PHYSICAL REVIEW D VOLUME 16, NUMBER 12 15 DECEMBER 1977

Many-body equilibrium of dual charged sources in general relativity

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(Received 23 June 1977)

Starting with the basic equations of Israel and Wilson, and Perjes, we give the explicit axially symmetric metric for n dual charged sources in equilibrium under their mutual electromagnetic and gravitational forces. We give the conditions for the removal of all connecting strut singularities and for an asymptotically flat space-time. In the case of two sources, we show that the strut-free asymptotically flat solutions having given values of electric and magnetic charge cannot have a separation of the sources which exceeds a certain maximum value.

I. INTRODUCTION

In 1972, Israel and Wilson,[1] and independently Perjes,[2] extended the static many-body solutions of the Einstein-Maxwell equations found by Majumdar[3] and Papapetrou[4] to the stationary case. This generalization allows one to discuss not only sources with spin but also sources with nontrivial dual (electric and magnetic) charge. Particular examples of the class of metrics found by Israel, Wilson, and Perjes (IWP) have been investigated in the literature.[5-11]

This paper extends the explicit axially symmetric metric of Ref. 9 to the case of n sources. We give the conditions necessary for the absence of strut singularities between the sources. In these solutions, the electric charge is equal to the mass, and the magnetic charge is equal to the Newman-Unti-Tamburino (NUT) parameter (in the case of zero duality angle). It was shown in Ref. 5 that the only IWP solutions with discrete sources which represent black holes, rather than naked singularities, are the static Majumdar-Papapetrou solutions (Refs. 3, 4). The Majumdar-Papapetrou solutions correspond to configurations of static sources having electric charge equal to the mass, held in equilibrium by the balance of electric and gravitational forces, while the IWP solutions in addition may have magnetic charge as well as spin. In the case of two sources, we show that the absence of struts in general imposes a limitation on the maximum separation. The IWP solution for n sources has also been considered by Spanos.[12]

II. THE n-BODY METRIC

The stationary line element is written in the standard form[13]

$$(ds)^2 = -f^{-1}\gamma_{mn}dx^m dx^n + f(\omega_m dx^m + dt)^2 , \tag{1}$$

where $\gamma_{mn}dx^m dx^n$ corresponds to Euclidean 3-space. Given any solution U of Laplace's equation in flat 3-space, IWP have shown that a valid metric is obtained by setting

$$f = |U|^{-2} \tag{2}$$

and

$$\epsilon^{npq}\partial_p\omega_q = -i\gamma^{1/2}f^{-1}\gamma^{mn}\partial_m[\ln(U/U^*)] , \tag{3}$$

where U satisfies the flat-space Laplace equation. An n-body solution corresponding to dually charged sources is generated by

$$U = 1 + \sum_{j=1}^{n} (m_j/R_j) \tag{4}$$

with

$$R_j{}^2 = x^2 + y^2 + (z - l_j\epsilon_j)^2 , \tag{5}$$

where

$$l_j = b_j + i\epsilon_j a_j , \quad m_j = M_j + iN_j . \tag{6}$$

As in Ref. 9, a_j represents an angular momentum per unit mass for the jth body, M_j the real part, and N_j the imaginary part of the mass parameter. The total mass of the system is $\sum M_j$. For the special case of zero duality angle (see Appendix), M_j can evidently be identified with the electric charge of the jth body and N_j with its magnetic charge (at least in the absence of singular struts). The approximate position of the jth source is given by $\epsilon_j b_j$. The symbol ϵ_j is a sign indicator; its value is +1 if the origin is below the jth body and −1 if above. Thus, b_j is always positive.

In spherical background coordinates,

$$x = r\sin\theta\cos\phi , \quad y = r\sin\theta\sin\phi , \quad z = r\cos\theta , \tag{7}$$

the expression for R_j becomes

$$R_j{}^2 = r^2 - 2l_j\epsilon_j r\cos\theta + l_j{}^2 . \tag{8}$$

The flat background metric is

$$\gamma_{11}=1,\quad \gamma_{22}=r^2,$$
$$\gamma_{33}=r^2\sin^2\theta,\quad \gamma\equiv \mathrm{Det}(\gamma_{mn})=r^4\sin^2\theta\,. \tag{9}$$

Because of axial symmetry one can set ω_r and ω_θ equal to zero as in Ref. 9; Eq. (3) reduces to

$$\partial_r\omega_\phi=2r\sin^2\theta\,\mathrm{Im}\left[\left(1+\sum_{j=1}^{n} m_j^*/R_j^*\right)\times\left(\sum_{i=1}^{n} l_i\epsilon_i m_i/R_i^{\,3}\right)\right], \tag{10}$$

$$\partial_\theta\omega_\phi=-2r^2\sin\theta\,\mathrm{Im}\left[\left(1+\sum_{j=1}^{n}\frac{m_j^*}{R_j^*}\right)\sum_{i=1}^{n}\left(\frac{m_i}{R_i^{\,3}}\right)\times(r-\epsilon_i l_i\cos\theta)\right]. \tag{11}$$

The integration of Eq. (10) yields

$$\omega_\phi=-\mathrm{Im}\left[\sum_{i=1}^{n}2m_i(\epsilon_i l_i-r\cos\theta)\left(\frac{1}{R_i}\right)-\sum_{i=1,j=1}^{n}\frac{\epsilon_i\epsilon_j m_i m_j^*}{(\epsilon_j l_i-\epsilon_i l_j^*)}\frac{(r^2-l_i^2)}{(r^2-\epsilon_i\epsilon_j l_i l_j^*)}\frac{R_j}{R_i}\right]+C(r)\,. \tag{12}$$

For the special case of two bodies, the above expression for ω_ϕ reduces to the metric of Ref. 6.

III. CONDITIONS FOR STRUT-FREE SOLUTION

The differential equations (10) and (11) require that ω_ϕ be constant for $\theta=0,\pi$. One can show that the limit of the first two terms in Eq. (12) is independent of r for $\theta\to 0,\pi$. Asymptotic flatness requires that ω_ϕ vanish for large r. Setting $\omega_\phi=0$ for r large and $\theta=0$, one obtains from Eq. (12)

$$C=-\mathrm{Im}\sum_{i=1}^{n}(2m_i)-\mathrm{Im}\sum_{i,j=1}^{n}\frac{m_i m_j^*\epsilon_i\epsilon_j}{\epsilon_j l_i-l_j^*\epsilon_i}\,. \tag{12a}$$

Expanding the above expression for ω_ϕ in the asymptotic region, one obtains

$$\omega_\phi\simeq\mathrm{Im}\left\{2\sum_{i=1}^{n}\left[m_i(\cos\theta-1)+\frac{2m_i\epsilon_i l_i}{r}\sin^2\theta\right]\right\}. \tag{13}$$

Thus, for the space to be asymptotically flat, we must further require that

$$\mathrm{Im}\sum_{i=1}^{n}(m_i)\equiv\sum_{i=1}^{n}(N_i)=0\,. \tag{14}$$

In addition to the above asymptotic considerations, let us expand ω_ϕ for small r in a neighborhood of the origin, which can be situated anywhere along the symmetry axis. One obtains

$$\omega_\phi=-\mathrm{Im}\sum_{i,j=1}^{n}\left[2m_i\epsilon_i\delta_{ij}-(1-\epsilon_i\epsilon_j)\frac{m_i m_j^*}{\epsilon_j l_i-\epsilon_i l_j^*}\right]+Ar^2\sin^2\theta+O(r^4)\,,$$

where A is a constant. Thus, ω_ϕ will not generally vanish at $r=0$ and strutlike singularities will connect the sources. These struts will not appear if the following condition is imposed:

$$\mathrm{Im}\sum_{i,j=1}^{n}\left[2m_i\epsilon_i\delta_{ij}-(1-\epsilon_i\epsilon_j)\frac{m_i m_j^*}{\epsilon_j l_i-\epsilon_i l_j^*}\right]=0\,. \tag{15}$$

This yields a separate condition for each choice of the ϵ_j or the origin. Note that Eq. (14) is included in Eq. (15) when the origin is above or below all the sources. One can show that closed timelike world lines circle the symmetry axis where ω_ϕ does not vanish. The above regularity conditions, Eq. (15), are needed to avoid this violation of causality.

As an example, consider the special case of three-body equilibrium. The singular struts will be removed one at a time by positioning the origin between pairs of sources. Positioning the origin between sources 1 and 2, one finds from Eq. (15) that

$$\mathrm{Im}\left[(m_1-m_2-m_3)+2\frac{m_1m_2^*}{l_1+l_2^*}+2\frac{m_1m_3^*}{l_1+l_3^*}\right]=0\,. \tag{16}$$

When the origin is positioned between sources 2 and 3, one obtains

$$\mathrm{Im}\left[(m_1+m_2-m_3)+2\frac{m_1m_3^*}{l_1+l_3^*}+2\frac{m_2m_3^*}{l_2+l_3^*}\right]=0\,. \tag{17}$$

If the origin is displaced from between the sources, all of the ϵ_j's will be of the same sign and Eq. (15) reduces to the condition for asymptotic flatness,

$$\mathrm{Im}(m_1+m_2+m_3)=0\,. \tag{18}$$

If one sets $a_j=0$, one finds that the determinant of the coefficients of the N_j is positive definite, which indicates that only the trivial solution $N_j=0$ is possible. Of course, when the a_j do not vanish there are solutions with nonvanishing N_j.

The previous expression for ω_ϕ, given in Eqs. (12) and (12a), can be expanded for large r [where Eq. (14) has been imposed],

$$\omega_\phi\simeq\frac{2\sin^2\theta}{r}\sum_{j=1}^{n}\epsilon_j(b_jN_j+\epsilon_ja_jM_j)\,. \tag{19}$$

Thus

$$\omega_\phi=2J\frac{\sin^2\theta}{r}\,, \tag{20}$$

where

$$J=\sum_{j=1}^{n}(b_j\epsilon_jN_j+a_jM_j) \tag{21}$$

can be identified with the angular momentum of the system. This expression is independent of origin; for example, it can be written as

$$J=\sum_{j=1}^{n-1}\sum_{l=j}^{n-1}A_{l,l+1}N_j+\sum_{j=1}^{n}a_jM_j\,, \tag{21a}$$

where $A_{j+1,j}=|\epsilon_{j+1}b_{j+1}-\epsilon_j b_j|$ is the coordinate separation.

The leading term in the asymptotic form of ω_ϕ in Eq. (13) is

$$\omega_\phi=2\sum_{i=1}^{n}N_i(1-\cos\theta)\,. \tag{22}$$

This is the same as the asymptotic form of the NUT solution[14] to the free-space Einstein equations with $\sum N_i$ acting as NUT parameter. It has been suggested that the NUT parameter corresponds to a dual mass formally analogous to the charge of a magnetic monopole.[15,16] In the present solution of the Einstein-Maxwell equations the correspondence is more than formal, with magnetic charge equaling dual mass when $\alpha=0$.

IV. PARTICLE SEPARATION LIMITATIONS

We will consider the special case of two bodies ($n=2$), and allow the separation parameter, b_1+b_2, to change through an infinite sequence of equilibrium positions while maintaining charge (electric and magnetic) conservation. For $n=2$, the regularity condition Eq. (15) becomes

$$\mathrm{Im}\left(m_1+\frac{m_1m_2^*}{l_1+l_2^*}\right)=0\,. \tag{23}$$

When N does not vanish, one obtains the following expression for $a\equiv a_1+a_2$:

$$a=\frac{M_1M_2-N^2}{2N}\pm\frac{1}{2N}\{(M_1M_2-N^2)^2 - 4N[N(M_1+M_2)b+Nb^2]\}^{1/2}\,. \tag{24}$$

For a to be real, $b=b_1+b_2$ must be such that

$$0\le b\le b_{\max}=-\tfrac{1}{2}(M_1+M_2) +\tfrac{1}{2}[(M_1+M_2)^2+N^{-2}(M_1M_2-N^2)^2]^{1/2}\,, \tag{25}$$

where the + sign on the square root was chosen because b is positive by definition. To interpret the limitation on b, consider the case when $b=b_{\max}$. Then Eq. (24) gives

$$a=(M_1M_2-N^2)/2N\,. \tag{26}$$

One can write

$$b_{\max}=-\overline{M}+(\overline{M}^2+a^2)^{1/2}\,, \tag{27}$$

where $\overline{M}=\frac{1}{2}(M_1+M_2)$ is the average mass (charge). The quantity $a=a_1+a_2$ is the average coordinate diameter of the two ring singularities associated with the sources. For $M^2\ll a^2$, one has $b_{\max}\simeq|a|$, while for $M^2\gg a^2$ one has $b_{\max}\simeq\frac{1}{2}(a/\overline{M})^2\overline{M}$. Thus, with N and M_j fixed, one cannot have a situation of strut-free equilibrium in which the separation of the sources is large with respect to both the dimensions of the sources and of the region of strong gravitational field.

This limitation on b appears to be related to the fact that the total angular momentum of the two sources,

$$J=M_1a_1+M_2a_2+Nb\,, \tag{28}$$

is not of the form that one would expect for two dual charged particles. For two such spinless particles, one has in flat space-time that

$$J=-(q_{e1}q_{m2}-q_{e2}q_{m1})\,, \tag{29}$$

where q_{ej}, q_{mj} are the electric and magnetic charge, respectively, of the jth particle ($j=1,2$). The expression Eq. (29) continues to hold even in curved axisymmetric space-time when the particles are point test particles,[17,18] and might be expected to hold in a suitable limit when the two sources are widely separated with respect to their geometrical and gravitational radii. In the present context, Eq. (29) would give $J=N(M_1+M_2)$, which differs from the last term of Eq. (28). When the separation is large, a strut appears between the sources, and this may be partly responsible for the difference in the two expressions.

APPENDIX

A duality rotation of the electromagnetic field is obtained by

$$U(r,\theta)\to U(r,\theta)e^{i\alpha}\,. \tag{A1}$$

where α is independent of r and θ, and the m_i appearing in U are unchanged. The electric and magnetic scalar potentials are given, for zero duality angle, by

$$A_4=\mathrm{Re}(1-U^{-1})\,, \tag{A2}$$

$$\phi=\mathrm{Im}(1-U^{-1})\,. \tag{A3}$$

If Eq. (30) is applied to the above potentials the corresponding expressions for the fields undergo a duality rotation. From Eqs. (2) and (3) one can see that the metric is invariant under this transformation.

When $\alpha=0$, examination of the asymptotic electromagnetic field shows that M_i can evidently be identified with the electric charge and N_i with the magnetic charge of the ith source (at least in the absence of struts). For nonzero α, the corresponding electric and magnetic charges are

$$M_i' = M_i \cos\alpha + N_i \sin\alpha, \tag{A4}$$

$$N_i' = N_i \cos\alpha - M_i \sin\alpha. \tag{A5}$$

The condition Eq. (14) for asymptotic flatness, $\sum N_i = \mathrm{Im}(\sum m_i) = 0$, remains unchanged; only the interpretation of N_i in terms of electric and magnetic charge changes.

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*Work supported by National Science Foundation under Grant No. PHY77-07111.

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