{,b}+\mathcal{B}^a)\ , \tag{4.11}$$

$$\hat{\mathcal{C}}^a=|S|(\mathcal{A}^{ab}|S|_{,b})_{,a}+|S|^2(-\mathcal{A}^{ab}\chi_{,a}\chi_{,b}+2i\mathcal{B}^a\chi_{,a}+\mathcal{C})\ .$$

The most physically meaningful entity in the system, the current $\vec{\mathcal{J}}$, is left unchanged,

$$\mathcal{J}^a\rightarrow\hat{\mathcal{J}}^a=\mathcal{J}^a \tag{4.12}$$

as is also the fundamental Lagrangian for which we obtain

$$\mathcal{L}\rightarrow\hat{\mathcal{L}}=\mathcal{L}\ . \tag{4.13}$$

However, the modified first-order Lagrangian $\mathcal{L}_\mathrm{I}$ is invariant *only* for pure phase transformations, i.e., transformations with $S=1$. (An analogous situation occurs for Einstein's equations of general relativity where the physically fundamental Lagrangian—the Ricci scalar—is of second order and can be replaced by an equivalent first-order Lagrangian only at the expense of sacrificing its covariance property.)

The gauge invariance property of the system may be used to transform it to various possible standard forms in which the external fields are simplified as far as possible. One usually wishes to express the external fields as strictly tensorial quantities rather than as tensor densities. There is a uniquely natural way of doing this provided that the matrix $\mathcal{A}$ with coefficients $\mathcal{A}^{ab}$ is nonsingular, which will necessarily be the case when the field equation (2.1) obtained from (4.1) is hyperbolic (as also if it were elliptic, though not of course in the special parabolic case exemplified by the *nonrelativistic* time-dependent Schrödinger equation). Since it is nonsingular the matrix $\mathcal{A}$ will have a nonzero determinant $|\mathcal{A}|$ and an inverse $\mathcal{A}^{-1}$ which allows us to construct a new matrix g given by

$$g=-\left(\frac{||\mathcal{A}||}{\hbar^4}\right)^{1/(n-2)}\mathcal{A}^{-1} \tag{4.14}$$

whose components g_{ab} will transform as those of a *covariant tensor* under a general coordinate transformation. The quantity $\hbar$ is an (essentially superfluous constant introduced in accordance with hallowed custom for dimensional convenience. (we recall that n denotes the space-time dimension which will normally be taken to be 4). We may go on to define *covariant vector* $\underline{A}$ with components given by

$$ieA_a=-\hbar^{-1}||g||^{-1/2}g_{ab}\mathcal{B}^b \tag{4.15}$$

and a *scalar* U given by

$$U=\hbar^{-1}||g||^{-1/2}(\hbar\mathcal{C}-ie\mathcal{B}^aA_a), \tag{4.16}$$

where e is another constant introduced for dimensional convenience. We now introduce a fundamental Hermitian structure—which in the present case must have just a single component, transforming as a *scalar density*—by setting

$$\beta=||g||^{1/2}. \tag{4.17}$$

Hence using (2.19) we may replace our original self-adjoint scalar density operator $\mathcal{H}$ by a strictly *scalar* operator H which will have the very simple form

$$H=-D^aD_a+U, \tag{4.18}$$

where we now treat g as a (pseudo-)Riemannian *metric tensor*, which is used for raising and lowering space-time indices and for the construction of a Riemannian covariant differentiation operator ∇ in terms of which the operator D is defined by

$$D_a=\hbar\nabla_a+ieA_a. \tag{4.19}$$

We may also introduce a fundamental current *vector* with components N^a given by

$$\begin{aligned}N^a&=\hbar^{-1}||g||^{-1/2}\mathcal{J}^a\\&=i(\Phi\overline{D}^a\overline{\Phi}-\overline{\Phi}D^a\Phi),\end{aligned} \tag{4.20}$$

which will satisfy

$$\nabla_aN^a=0 \tag{4.21}$$

when the field equation (2.17) is satisfied.

The effect of the GL(1, C) gauge transformations can now be represented most conveniently by making the decomposition

$$U=\frac{\hbar^2}{4}\left(\frac{n-2}{n-1}\right)R+\mu^2, \tag{4.22}$$

where

$$R=R_a{}^a \tag{4.23}$$

is the scalar contraction of the Ricci tensor associated with the metric g, and μ is a new external scalar field. Then on making the *phase gauge* transformation

$$A_a\to\hat{A}_a=A_a+\frac{\hbar}{e}\chi_{,a} \tag{4.24}$$

and the *conformal gauge* transformation

$$\begin{aligned}g_{ab}&\to\hat{g}_{ab}=\lambda^2g_{ab},\\ \mu&\to\hat{\mu}=\lambda^{-1}\mu\end{aligned} \tag{4.25}$$

with

$$\lambda=|S|^{2/(n-2)}, \tag{4.26}$$

we obtain (with the aid of formulas given e.g. by Schouten[37])

$$H\to\hat{H}=\overline{S}HS \tag{4.27}$$

with

$$\hat{H}=-\hat{D}^a\hat{D}_a+\frac{\hbar^2}{4}\left(\frac{n-2}{n-1}\right)\hat{R}+\hat{\mu}^2, \tag{4.28}$$

where all careted quantities are defined in terms of the new metric λ^2g. The corresponding transformation of the current vector is

$$J^a\to\hat{J}^a=\lambda^{-n}J^a. \tag{4.29}$$

As can be seen more clearly from the earlier form (4.12), this last condition is precisely what is required to ensure that the total charge flux across a given hypersurface segment remains constant under the gauge transformation. Since it is the charge flux rather than the amplitude or phase of Φ that is of direct physical significance, one is in practice free to adjust the gauge according to convenience. In practical applications to relativistic free particle wave equations the most obviously natural and convenient way to choose the conformal gauge is to require that the "effective mass" field, μ, be *uniform*, i.e.,

$$\mu_{,a} = 0 \tag{4.30}$$

(see, however, the discussion of Bekenstein,[38] who considers an alternative possibility which has certain advantages when *active* gravitational effects are being treated). Except when the gauge-invariant "electromagnetic field" F defined by

$$F_{ab} = 2A_{[a,b]} \tag{4.31}$$

is zero (in which case $\underline{A}$ can be adjusted to vanish), there is no correspondingly unique way of fixing the phase gauge, but it can at least be restricted partially in a convenient way by a gauge condition of the much used Lorentz form

$$\nabla_a A^a = 0 \,. \tag{4.32}$$

In so far as gauge conditions are permissible, we see that the 15 independent external field components ($\mathcal{A}^{ab}$, $\mathcal{B}^a$, $\mathcal{C}$) with which we started out are partially redundant, and that the invariant properties of the wave equation can be completely characterized by just two tensor fields which may be taken to be g and F, where the latter is subject to the further Maxwell restriction

$$F_{[ab,c]} = 0 \,; \tag{4.33}$$

these fields are defined in a uniform-mass gauge to within *constant* scalar multiples which are adjustable by the choice of e and of the constant value of μ. What we have shown so far in this section may be summed up as follows:

The laws of motion of a particle in an external field, as determined by a single self-adjoint partial differential equation in the first-quantized passive-test-particle limit (wherein self-interactions and internal spin or multiple structure are ignored), depend on just two constants, e and μ say, and just *two* tensor fields, namely a symmetric pseudo-Riemannian metric tensor g and an antisymmetric Maxwell tensor F subject to (4.34).

We are now in a position to state what may be called the *extended equivalence principle* to the effect that we have the following.

The metric tensor g and the Maxwell tensor F, as specified in the preceding paragraph, are *universal*, in the sense that the *same* g and F may be used for the description of *any* free passive-test-particle motion (so that only the constants e and μ depend on the particular kind of particle involved). This principle is an immediate corollary of the fact that the graviton and the photon are—as far as is known at present—the *only zero-mass bosons*, i.e., the only particles that can give rise to long-range effectively "external" fields governing the background in which local processes take place. (If a violation of this principle were one day detected experimentally, the immediate conclusion to be drawn would be that another kind of zero-mass boson exists.)

The extended equivalence principle expounded in the previous paragraph is a generalization of the better-known (strong) Einstein equivalence principle which refers only to the metric tensor g but not to the Maxwell tensor. In other words, the Einstein version of the principle applies directly only to particles that are *electrically neutral* in the sense that the quantities $\mathcal{B}^a$ (or A_a) have no influence on their motions which may therefore be characterized by setting $e = 0$ in the foregoing formalism. It should now be remarked that this Einstein equivalence principle is merely a slight reinforcement of the principle of local Lorentz invariance as applied within the present framework. According to this latter principle, the metric g determined by the wave equation describing the motion of any particular kind of particle must be invariant under the Lorentz group action that leaves invariant the metric g' say determined by the wave equation describing the motion of a second kind of particle, which implies that g may differ from g' at most by an overall conformal scale factor.

To establish the Einstein equivalence principle itself we need only to know that the conformal factor relating g and g' is a *constant* (which will then be able to be absorbed into the definition of the effective mass μ in such a way as to render g identical to g'). Thus to prove the validity of the Schiff's conjecture subject to local Lorentz invariance, we need only to demonstrate that the constancy of this conformal factor is a consequence of the "weak" equivalence principle, i.e., of the universality of the projective affine connection determining the classical trajectories of neutral particles.

To obtain an explicit demonstration of this almost obvious result, one takes the classical limit by the standard Ehrenfest method, using the fact that the trajectories of localized *high-frequency wave packets* obeying any *linear* (first- or higher-order) scalar wave equation may be obtained as *bicharacteristics* of the corresponding nonlinear *first-order* Hamilton-Jacobi equation by applying the *eikonal approximation* to the scalar wave equation

$$(H\Phi) = 0 \,. \tag{4.34}$$

This approximation consists of ignoring all derivatives of background tensor fields in the scalar wave operator H, while replacing the operator components ∇_a when they come to act on Φ by gradient components $(i/\hbar)S_{,a}$, a so that the real unknown S (the action) replaces the original unknown Φ which factors out leaving an equation of the Hamilton-Jacobi form

$$H(x, \nabla S) = 0 , \tag{4.35}$$

where H is a scalar function of the space-time position x, and of the covariant vector ∇S. This function will be real as a consequence of the self-adjointness property of βH. The bicharacteristics of Eq. (4.35) are now obtainable immediately as solutions of the corresponding Hamiltonian system of ordinary differential equations

$$\frac{dx^a}{d\tau} = \frac{\partial H}{\partial \pi_a} , \tag{4.36}$$

$$\frac{d\pi_a}{d\tau} = -\frac{\partial H}{\partial x^a} \tag{4.37}$$

subject to the "proper mass-shell" *constraint*

$$H = 0 , \tag{4.38}$$

where π_a are the components of a generalized momentum covector $\underline{\pi}$, and the function $H(x, \underline{\pi})$ is obtained by substituting $\underline{\pi}$ in place of ∇S in the left-hand side of the eikonal equation (4.35).

In our particular case we obtain from (4.18) and (4.22) a Hamilton-Jacobi–type equation given by

$$(S_{,a} + eA_a)g^{ab}(S_{,b} + eA_b) + \mu^2 = 0 , \tag{4.39}$$

and the directly corresponding *constrained Hamiltonian system* is specified by

$$H(x, \underline{\pi}) \equiv (\pi_a + eA_a)g^{ab}(\pi_b + eA_b) = 0 . \tag{4.40}$$

When e is zero (and assuming of course that we are using the conformal gauge in which μ is uniform) the resulting Hamilton equations are simply those for the geodesics of the connection derived from the metric g. The effect of a conformal variation of g can be represented by allowing μ instead to vary, which would have the effect of introducing an extra force proportional to the gradient of μ, and thus would cause the trajectories to deviate from the geodesics of the universal connection in violation of the weak equivalence principle. Thus this latter principle allows us to conclude (in accordance with Schiff's conjecture and subject to local Lorentz invariance) that the metric is indeed universal as required by the strong equivalence principle.

We note here for future reference that since the eikonal equation (4.35) is unaffected by multiplication of the left-hand side by an arbitrary positive scalar field, λ^2 say, it follows that the trajectories are also unaffected by a transformation of the form

$$H \to H' \equiv \lambda^2 H . \tag{4.41}$$

It can be seen directly from the Hamiltonian equations (4.36) and (4.39)—bearing in mind the restraint (4.38)—that the effect of such a transformation is merely to bring about a reparametrization of the trajectories given by

$$\tau \to \tau', \quad d\tau'/d\tau = \lambda^{-2} . \tag{4.42}$$

Thus—provided μ is nonzero—we may replace (4.40) by the equivalent systems

$$H' \equiv (\pi_a + eA_a) \frac{g^{ab}}{2\mu} (\pi_b + eA_b) + \tfrac{1}{2}\mu = 0 \tag{4.43}$$

or

$$H'' \equiv (\pi_a + eA_a) \frac{g^{ab}}{\mu^2} (\pi_b + eA_b) = 1 , \tag{4.44}$$

the last of these having the advantage of maximum brevity of expression. When dealing with relativistic *free* particle wave equations in the standard conformal gauge for which μ is uniform the distinction between these three alternatives is immaterial. However, an important relativistic application in which it is preferable to retain a variable μ occurs in the treatment of *perfect fluids* (see Carter,[39] work in preparation). In such cases the *second* alternative (4.43) has the attractive feature that it corresponds to a parametrization which contains the *proper time* with respect to the metric g. Nevertheless the original form (4.40) remains the most satisfactory for many purposes since, although it implies improper parametrization, it is the only one of the three alternatives that remains valid for *null* trajectories.

V. CONSERVATION LAWS RESULTING FROM ORDINARY SPACE-TIME INVARIANCE GROUPS

Before returning to consider the first-quantized system described by (4.18) and (4.34) we shall recapitulate some well-known results concerning the constants of motion in the classical limit. We start with the familiar fact that a general function K say of position x and of momentum $\underline{\pi}$ will be a constant of any motion satisfying (4.36) and (4.37) if and only if the Poisson bracket, as defined by

$$[H,K]_P \equiv \frac{\partial H}{\partial x^a}\frac{\partial K}{\partial \pi_a} - \frac{\partial K}{\partial x^a}\frac{\partial H}{\partial \pi_a} , \tag{5.1}$$

is zero, i.e.,

$$[H,K]_P = 0 ; \tag{5.2}$$

this is the classical analog of the quantum commutation condition (3.16).

The most obvious kind of constant of motion is the kind that arises from a straightforward space-time symmetry of the system. Clearly the Hamiltonian *function* $H(x, \underline{\pi})$ will be invariant under a space-time action generated by a vector $\vec{k}$ say if, and only if, for any space-time covector field $\underline{\pi}$ (consistent with the restraint) that is invariant under the action generated by $\vec{k}$, the corresponding *scalar field* $H(x, \underline{\pi})$ is also invariant, i.e., if and only if

$$\vec{k}\mathcal{L}\underline{\pi} = 0 \quad \vec{k}\mathcal{L}H(x, \underline{\pi}) = 0 \,, \tag{5.3}$$

where the symbol $\vec{k}\mathcal{L}$ denotes Lie differentiation with respect to the vector field $\vec{k}$. Using the explicit formulas

$$\vec{k}\mathcal{L}\pi_a \equiv \pi_{a,b}k^b + \pi_b k^b{}_{,a} \,, \tag{5.4}$$

$$\vec{k}\mathcal{L}H(x, \underline{\pi}) \equiv \left(\frac{\partial H}{\partial x^a} + \frac{\partial H}{\partial \pi_b}\pi_{b,a}\right)k^a \,, \tag{5.5}$$

one obtains the identity

$$\vec{k}\mathcal{L}H(x, \underline{\pi}) \equiv [H, k^a\pi_a]_P + \frac{\partial H}{\partial \pi_a}\vec{k}\mathcal{L}\pi_a \,, \tag{5.6}$$

from which the following theorem is an obvious deduction:

For a *space-time field* $\vec{k}$ to generate a symmetry of the Hamiltonian function it is not only necessary (as is well known from the theory of ignorable coordinates) but *also sufficient* that the scalar contraction $k^a\pi_a$ be a constant of the motion.

We are concerned here with the particular case when H has the form (4.40), for which, by introducing

$$u^a = \frac{1}{2\mu}\frac{dx^a}{d\tau} \tag{5.7}$$

the corresponding Hamiltonian equations (4.36) and (4.37) may be expressed as

$$\pi_a = \mu u_a - eA_a \,, \tag{5.8}$$

$$u_{a;b}u^b = \frac{e}{\mu}F_{ab}u^b \,, \tag{5.9}$$

while the constraint (4.38) takes the form

$$u^a u_a = -1 \,. \tag{5.10}$$

Since the only fields involved are g and $\underline{A}$ we deduce that the necessary and sufficient condition for $k^a\pi_a$ to be a constant of motion will be

$$\vec{k}\mathcal{L}g = 0 \tag{5.11}$$

(which will be *sufficient* in the neutral case) and

$$\vec{k}\mathcal{L}\underline{A} = 0 \tag{5.12}$$

(which will only be relevant when $e \neq 0$). This may be verified directly by evaluating the Poisson bracket, which has the explicit form

$$[H, k^a\pi_a]_P \equiv -2k_{a;b}\mu^2 u^a u^b + 2(A_{a;b}k^b + k^b{}_{;a}A_b)e\mu u^a \,. \tag{5.13}$$

Since μu^a may have an arbitrary initial value the coefficient of the linear term and the (symmetrized) coefficient of the term quadratic in μu^a must vanish separately if the quantity in brackets is to be zero always. Hence, using the identities

$$\vec{k}\mathcal{L}g_{ab} \equiv 2k_{(a;b)} \,, \quad \vec{k}\mathcal{L}A_a = A_{a;b}k^b + A_b k^b{}_{;a} \,, \tag{5.14}$$

we immediately obtain the conditions (5.11) and (5.12).

Now let us consider the analogous situation for a corresponding first-quantized system. The condition that the system be invariant under the space-time action generated by the field $\vec{k}$ is equivalent to the requirement that all the various operators of the system should commute with the operation $\vec{k}\mathcal{L}$. In particular, this must apply to the fundamental Hermitian structure β which in the scalar case is simply equal to the measure $\|g\|^{1/2}$, and hence we obtain as a preliminary requirement that

$$\vec{k}\mathcal{L}\,\|g\|^{1/2} \equiv \|g\|^{1/2}k^a{}_{;a} = 0 \,. \tag{5.15}$$

Provided this condition is satisfied, the requirement of invariance of the system under the space-time action generated by $\vec{k}$ is equivalent to the requirement that the system be invariant under the action on the fundamental scalar field Φ generated by the operator

$$iK = -k^a\nabla_a \,. \tag{5.16}$$

By (3.15) we may express this in the equivalent form

$$K = \tfrac{1}{2}i[k^a\nabla_a + \nabla_a k^a] \,, \tag{5.17}$$

wherein it is manifest that the self-adjointness requirement (2.20) will be satisfied. The corresponding commutator with the scalar wave operator (4.18) subject to (4.22) will be given by

$$[H, k^a\nabla_a] \equiv -2k_{(a;b)}D^aD^b - [\hbar(2k_{(a;b)}{}^{;b} - k^b{}_{;b;a}) - ie(A_{a;b}k^b + A_b k^b{}_{;a})]D^a + \frac{\hbar^2}{4}\left(\frac{n-2}{n-1}\right)k^a R_{,a} + ie\hbar(A_{a;b}k^b + A_b k^b{}_{;a})^{;a} \,. \tag{5.18}$$

Since the symmetrized derivatives of Φ may be chosen arbitrarily at any initial point, it follows that such an operator can only vanish identically if the coefficients of each of the symmetrized operators D^a, $D^{(a}D^{b)}$, etc. appearing within it are zero separately.

Applying this to the first line in (5.18) we obtain the familiar Killing condition

$$k_{(a;b)}=0\,, \tag{5.19}$$

which is equivalent to (5.11) by (5.14) and implies at once that the first term of both the second and third lines will drop out. From what remains of the second line we obtain

$$A_{a;b}k^b+A_bk^b{}_{;a}=0\,, \tag{5.20}$$

which is equivalent to (5.12) by (5.14) and implies that the remaining term on the third line drops out. Thus we confirm that the *same* conditions (5.11) and (5.12) that were necessary and sufficient for the function $K=k^a\pi_a$ to be a constant of motion in the classical limit are also necessary and sufficient for the operator $K=-ik^a\nabla_a$ to be a constant of motion—in the sense of Sec. III—in the quantized theory.

Let us now examine the form of the corresponding conserved current obtained by substituting the field $(K\Phi)$ as given by (5.17) in place of Φ (but not $\overline{\Phi}$) in (4.20) which gives

$$N^a(K\Phi,\overline{\Phi})=\hbar k^b(-\overline{\Phi}\Phi_{,b}{}^{;a}+\overline{\Phi}^{;a}\Phi_{,b})$$
$$+\hbar k^{a;b}\overline{\Phi}\Phi_{,b}-iek^bA^a\overline{\Phi}\Phi_{,b}\,. \tag{5.21}$$

This can be written in a more meaningful form if we introduce a covariant tensor W with components given by

$$W^{ab}=2\hbar(\overline{\Phi}k^{[a}D^{b]}\Phi) \tag{5.22}$$

and a mixed tensor $\mathcal{T}$ with components given by the expression

$$\mathcal{T}^a{}_b=\hbar(\overline{\Phi}_{,b}D^a\Phi+\Phi_{,b}\overline{D}^a\overline{\Phi})$$
$$-g^b_a[(D_c\Phi)\overline{D}^c\overline{\Phi}+\overline{\Phi}D_cD^c\Phi] \tag{5.23}$$

in which all the terms except the last are manifestly real, and where the last term also will be real whenever the field equations are satisfied, so that we shall have the properties

$$W^{ab}\equiv W^{[ab]}\,, \tag{5.24}$$

$$\mathcal{T}^a{}_b=\overline{\mathcal{T}}^a{}_b\,. \tag{5.25}$$

In terms of these quantities we obtain

$$\hbar N^a(K\Phi,\overline{\Phi})=\mathcal{T}^a{}_bk^b+\nabla_bW^{ab}\,. \tag{5.26}$$

Since it is the divergence of an antisymmetric quantity, the second term is trivially conserved, i.e.,

$$\nabla_a(\nabla_bW^{ab})=0\,, \tag{5.27}$$

so that the complex conservation condition

$$\nabla_aN^a(K\Phi,\overline{\Phi})=0 \tag{5.28}$$

finally yields only a single nontrivial conservation law for the real vector field with components $\mathcal{T}^a{}_bk^b$, which must satisfy

$$\nabla_a(\mathcal{T}^a{}_bk^b)=0\,. \tag{5.29}$$

The effective "energy-momentum tensor" brought to light in this way is in fact just the "canonical" energy-momentum tensor defined in the standard way by applying the formula

$$\mathcal{T}^a{}_b=-\frac{\partial\Lambda}{\partial\Phi_{,a}}\Phi_{,b}-\frac{\partial\Lambda}{\partial\overline{\Phi}_{,a}}\overline{\Phi}_{,b}+\Lambda g^a_b \tag{5.30}$$

to the first-order Lagrangian scalar Λ defined from (4.6) by

$$\mathfrak{L}_{\rm I}=\|g\|^{1/2}\Lambda\,, \tag{5.31}$$

which gives the explicit expression

$$\Lambda\equiv-(D_c\Phi)(\overline{D}^c\overline{\Phi})-U\Phi\overline{\Phi}\,. \tag{5.32}$$

This may be expressed alternatively by

$$\mathcal{T}^a{}_b=T^a{}_b-eN^aA_b\,, \tag{5.33}$$

where

$$T^{ab}=2\frac{\partial\Lambda}{\partial g_{ab}}+\Lambda g^{ab} \tag{5.34}$$

is the ordinary geometric energy-momentum tensor which satisfies

$$\nabla_bT^{ab}=eN_bF^{ab}\,. \tag{5.35}$$

VI. GENERAL FIRST- AND SECOND-ORDER CONSTANTS OF MOTION

The canonical quantity $k^a\pi_a$ and its quantum analog that arose in the preceding section as constants of motion associated with a space-time symmetry have the—for some purposes undesirable—property of being phase-gauge dependent. We shall start our investigation of constants associated with more general kinds of symmetry by considering the necessary and sufficient conditions to have a constant of the classical motion of the general linear form

$$K\equiv\mu k^au_a+c\,, \tag{6.1}$$

where $\vec{k}$ is a vector field and c is a scalar field, and where we now choose to work with the gauge-independent velocity $\vec{u}$ rather than the gauge-dependent momentum $\underline{\pi}$.

On working out the Poisson bracket with H as given by (4.40) we obtain

$$[H,\mu k^au_a+c]_P\equiv-2K_{a;b}\mu^2u^au^b$$
$$+2(eF_{ab}k^b-c_{,a})\mu u^a\,, \tag{6.2}$$

from which we see that the necessary and sufficient conditions for K as given by (6.1) to be a constant of the classical motion are

$$k_{(a;b)}=0\,, \tag{6.3}$$

$$eF_{ab}k^b=c_{,a}\,, \tag{6.4}$$

of which the first is once more the Killing condition for $\vec{k}$ to generate a symmetry of the metric, while the second may, if $\vec{k}$ is timelike, be interpreted as a requirement that c play the role of an effective potential for the electric component of the Maxwell field as defined relative to the rest frame determined by $\vec{k}$. A field c of the required form can exist locally if and only if the exterior derivative of the left-hand side of (6.4) is zero, which is equivalent (by Cartan's identity) to the condition

$$\vec{k}\,\pounds\, F_{ab}=0\,. \tag{6.5}$$

Thus we see that there can be a linear constant of the motion of the form (6.1) *only* if $\vec{k}$ generates a symmetry both of the metric g and of the *gauge-independent* electromagnetic field F, and that when this requirement is satisfied the required field c will be determined uniquely modulo an additive constant. A noteworthy special case is that when the field is purely *magnetic* relative to a, the frame determined by $\vec{k}$—if it is timelike—in the sense that $F_{ab}k^b$ is zero, in which case we obtain a constant of the motion of the simple *homogeneous* form $K^a u_a$.

Having thus seen that a linear constant of classical motion is always related to a symmetry of g and F under an ordinary space-time transformation group, let us now move on to consider the requirements for the existence of a *quadratic* constant of the classical motion, starting with the case of a constant of the *homogeneous* form

$$K\equiv \mu^2 a^{ab}u_a u_b\,, \tag{6.6}$$

where a^{ab} is a *symmetric tensor* field. Working out the Poisson bracket with H as given by (4.40) yields

$$[H,\mu^2 a^{ab}u_a u_b]_P \equiv -\,2a^{ab;c}\mu^3 u_a u_b u_c + 4a_c{}^{(a}F^{b)c}e\mu^2 u_a u_b\;. \tag{6.7}$$

The requirement that the coefficients of the terms cubic and quadratic in $\vec{u}$ must vanish separatly gives

$$a^{(ab;c)}=0\,, \tag{6.8}$$

$$a_c{}^{(a}F^{b)c}=0 \tag{6.9}$$

as necessary and sufficient conditions for the homogeneous quadratic function (6.7) to be a constant of the classical motion for general neutral and charged particles. It was the discovery[13] in the Kerr and Kerr-Newman spaces of a constant of precisely this kind that provided the initial stimulation for the present study. If we are interested in functions of the more general nonhomogeneous quadratic form

$$K=\mu^2 a^{ab}u_a u_b+\mu b^a u_a+c\,. \tag{6.10}$$

then it is evident that by combining (6.2) and (6.7) we shall obtain as necessary and sufficient conditions three equations consisting of *firstly* (6.8) as it stands, *secondly* the combination

$$2ea_c{}^{(a}F^{b)c}=b^{(a;b)}\,, \tag{6.11}$$

which replaces the separate equations (6.3) and (6.9), and *thirdly* the equation (6.4) as it stands.

If we are only interested in the *geodesic* trajectories of *neutral* particles then Eq. (6.8) alone is clearly a necessary and sufficient condition for the existence of a quadratic constant of motion. Analogous conditions for cubic and higher-order constants of geodesic motion have been discussed by Eisenhart[25] and Woodhouse.[27] Walker and Penrose[22] have suggested that because (6.8) is the analog of the Killing condition (6.3) for the existence of a linear constant, then symmetric tensor a satisfying (6.8) should be described as a "Killing" tensor; but since the earliest studies of conditions for the existence of such a quadratic constant were made by Stackel,[40] it would seem that the term *Stackel tensor* is more appropriate. I prefer to reserve the description *Killing tensor* for the case to be described below where the symmetric tensor a gives rise to an operator generating a (hidden) *symmetry of the quantized system* in an analogous manner to that in which a (nonhidden) symmetry operator arises (as described in the preceding section) from an ordinary Killing vector. In the particular case of the Kerr and Kerr-Newman solutions the distinction does not arise since the Stackel tensor turns out[14] to be also a Killing tensor in this strong sense. A primary purpose of the present work is to account for this fact by showing that for a vacuum Einstein or Einstein-Maxwell space, though not in general otherwise, a Stackel tensor will automatically be a Killing tensor.

Let us now move on to consider the analogous conditions for the existence of corresponding constants of the motion in the first-quantized system. Subject to the self-adjointness condition (2.20) the most general first-order differential operator may be expressed in the gauge-invariant form

$$K=-\tfrac{1}{2}(k^a D_a+D_a k^a)+c\,, \tag{6.12}$$

and hence substituting the expression (4.18) for H we obtain the commutator analogous to (6.2) in the form

$$[H,-\tfrac{1}{2}i(k^a D_a+D_a k^a)+c]\equiv 2i\hbar k^{(a;b)}D_a D_b+2\hbar(i k^{(a;b)}{}_{;b}-ek_bF^{ba}-c^{;a})D_a+\hbar^2(\tfrac{1}{2}i\hbar k_b{}^{b;a}-ek_bF^{ba}-c^{;a})_{;a}+i\hbar k^a U_{,a}\,. \tag{6.13}$$

In order for such a commutator to be zero, the (symmetrized) coefficients of the symmetric gauge-invariant differentiation operators of each order must vanish separately. From the first (highest-order) coefficient we obtain our previous (classical) condition (6.3), and when this is satisfied the condition obtained from the second term reduces to (6.4).

Finally when both (6.3) and (6.4) are satisfied the last (nondifferential) term yields simply

$$\vec{k}\pounds\, U=0\,, \tag{6.14}$$

and if U has the form (4.22) this will be satisfied automatically by consequence of the isometry implied by (6.3). Thus for the first-order quantum operator (6.2) to be a constant of motion in the sense of Sec. III the necessary and sufficient conditions are precisely the same as for its classical analog (6.1). We shall see now, however, that for a second-order quantum operator the analogous property does *not* exist, in so much as such requirements for it to be a constant of motion are in general more severe than in the classical limit.

The simplest gauge-invariant form for a second-order operator satisfying the self-adjointness condition (2.20) may be given by

$$K=-D_a a^{ab} D_b\,, \tag{6.15}$$

which is the quantum analog of the homogeneous quadratic function (6.6). The most general second-order operator satisfying (2.20) may be written as a constant coefficient linear combination of (6.12) and (6.15). The commutator of (6.15) with H as given by (4.18) may be evaluated in two steps by working out

$$\begin{aligned}[D_cD^c,\, D_a a^{ab}D_b] \equiv{} & 2\hbar\, a^{ab;c} D_{(a}D_bD_{c)} + \hbar\,(3\hbar\, a^{(ab;c)}{}_{;c} - 4ie a_c{}^{(a}F^{b)c})D_{(a}D_{b)} \\ & + \left\{\frac{\hbar^3}{2} g_{cd}(a^{(cd;a);b} - a^{(cd;b);a}) - \frac{4\hbar^3}{3} a_c{}^{[a}R^{b]c} - 4ie\hbar^2 a_c{}^{(a}F^{b)c}\right\}_{;b} D_a \\ & + \frac{ie\hbar^3}{3}\{2R_{ab}a_c{}^a F^{bc} - 2(a_c{}^a F^{bc})_{;a;b} + (a_a{}^b F^{ac}{}_{;c})_{;b}\} + \frac{ie\hbar^3}{2} g_{cd}a^{(cd;a);b}F_{ab}\end{aligned} \tag{6.16}$$

and

$$[U,-D_a a^{ab}D_b] \equiv 2\hbar\, a^{ab}U_{,b}D_a + (\hbar^2 a^{ab}U_{,b})_{;a}\,, \tag{6.17}$$

and then taking the sum

$$[H,-D_a a^{ab}D_b] \equiv [D_cD^c, D_a a^{ab}D_b] + [U,-D_a a^{ab}D_b]\,. \tag{6.18}$$

By setting equal to zero the symmetrized coefficients of the differential operators of each order separately, we obtain as necessary and sufficient conditions for the operator (6.15) to be a constant of motion *four* separate equations consisting *firstly* and *secondly* of the corresponding classical equations (6.8) and (6.9), together with *thirdly* the equation

$$\hbar^2\{a_c{}^{[a}R^{b]c}\}_{;b} = \tfrac{3}{2}a^{ab}U_{,b} \tag{6.19}$$

[obtained by setting equal to zero the coefficient of D_a after (6.8) and (6.9) have been satisfied] and finally the equation

$$\hbar^2\{a_a{}^b F^{ac}{}_{;c}\}_{;b}=0 \tag{6.20}$$

[obtained by setting the nondifferential term equal to zero after (6.8), (6.9), and (6.19) have been satisfied], where we retain the multiplying factor $\hbar^2$ as a reminder that this condition is redundant in the classical limit. This last equation (6.20) will be irrelevant if we are concerned only with neutral-particle motions. However, it would seem reasonable to stipulate that a^{ab} should at least satisfy (6.19) in addition to (6.8) in order to merit description as a Killing tensor.

It is now apparent that the first auxiliary quantum requirement (6.19) will automatically be satisfied when the Ricci tensor is zero, bearing in mind that by (4.22) we shall have

$$U_{,b}=\frac{\hbar^2}{4}\left(\frac{n-2}{n-1}\right)R_{,b}\,. \tag{6.21}$$

Thus we see that a Stackel tensor will necessarily be a Killing tensor when the *vacuum Einstein* equations are satisfied. It is also apparent that the second auxiliary quantum requirement (6.20) (which will in any case be relevant only if we are concerned with *charged*-particle motions) will be satisfied automatically if the *source-free Maxwell equation*

$$F^{ac}{}_{;c}=0 \tag{6.22}$$

is satisfied. Furthermore, it can easily be seen that if the source-free *Einstein-Maxwell equations* are satisfied the auxiliary conditions will still be redundant, since when the Ricci tensor is proportional to the Maxwell energy tensor we shall have not only

$$R=0 \tag{6.23}$$

but also—as a consequence of the classical condition (6.9)—

$$R_c{}^{[a}a^{b]c}=0 \tag{6.24}$$

which is clearly sufficient for (6.19) to be satisfied. (It has been pointed out to me by Gibbons that the automatic satisfaction of the quantum auxiliary conditions in the vacuum Einstein-Maxwell spaces could have been predicted from the corresponding result for pure Einstein vacuum spaces by using the five-dimensional formulation of Einstein theory wherein the field equations take the same form—namely vanishing Ricci tensor—as in those of pure Einstein theory in four dimensions.)

This completes our demonstration of how the existence of a quantum constant of motion of the form (6.15) in the Kerr and Kerr-Newman spaces follows automatically from the existence of a classical constant of the form (6.6) by consequence of the fact that the source-free Einstein-Maxwell equations are satisfied.

Analogous conclusions apply if we go on to consider the most general possible second-order operator subject to (2.20), which may be expressed as

$$K=-D_a a^{ab}D_b-\tfrac{1}{2}[b^aD_a+D_ab^a]+c\,. \tag{6.25}$$

By combining (6.13), (6.16), and (6.17) and requiring that the symmetrized coefficients of the differential operators of each separate order be equal to zero, working systematically from the highest-order downward we obtain in succession, as necessary and sufficient conditions for this operator K given by (6.25) to commute with H as given by (4.18), a set of *four* equations consisting of *firstly* and *secondly* the corresponding classical conditions (6.8) and (6.11), followed by *thirdly* the equation

$$eb_bF^{ba}+c^{:a}=a^{ab}U_{,b}-\tfrac{2}{3}\hbar^2\{a_c{}^{[a}R^{b]c}\}_{;b} \tag{6.26}$$

and finally the equation

$$\{e\hbar^2a_a{}^bF^{ac}{}_{;c}+3b^a(U-\tfrac{1}{6}\hbar^2R)\}_{;a}=0 \tag{6.27}$$

It is noted that (6.11) implies that the vector $\vec{b}$ must necessarily satisfy

$$b^a{}_{;a}=0 \tag{6.28}$$

so that although it need not be the generator of an isometry, it must at least generate a symmetry of the fundamental measure β. Bearing this in mind we see that the condition (6.27) reduces rather miraculously to the form (6.20) if we substitute for U the expression obtained from (4.22) in the particular case when the space-time dimension n is set equal to *four*, which gives simply

$$U-\tfrac{1}{6}\hbar^2R=\mu^2\,, \tag{6.29}$$

which is just a constant. In so far as we are concerned only with the four-dimensional case, our results may be summarized, in terms of the gravitational and electromagnetic source equations

$$R^{ab}-\tfrac{1}{2}Rg^{ab}=\frac{1}{4\pi}(F^a{}_cF^{cb}-\tfrac{1}{4}F_{cd}F^{cd}g^{ab}-\tfrac{1}{2}T_M{}^{ab})\,, \tag{6.30}$$

$$F^{ac}{}_{;c}=4\pi j^a\,, \tag{6.31}$$

where $\vec{j}$ is the electromagnetic current vector and $T_M{}^{ab}$ is the nonelectromagnetic part of the energy-momentum tensor, as in the following theorem:

$$[-D_aD^a+\tfrac{1}{6}\hbar^2R+\mu^2,\ -D_aa^{ab}D_b-\tfrac{1}{2}[b^aD_a+D_ab^a]+c]=0$$

if and only if the following is true:

	classical terms	auxiliary terms
I	$a^{(ab;c)}=0$,	
II	$b^{(a;b)}=2ea_c{}^{(a}F^{b)c}$	
III	$c^{:a}=-eb_bF^{ba}$	$-\dfrac{4\pi\hbar^2}{3}\{a^{ab}T_M{}^c{}_{c;b}+4(a_c{}^{[a}T_M{}^{b]c})_{;b}\}$,
IV	0	$=\hbar^2(a_c{}^bj^c)_{;b}$, (6.32)

It is now evident that when the *source-free Maxwell equations*

$$j^a=0 \tag{6.33}$$

are satisfied, the fourth condition will be redundant, and that when in addition

$$T_M{}^{ab}=0 \tag{6.34}$$

so that the *source-free Einstein-Maxwell equations* hold, the remaining three conditions will reduce to their classical form.

VII. SIMPLE EXAMPLES

We start by remarking that the metric tensor will always be a Stackel tensor, in the sense that if

$$a_{ab}=g_{ab}. \tag{7.1}$$

then (6.8) will hold; but the metric will not in general be a Killing tensor, since the auxiliary condition (6.19), which would be required for the operator

$$\Box\equiv\nabla_ag^{ab}\nabla_b \tag{7.2}$$

to commute with H, as given by (4.18) and (4.22), will be satisfied only if the Ricci scalar is a constant, i.e., if

$$R_{,a}=0 \tag{7.3}$$

However, when there are geometric symmetries, generated by not necessarily distinct Killing vectors $\vec{k}_{(1)}$ and $\vec{k}_{(2)}$ say, then their symmetrized product $\vec{k}_{(1)}\odot\vec{k}_{(2)}$ will necessarily be not only a Stackel tensor but also a Killing tensor in the strong sense: If we set

$$a^{ab} = k_{(1)}{}^{(a}k_{(2)}{}^{b)} \tag{7.4}$$

we shall not only satisfy the Stackel condition (6.8) but will also have both

$$a^{ab}R_{,b} = 0 \tag{7.5}$$

and

$$R_c{}^{[a}a^{b]c} = \tfrac{3}{4}\left(k_{(1)}{}^{[a;b}k_{(2)}{}^{c]} + k_{(2)}{}^{[a;b}k_{(1)}{}^{c]}\right)_{;c}, \tag{7.6}$$

whence

$$\left(R_c{}^{[a}a^{b]c}\right)_{;c} = 0 \tag{7.7}$$

(since the right hand side of (7.6) is divergence of a trivector), so that both the left- and right-hand sides of the auxiliary Killing tensor condition (6.19) will be zero.

A trivial but instructive special case is that of flat space, for which the metric may be expressed in terms of a particular system of flat coordinates r^a say by

$$ds^2 = g_{AB}dr^A dr^B, \tag{7.8}$$

where g_{AB} are *constants* and the Cartesian capital indices A, B run from 1 to n. In such a space there will not only be an n parameter family of translational Killing vectors $\vec{e}_{(A)}$ with flat coordinate components given by

$$e_{(A)}{}^B = g^B_A \tag{7.9}$$

but also a $\frac{1}{2}n(n-1)$ parameter set of independent rotational Killing vectors $\vec{m}_{(A)(B)}$ given by

$$m_{(A)(B)}{}^c = 2r^D g_{D[A}g_{B]}{}^c. \tag{7.10}$$

Among the very large [actually $(n^2+n)(n^2+n+2)/8$ parameter] family of flat-space Killing tensors that can be constructed from these by taking linear combinations of symmetrized tensor products, there is a particularly important $(n+1)$-parameter subset of the form

$$a = \tfrac{1}{2}\alpha g^{AC}g^{DF}\vec{m}_{(A)(B)} \odot \vec{m}_{(C)(D)} + \beta^A g^{BC}\vec{m}_{(A)(B)} \odot \vec{e}_{(C)}, \tag{7.11}$$

where the parameters α and β^A may be regarded as a scalar and as the components of a *Cartesian vector*, respectively.

Now although the constants of the motion obtained from these Killing tensors are essentially trivial when the Hamiltonian operator has the canonical relativistic form specified by (4.18), (4.22), and (4.30), the formalism of the preceding section enables them to be used for the construction of nontrivial constants of the motion in certain important special cases when the flat-space symmetry is *broken* by taking U to be an appropriate potential *not* of the form specified by (4.22) and (4.30).

These cases arise in the context of the nonrelativistic limit in ordinary Euclidean flat space where the dimension n is taken equal to 3 (instead of equal to 4 or 5 as in the physical examples that we have been considering hitherto); this is sufficient for treating cases where the background fields are stationary. While abandoning the expression (4.22) for U (it would diverge in any case in three dimensions), we may still retain the general form (4.18) for what is now to be interpreted as an ordinary nonrelativistic Schrödinger operator, but we must replace the expression (4.40) by the more general form

$$H = g^{ab}(\pi_a + eA_a)(\pi_b + eA_b) + U \tag{7.12}$$

for the corresponding classical Hamiltonian.

In the physical interpretation of (7.12) the covector $\underline{A}$ is now to be thought of as the three-dimensional potential for the magnetic part only of the Maxwell field, and U is a combined potential for both the electrostatic and Newtonian gravitational fields. Using those formulas of the preceding section which were left in their general form, it is easy to see that the necessary and sufficient conditions for a general quadratic function of the form (6.10) to be a constant of the classical motion determined by (7.12) are

$$\begin{aligned} &a^{(ab;c)} = 0, \quad eb_b F^{ba} + c^{;a} = a^{ab}U_{,b}, \\ &b^{(a;b)} = 2ea_c{}^{(a}F^{b)c}, \quad b^a U_{,a} = 0 \end{aligned} \tag{7.13}$$

(where we are now to interpret F^{ba} as being related to the *magnetic* field components B_a by $F^{ab} = \epsilon^{abc}B_c$, where ϵ^{abc} is the three-dimensional alternating tensor). Moreover, since we are now restricting our attention to *flat* space, the only additional requirement for the corresponding quantum operator of the form (6.25) to commute with the Hamiltonian (4.18) will be the condition

$$(a_a{}^b j^a)_{,b} = 0, \tag{7.14}$$

where j^a is now to be interpreted as the ordinary three-dimensional electric current vector.

I know of two cases where nontrivial constants of the motion can be constructed in addition to those whose presence is obvious from ordinary space symmetries. Both occur when the magnetic field is absent, in which case the restrictions on a and c in (7.13) decouple from the restrictions on $\vec{b}$, which is merely required to be the generator of an ordinary space symmetry of the system.

The remaining classical restrictions on a and c thus reduce to

$$a^{(ab;c)} = 0, \quad a^{ab}U_{,b} = c^{;a}. \tag{7.15}$$

Moreover, since the absence of a magnetic field implies by the relevant Maxwell equation that there can be no current, the auxiliary condition (7.14) will be redundant, so that the classically necessary conditions (7.15) will also be *sufficient* for the corresponding quantum operator

$$K = -\hbar^2 \nabla_a a^{ab} \nabla_b + c \tag{7.16}$$

to commute with the Hamiltonian operator (4.18).

The most famous example of a nontrivial constant of the motion of the form (7.16) arises when the translational symmetry generated by the vectors (7.9) is broken by the introduction of a simple spherically symmetric potential of the standard monopole form

$$U = Q/r, \tag{7.17}$$

where Q is a constant and

$$r = (g_{AB} r^A r^B)^{1/2}. \tag{7.18}$$

Since the rotational subgroup generated by (7.10) is left intact, it is obvious that the corresponding Casimir operator (i.e., the total squared angular momentum) obtained by setting β^A equal to zero in the special form (7.11) will still give rise to a constant of the motion. However, since the potential (7.17) is not invariant under the action generated by the vectors (7.9), it is no longer at all obvious that a Killing tensor of the form (7.11) will still give rise to a constant of the motion in the more general case when the β^A are nonzero. It turns out nevertheless that a constant of the form (7.16) can in fact be constructed for a tensor a of the form (7.11) for an arbitrary set of constants $\beta_A = g_{AB}\beta^B$ simply by setting

$$c = Q\frac{\beta_A r^B}{2r}. \tag{7.19}$$

It can easily be checked directly that this does in fact satisfy the condition (7.15). Thus in addition to the trivial total angular momentum constant, obtained when the β^A are zero, we have found a three-parameter family of nontrivial constants, for which we may take as a basis the particular members obtained when α and any two of the three β^A are zero. The three independent constants found in this way are in fact well known as the components of the famous Runge-Lenz vector whose existence can be regarded as accounting for the closure of the orbits in the classical Keppler problem. The existence of the corresponding quantum constants, which were discovered by Pauli, accounts for the degeneracy of energy levels in the nonrelativistic hydrogen atom problem. [Although these operators do not commute with each other, they do not thereby give rise to any further nontrivial linearly independent constants of the motion since they combine with the ordinary angular momentum operators to form a closed O(4) or O(3.1) algebra when acting respectively on negative- or positive-energy eigenstates—see e.g. Barut and Bornzin[41] for a recent discussion.]

The second example of a nontrivial constant of the form (7.16) arises when even the rotational part of the flat-space symmetry group is broken by adding a second monopole potential with the center displaced from the origin at a point with Cartesian coordinates z^A say, so that the total potential takes the form

$$U = \frac{Q}{r} + \frac{Q'}{r'}, \tag{7.20}$$

where

$$r' = [g_{AB}(r^A - z^A)(r^B - z^B)]^{1/2} \tag{7.21}$$

and Q' is a second constant. Under these circumstances even among the angular momentum Killing vectors the only linearly independent one whose action leaves the system invariant is the generator, $z_A \epsilon^{ABC} \vec{m}_{(B)(C)}$, of rotations about the connecting axis. Despite the fact that the obvious space symmetries have been thus cut down to a single-parameter group, there remains a *second* independent but *nontrivial* constant of the motion based on a Killing tensor of the form (7.11) with

$$\beta^A = \alpha z^A. \tag{7.22}$$

To see how this arises, we start by remarking that a Killing tensor of the form (7.11) may be considered as being uniquely specified by the *choice of the origin*, with $r^A = 0$, of the Cartesian coordinate system and by *the choice* of the *second point*, with $r^A = z^A$, modulo an arbitrary scalar multiplier α used to relate β^A to z^A via the formula (7.22). Furthermore—and this is the key point—the *same* Killing tensor a would have been obtained (with the same value of α) if we had *started* by taking our origin at the *second* point (with coordinates $r^A = z^A$ in the original coordinates system), thus introducing a new system of coordinates r'^A defined by

$$r'^A = r^A - z^A \tag{7.23}$$

and *then* had chosen as a second point the *original origin* with new coordinates given by $z'^A = -z^A$. In brief we shall have

$$a(\alpha, \vec{z}) = a'(\alpha, \vec{z}'). \tag{7.24}$$

It follows from this involuted symmetry (which, if not quite obvious, can easily be seen by direct algebraic substitution) that since a single monopole at the original origin $r^A = 0$ could be allowed for by a term c of the form (7.19), then a monopole at the new origin $r'^A = 0$ can be allowed for by a term defined analogously in terms of the displaced coordinate system. Hence by the linearity, a potential of the double monopole form (7.20) can be allowed for immediately just by replacing (7.19) by

$$c = \frac{\alpha}{2} g_{AB} z^A \left(\frac{Q r^B}{r} - \frac{Q' r'^B}{r'} \right). \tag{7.25}$$

Since the operator K thus obtained [by substituting (7.11), (7.22), and (7.25) in (7.16)] commutes with the only other independent constant of the motion, (namely the axial angular momentum operator), we do not obtain an interesting noncommutative algebra as in the previous example; but its role is even more vital because without it one would not have a "complete commuting set of good quantum numbers" for the problem. The existence of the constant constructed in this way is in fact well known (see e.g. Coulson and Walmsey[42]), but it is normally derived by separation of variables in ellipsoidal polar coordinates, in contrast with the present approach based only on the use of ordinary Cartesian coordinates. It is apparent that the more widely known Runge-Lenz constants may be thought of as a limiting case arising when the second monopole is set equal to zero. Another physically interesting limit is that where the two monopoles becomes infinitesimally close, so as to give a pole-dipole potential of the form

$$U = \frac{Q}{r} + \frac{P_A r^A}{r^3} \tag{7.26}$$

(where the constants P^A are the components of the dipole moment), which can be allowed for by setting

$$\beta_A = 0, \quad c = \frac{\alpha P_A r^A}{r} \tag{7.27}$$

in (7.11) and (7.16). It was this last example (drawn to my attention by Misner in 1966) that provided the original encouragement for seeking the closely analogous constant of the motion that turned out to occur in the Kerr and Kerr-Newman solutions.

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