Gravitational and electromagnetic wave flux compared and contrasted

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The ratio of Poynting flux to density of electromagnetic energy in the generic case tells the direction and magnitude of the boost required to reduce the flux to zero (parallel $\vec{E}$ and $\vec{B}$); but the ratio of flux to density of gravitational superenergy in the generic case gives the wrong direction and magnitude for the boost required to reduce that flux to zero (3×3 symmetric traceless tensors $\mathcal{E}$ and $\mathcal{B}$ simultaneously diagonalizable). This difference is established and illustrated by an example, and made reasonable by comparing and contrasting gravitation and electromagnetism.

Familiar electromagnetism helps one predict or elucidate effects in less familiar gravitation theory, but in one respect it is misleading. It raises false expectations about that speed with which an observer must travel to reduce the generic local field to simple canonical form. This anomaly, briefly described in a recent abstract,[1] is spelled out here in more detail for the light it may cast on the gravitational field.

Given that the generic local electromagnetic field *in vacuo* is described by the vectors $\vec{E}$ and $\vec{B}$, the observer has only to climb onto a frame moving with velocity

$$\vec{v}=\vec{n}\tanh\alpha \tag{1}$$

to reduce the electromagnetic field to a canonical form in which the transformed components $\vec{E}'$ and $\vec{B}'$ are parallel. Moreover, the necessary velocity is given by a simple formula[2] containing the ratio of Poynting flux to energy density,

$$\vec{n}\tanh 2\alpha=\frac{(\text{Poynting flux})}{(\text{energy density})}=\frac{2\vec{E}\times\vec{B}}{\vec{E}^2+\vec{B}^2}\,. \tag{2}$$

Therefore, it is not surprising that instructive recent computer studies of the gravitational radiation given out in the collision between two black holes[3] should have supposed that a formula similar to (2) would apply to gravitational radiation. At first sight the following logic seems reasonable: (1) Evaluate the flux of gravitational superenergy at the point in question. (2) Evaluate there the density of gravitational superenergy. (3) Evaluate the ratio of these two expressions, a directed quantity. (4) Let the observer travel in this direction with the appropriate velocity. (5) Then, in this frame, the flux of gravitational superenergy will be zero. (6) The vanishing of this flux, according to a well-known and long-established result of Bel[4] is a necessary and sufficient condition for the simultaneous reducibility of the "electric" and "magnetic" parts of the gravitational field to diagonal 3×3 tensors; or, in other words, it is sufficient for expressing the generic local vacuum gravitational field in Petrov's canonical form.[5]

It is enough to give one counterexample to show that this line of reasoning is wrong and why it is wrong: point (5) is mistaken. It is not in general true that the boost needed to reduce the flux of gravitational superenergy to zero points in the direction of that flux.

A few words on notation and on the statement and the significance of the problem may be appropriate before the example is laid out. We are concerned only with local values of the field, and therefore find it most convenient to express quantities always in a local Lorentz frame. We raise and lower indices with a flat-space metric tensor in the Landau-Lifshitz $-+++$ convention. We are interested, not in the part of the local curvature that is of local origin, the Einstein curvature

$$G_{\mu\nu}=8\pi T_{\mu\nu}\,, \tag{3}$$

but the part of the local curvature that is of distant origin, the conformal curvature,

$$\begin{gathered}C^{\alpha\beta}{}_{\gamma\delta}=R^{\alpha\beta}{}_{\gamma\delta}-2\delta^{[\alpha}{}_{[\gamma}R^{\beta]}{}_{\delta]}+\tfrac{1}{3}\,\delta^{[\alpha}{}_{[\gamma}\delta^{\beta]}{}_{\delta]}R\,,\\ (C^{\alpha\beta}{}_{\gamma\delta}=R^{\alpha\beta}{}_{\gamma\delta}\quad\text{in source-free space})\,,\end{gathered} \tag{4}$$

a distinction that is compared and contrasted in Table I with the corresponding distinction in electromagnetism.

When one deals with a moving charge-bearing fluid and the electromagnetic field produced by that field, one presents the local physics of the fluid most simply in one Lorentz frame, comoving with the fluid; and the local physics of the electromagnetic field most simply in quite another Lorentz frame, in which $\vec{E}$ and $\vec{B}$ have the canonical configuration of mutual parallelism. Likewise the two parts of the curvature stand out most clearly and simply when one describes them in two distinct Lorentz frames. It only makes for complication to look at both within the straightjacket of a single inertial reference system. It is trivial in

TABLE I. Electromagnetic and gravitational fields compared and contrasted with special attention to the new components that come into play in a region containing a continuous distribution of sources ($\vec{D}$ versus $\vec{E}$ and $R_{\alpha\beta\gamma\delta}$ versus $C_{\alpha\beta\gamma\delta}$, table taken from Wheeler[6]).

	Electromagnetism	Gravitation or tidal action of space-time geometry
Characterization in empty space	$\vec{E}$, $\vec{B}$ or $F_{\alpha\beta}$	$C_{\alpha\beta\gamma\delta}$
Independent components	6	10
Number of combinations of these invariant under Lorentz transformations	2	4
Characterization in region containing sources	$\vec{E},\vec{B},\vec{D},\vec{H}$	$R_{\alpha\beta\gamma\delta}$
Independent components	12	20
Normal connection between the additional components and the components that are relevant in source-free space	Multiplicative (dielectric constant, magnetic permeability, etc.)	Additive (governed exclusively by local source density)
Is any component of the field at a point fixed exclusively by the source density at that point?	No. (Example: $\vec{\nabla}\cdot\vec{E}=4\pi\rho$ gives derivative of field, not the field itself)	Yes. (The 10 $G_{\alpha\beta}$ at a point are completely determined by the 10 $T_{\alpha\beta}$ at that very point)
Are any components of the field appropriately described as of "distant origin"?	Yes. (All 12)	Yes. (The 10 $C_{\alpha\beta\gamma\delta}$, as compared and contrasted to the 10 $G_{\alpha\beta}$. It takes all 20 of these quantities—"local" plus "distant"—to reconstitute the full gravitational field $R_{\alpha\beta\gamma\delta}$)
Method to measure field	Acceleration of moving charged, test particles	Relative acceleration of two nearby moving, uncharged, test particles
Distinction between new components—in region containing sources—and the components which suffice for source-free space	$\vec{H}$: put test particle in needle-shaped slot cut out of medium parallel to field $\vec{B}$: penny-shaped slot normal to field $\vec{D}$: penny-shaped slot $\vec{E}$: needle-shaped slot	$C_{\alpha\beta\gamma\delta}$: two test particles in a small hole cut out of medium, with their separation small compared to the size of the hole $R_{\alpha\beta\gamma\delta}$: the two test particles are in two separate holes small in comparison to the separation between them

the generic case to find the frame in which the Einstein curvature—or the stress-energy tensor—is diagonal, with four numbers running down the diagonal [4 = 10 (the number of independent components in $T_{\mu\nu}$) −6 (the number of parameters in the general Lorentz transformation)]. But how does it stand with the conformal curvature, the part of the gravitational field of distant origin?

As the electromagnetic field allows for simple presentation in the frame in which the two parts of that field,

$$\vec{E}=(E_x,E_y,E_z)=(F^{01},F^{02},F^{03})$$
and
$$\vec{B}=(B_x,B_y,B_z)=(F^{23},F^{31},F^{12}), \tag{5}$$

are parallel, so the conformal curvature, Matte[7] notes, lets itself be split into "electric" and "magnetic" parts, each a symmetric zero-trace tensor,

$$C^{\alpha\beta}{}_{\gamma\delta} = \begin{array}{c|c} & \gamma\delta \longrightarrow \\ \alpha\beta\downarrow & \begin{array}{cccccc} 01 & 02 & 03 & 23 & 31 & 12 \end{array} \\ \begin{array}{c} 01\\02\\03\\23\\31\\12 \end{array} & \begin{bmatrix} \mathcal{E}_{xx} & \mathcal{E}_{xy} & \mathcal{E}_{xz} & -\mathcal{B}_{xx} & -\mathcal{B}_{xy} & -\mathcal{B}_{xz} \\ \mathcal{E}_{yx} & \mathcal{E}_{yy} & \mathcal{E}_{yz} & -\mathcal{B}_{yx} & -\mathcal{B}_{yy} & -\mathcal{B}_{yz} \\ \mathcal{E}_{zx} & \mathcal{E}_{zy} & \mathcal{E}_{zz} & -\mathcal{B}_{zx} & -\mathcal{B}_{zy} & -\mathcal{B}_{zz} \\ \mathcal{B}_{xx} & \mathcal{B}_{xy} & \mathcal{B}_{xz} & \mathcal{E}_{xx} & \mathcal{E}_{xy} & \mathcal{E}_{xz} \\ \mathcal{B}_{yx} & \mathcal{B}_{yy} & \mathcal{B}_{yz} & \mathcal{E}_{yx} & \mathcal{E}_{yy} & \mathcal{E}_{yz} \\ \mathcal{B}_{zx} & \mathcal{B}_{zy} & \mathcal{B}_{zz} & \mathcal{E}_{zx} & \mathcal{E}_{zy} & \mathcal{E}_{zz} \end{bmatrix} \end{array} \tag{6}$$

simple in the generic case in that frame in which $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are simultaneously diagonal. As for electromagnetism, the energy density and Poynting flux are given in terms of the squares and products of vectors,

$$T^{00} = (\vec{E}^2 + \vec{B}^2)/8\pi \tag{7}$$

and

$$T^{0i} = (\vec{E}\times\vec{B})^i/4\pi = \epsilon_{ijk}E_jB_k/4\pi\ , \tag{8}$$

so for gravitation the Debever-Bel-Robinson density and flux of superenergy are given in terms of the squares and products of 3×3 matrices,

$$T^{0000} = \mathrm{Tr}(\mathcal{E}^2 + \mathcal{B}^2)\ , \tag{9}$$

$$T^{000i} = 2\epsilon_{ijk}(\mathcal{E}\mathcal{B})_{jk}\ . \tag{10}$$

(See Zakharov[8] for references and Sejnowski[9] for a start towards a physical interpretation of superenergy.) As the vanishing of T^{0i} is the necessary and sufficient criterion for being in a frame in which—by a suitable rotation in three-space—$\vec{E}$ and $\vec{B}$ are both reduced to parallelism with the x axis (1 component each), so the vanishing of T^{000i} is the necessary and sufficient condition (Bel[4]) for being in a frame in which—by a suitable rotation in three-space—$\mathcal{E}$ and $\mathcal{B}$ are simultaneously diagonalized. More specifically, the conformal part of the gravitation field has to begin with 10 independent components, out of which, however, three were taken away by the boost and three were taken away by the rotation, leaving four, distributed two in $\mathcal{E}$ and two in $\mathcal{B}$. Thus in each 3×3 matrix there are, after diagonalization, only three numbers down the diagonal, linked by the one condition of zero sum (tracelessness). It is not surprising that in the original frame the six independent components of $\mathcal{E}_{ij}$ let themselves be represented in terms of second derivatives,

$$\mathcal{E}_{ij} = \partial^2\Phi/\partial x^i\,\partial x^j\ , \tag{11}$$

of a scalar function that satisfies Laplace's equation; but it is beautiful that the same kind of representation is valid in every Lorentz frame, despite the mixing of $\mathcal{B}$ and $\mathcal{E}$ in the general Lorentz transformation.

As in electromagnetism, a boost of the observer (not the source) in the z direction with the velocity $v = \tanh\alpha = \sinh\alpha/\cosh\alpha \equiv s/c$ changes his perception of electric and magnetic fields from the laboratory values $\vec{E}, \vec{B}$ to the "rocket frame" values

$$\begin{aligned} E_x' &= cE_x - sB_y\ ,\\ E_y' &= cE_y + sB_x\ ,\\ E_z' &= E_z\ ,\\ B_x' &= cB_x + sE_y\ ,\\ B_y' &= cB_y - sE_x\ ,\\ B_z' &= B_z\ , \end{aligned} \tag{12}$$

so it changes his perception of the "electric" and "magnetic" parts of the gravitation field to the rocket values

$$\begin{aligned} \mathcal{E}_{11}' &= c^2\mathcal{E}_{11} - s^2\mathcal{E}_{22} - 2sc\mathcal{B}_{12}\ ,\\ \mathcal{E}_{12}' &= (c^2+s^2)\mathcal{E}_{12} + sc\mathcal{B}_{11} - sc\mathcal{B}_{22}\ ,\\ \mathcal{E}_{22}' &= -s^2\mathcal{E}_{11} + c^2\mathcal{E}_{22} + 2sc\mathcal{B}_{12}\ ,\\ \mathcal{E}_{13}' &= c\mathcal{E}_{13} - s\mathcal{B}_{23}\ ,\\ \mathcal{E}_{23}' &= c\mathcal{E}_{23} + s\mathcal{B}_{13}\ ,\\ \mathcal{E}_{33}' &= \mathcal{E}_{33}\ ,\\ \mathcal{B}_{11}' &= c^2\mathcal{B}_{11} - s^2\mathcal{B}_{22} + 2sc\mathcal{E}_{12}\ ,\\ \mathcal{B}_{12}' &= (c^2+s^2)\mathcal{B}_{12} - sc\mathcal{E}_{11} + sc\mathcal{E}_{22}\ ,\\ \mathcal{B}_{22}' &= -s^2\mathcal{B}_{11} + c^2\mathcal{B}_{22} - 2sc\mathcal{E}_{12}\ ,\\ \mathcal{B}_{13}' &= c\mathcal{B}_{13} + s\mathcal{E}_{23}\ ,\\ \mathcal{B}_{23}' &= c\mathcal{B}_{23} - s\mathcal{E}_{13}\ ,\\ \mathcal{B}_{33}' &= \mathcal{B}_{33}\ . \end{aligned} \tag{13}$$

To see why gravitation "goes wrong," ask why electromagnetism "goes right." Let $\vec{E}$ and $\vec{B}$ be what they will, provided only that they are generic, in the sense that the field is not "null," i.e., *excluding* the case where simultaneously

$$\vec{E}^2 - \vec{B}^2 = 0 \ \text{ and } \vec{E}\cdot\vec{B} = 0\ . \tag{14}$$

If $\vec{E}$ and $\vec{B}$ are already parallel in the laboratory frame, no task remains. If they are not parallel, they define a Poynting flux. Let the z axis be oriented in the direction of this flux. Then $\vec{E}$ and $\vec{B}$ lie in the (x, y) plane, and have no z components. As sensed in the rocket frame, the fields have the values (12)—still with no components in the z direction—and therefore the fields generate a Poynting flux that points exclusively in the z direction,

$$\begin{aligned} (T^{03})' &= (E_x'B_y' - E_y'B_x')/4\pi\\ &= (c^2+s^2)(E_xB_y - E_yB_x)/4\pi\\ &\quad - 2sc(E_x{}^2 + E_y{}^2 + B_x{}^2 + B_y{}^2)/8\pi\ . \end{aligned} \tag{15}$$

Recognizing that $c^2+s^2 = \cosh 2\alpha$ and $2sc = \sinh 2\alpha$, one only has to equate expression (15) to zero to arrive at formula (2) for the boost that makes $\vec{E}$ and $\vec{B}$ parallel.

What is going on behind the scenes shows more clearly when one translates the transformation of the Poynting vector from the language of field components to the following language of stress-energy components:

$$(T^{03})' = -csT^{00} + ccT^{03} + ssT^{30} - scT^{33}\ . \tag{16}$$

We have only to recognize that T^{33} is identical with T^{00} to recover Eq. (2). However, this is exactly where we encounter accidents that do not happen and that cannot be expected to happen in the case of the superenergy of gravitation, with its more complicated tensorial structure. Thus the identity of

$$8\pi T^{33} = E_x{}^2 + E_y{}^2 - E_z{}^2 + B_x{}^2 + B_y{}^2 - B_z{}^2$$

and (17)

$$8\pi T^{00} = E_x{}^2 + E_y{}^2 + E_z{}^2 + B_x{}^2 + B_y{}^2 + B_z{}^2$$

is essential for deriving the boost formula (2). And even more is required. It is not enough that the components T^{01} and T^{02} of the Poynting flux should vanish in the laboratory frame. Also, in the rocket frame the corresponding components,

$$\begin{aligned}(T^{01})' &= cT^{01} - sT^{31},\\ (T^{02})' &= cT^{02} - sT^{32},\end{aligned} \tag{18}$$

must vanish. The first terms on the right are already zero, but what about the other terms on the right? The new components of the Maxwell stress-energy tensor that appear here are

$$\begin{aligned}T^{31} &= -(E_z E_x + B_z B_x)/4\pi,\\ T^{32} &= -(E_z E_y + B_z B_y)/4\pi.\end{aligned} \tag{19}$$

It would be impossible to drop these terms, just as it would be impossible to equate T^{33} in (17) to T^{00}, were it not that both E_z and B_z are zero. The ultimate "miracle" is then this: that from the Poynting flux pointing in the z direction in the laboratory, the vanishing of E_z and B_z unambiguously follows. In other words, the vanishing of two components, T^{01} and T^{02}, of the Poynting flux, rather than discharging diffuse buckshot at all six components of the electromagnetic field, pinpoints its fire so as to eliminate two.

In gravitation the conditions $T^{0001} = 0$ and $T^{0002} = 0$, even if it were right to impose them—and it is not—would in the generic case find their fire diluted by the larger number of targets, the 10 components of $C^{\alpha\beta}{}_{\gamma\delta}$ versus the 6 components of $F^{\alpha\beta}$. Therefore it is not surprising that no component of $\mathcal{E}$ or $\mathcal{B}$ would thereby be eliminated—much less any component of $(T^{000i})'$. Complexity would remain complexity.

At this point an important distinction must be recognized between electromagnetism and gravitation. Many alternative boosts reduce the generic Maxwell field to canonical form; but only one boost allows the generic conformal curvature to be put in canonical form. With many choices open for the boost in electromagnetism (see Appendix), we do not wonder that a boost can be found that is given by the rather simple analytical prescription (2). With only one boost that "works" in the generic case for the part of the gravitational field of distant origin, we are not surprised that the order of the equations to be solved to find that boost put them beyond the power of all but numerical methods. To expect the direct opposite, a simple answer, would be natural from the following unjustified and incorrect line of extrapolation:

(1) Given a four-vector field, A^μ, find the boost,

$$\vec{v} = \vec{n}\tanh\alpha, \tag{20}$$

that reduces it to the canonical form of a pure timelike vector. Correct answer:

$$n^i \tanh\alpha = A^i/A^0. \tag{21}$$

(2) Given the same kind of information for the generic Maxwell field, $F^{\mu\nu}$. Correct answer:

$$n^i \tanh 2\alpha = T^{0i}/T^{00}. \tag{22}$$

(3) Given the same kind of information for the generic conformal curvature, $C^{\mu\nu}{}_{\rho\sigma}$. Wrong answer:

$$n^i \tanh 4\alpha = T^{000i}/T^{0000}. \tag{23}$$

Now, for our example, we have

$$\mathcal{E} = \begin{bmatrix} -15.91 & +4.39 & -0.45 \\ +4.39 & +16.28 & +0.02 \\ -0.45 & +0.02 & -0.38 \end{bmatrix}, \tag{24}$$

$$\mathcal{B} = \begin{bmatrix} -4.90 & -16.26 & 0.29 \\ -16.26 & 3.53 & 0.94 \\ 0.29 & 0.94 & 1.37 \end{bmatrix}, \tag{25}$$

$$T^{0000} = \mathrm{Tr}\,\mathcal{E}^2 + \mathrm{Tr}\,\mathcal{B}^2 = 569.3 + 557.3 = 1126.6, \tag{26}$$

$$T^{000i} = 2\epsilon_{ijk}(\mathcal{E}\mathcal{B})_{jk} = (19.2, 5.8, 1120.2). \tag{27}$$

That prescription (23) for the boost is wrong can be confirmed either numerically or by use of the analytical transformation formula,

$$\begin{aligned}\mathcal{B}'_{ab} = {}& c^2\mathcal{B}_{ab} - c(c-1)n_a n_i \mathcal{B}_{ib} - c(c-1)\mathcal{B}_{aj}n_j n_b\\ &+ (c-1)^2 n_a n_i \mathcal{B}_{ij} n_j n_b + s^2 \epsilon_{ari} n_r \mathcal{B}_{ij} n_s \epsilon_{jsb}\\ &- s(c-1)n_a n_i \mathcal{E}_{ij}\epsilon_{bjs}n_s + sc\,\mathcal{E}_{aj}\epsilon_{bjs}n_s\\ &- s(c-1)\epsilon_{air}n_r \mathcal{E}_{ij}n_j n_b + sc\,\epsilon_{air}n_r\mathcal{E}_{ib},\end{aligned} \tag{28}$$

and a corresponding formula for $\mathcal{E}'_{ab}$, where on the right everywhere that a $\mathcal{B}_{pq}$ appears we insert an $\mathcal{E}_{pq}$, and everywhere that an $\mathcal{E}_{pq}$ appears we insert a $-\mathcal{B}_{pq}$. The right boost does not point in the direction T^{000i}, but in the z direction. The magnitude of the right boost is not given by the formula

$$\begin{aligned}\tanh 4\alpha &= [(T^{0001})^2 + (T^{0002})^2 + (T^{0003})^3]^{1/2}/T^{0000}\\ &= 0.994\,478,\\ 4\alpha &= 0.9449,\\ \tanh\alpha &= 0.627, \quad \sinh\alpha = 0.805,\\ \cosh\alpha &= 1.284,\end{aligned} \tag{29}$$

but by

$$\tanh\alpha = 0.8000, \quad \sinh\alpha = \tfrac{4}{3}, \quad \cosh\alpha = \tfrac{5}{3}, \tag{30}$$

these simplicities having been prearranged. The correct boost in the correct direction transforms the part of the gravitation field that is of distant origin to the form

$$\mathcal{E}' = \begin{bmatrix} -0.856 & +1.251 & -2.013 \\ +1.251 & +1.231 & +0.424 \\ -2.013 & +0.424 & -0.375 \end{bmatrix}, \qquad \mathcal{B}' = \begin{bmatrix} -0.392 & -2.552 & 0.514 \\ -2.552 & -0.982 & 2.176 \\ 0.514 & 2.176 & 1.374 \end{bmatrix}, \tag{31}$$

with

$$\mathrm{Tr}(\mathcal{E}')^2 = 14.00, \quad \mathrm{Tr}(\mathcal{B}')^2 = 26.00, \qquad (T^{000i}) = 2\epsilon_{ijk}(\mathcal{E}'\mathcal{B}')_{jk} = 0.00\,; \tag{32}$$

the latter is the sign that $\mathcal{E}$ and $\mathcal{B}$ are now simultaneously diagonalizable. Finally, a space rotation described by familiar Euler angles with the values $\psi = 30°$, $\theta = 45°$, $\phi = 30°$ gives

$$\mathcal{E}'' = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathcal{B}'' = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \tag{33}$$

This example obviously was set up by working backwards from the answer to the problem. Had the problem been known without the answer, how would one have proceeded? Perhaps one would have proceeded analytically, taking the three components of $n^i \sinh\alpha$ to be the unknowns, and writing down the three simultaneous equations,

$$(T^{000i})' = 0\,, \tag{34}$$

of the eighth order in the three unknowns. An inspection of these equations is enough to convince one that that approach is hopeless. An alternative, and the only alternative that this investigator was able to imagine, is numerical: (1) try a set of values for the three quantities $n^i \sinh\alpha$; (2) calculate the resultant three flux components $(T^{000i})'$; (3) square the flux components and add them together; (4) take the resultant number as criterion of merit for the choice of the original $n^i \sinh\alpha$; and (5) by standard trial and error procedures keep improving this choice until the squared flux is "reduced to zero" within any preassigned positive bound.

All of the considerations presented here refer to the generic conformal curvature, otherwise known as Petrov class N or Penrose class [1111]. The curvature outside two colliding black holes or outside a collapsing object of irregular shape has this character everywhere except at a set of points of measure zero. As one approaches closer and closer to an exceptional point, the calculated speed for the boost required to reduce the curvature to canonical form will approach closer and closer to the speed of light. This is no surprise, and occasions no alarm. One has only to bring a point charge a little distance from the north pole of a long thin bar magnet to have for illustration a field that is utterly tame. However, there is a whole set of points where the conditions (14) for a so-called "pure radiation field" are fulfilled. These points lie on a circle centered on the axis that connects the charge with the pole. At each point the electric and magnetic fields are perpendicular in direction and equal in magnitude. No one would say that there is anything at any one of these points that is moving with the speed of light. There is no reason to think of anything physically exceptional taking place at points where the conformal curvature, the part of the gravitational field of nonlocal origin, is "nongeneric" in the sense discussed here. It may be an inconvenience in the mathematical description of the field at such points that no finite boost enables one simultaneously to diagonalize the "electric" and "magnetic" parts of the conformal curvature. However, it is nothing more than an inconvenience. The true distinction between a gravitational field that is "radiative" and one that is not is not a local distinction. It is marked by global signs such as the integrated outward flux across a closed surface.

Nothing said here is meant in the least to detract from the beautiful and even central part played in the analysis of the conformal curvature tensor by determination of its characteristic null directions.[10,11] However, for the physics of the generic conformal curvature these null directions have as little direct significance in gravitation as in electromagnetism. In both cases the observer has to climb onto a Lorentz frame moving with a finite velocity to see the field reduced to a simple canonical form.

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here for determining the "catch up boost" and (b) the geometry of the four null vectors. (Work in preparation for publication by Professor Debever.)

APPENDIX

The many boosts that reduce the electromagnetic field to parallelism: Given a generic $\vec{E}$, $\vec{B}$, employ formula (2) to determine a boost of velocity parameter α that reduces $\vec{E}$ and $\vec{B}$ to parallelism. Then make a further boost of velocity parameter μ, either positive or negative, and of any magnitude in the common direction (call it x) of $\vec{E}$ and $\vec{B}$. This leaves the fields parallel. But one could have gone from the original frame to the final frame in a single Lorentz transformation, describable as the product of a rotation R and boost B. Therefore, there is an infinity of boosts that reduce original fields to parallelism. Among all these, the original boost, given by formula (2), is the most economical; for the original boost the velocity parameter is least. To see this point, make an obvious choice of axes and write

$$(\cosh\tfrac{1}{2}\mu + \sigma_x \sinh\tfrac{1}{2}\mu)(\cosh\tfrac{1}{2}\alpha + \sigma_z \sinh\tfrac{1}{2}\alpha) = BR = [\cosh\tfrac{1}{2}\beta + (\vec{n}_\beta\cdot\vec{\sigma})\sinh\tfrac{1}{2}\beta][\cos\tfrac{1}{2}\theta - i(\vec{n}_\theta\cdot\vec{\sigma})\sin\tfrac{1}{2}\theta]\,. \qquad (35)$$

Comparing the real and imaginary parts of the coefficients on both sides of this formula and doing a little simplification, one arrives at an equation for determining the boost parameter β of that single boost which, along with the rotation, produces the same effect as the two combined boosts:

$$\sinh^2(\tfrac{1}{2}\beta) = \sinh^2(\tfrac{1}{2}\mu)\cosh^2(\tfrac{1}{2}\alpha) + \cosh^2(\tfrac{1}{2}\mu)\sinh^2(\tfrac{1}{2}\alpha)\,. \qquad (36)$$

It is clear from this result that β is least when μ is zero.

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