Grand partition function of hadronic bremsstrahlung*

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The grand partition function of hadronic bremsstrahlung is obtained using saddle-point procedures. Several levels of approximation are considered. The results are qualitatively consistent with earlier simple approximations.

I. INTRODUCTION

In the last few years production of hadrons at high energies and with large transverse momentum (p_T) has been a subject of extensive theoretical and experimental study. Models based on hadronic bremsstrahlung[1-3] have been found to provide a useful description of the main experimental facts concerning single-particle inclusive distributions and associated multiplicities[4,5] as well as two-particle correlations for hadrons produced with large p_T.[6]

The principal results of these models in connection with high-p_T processes are the following:

(a) Hadronic bremsstrahlung provides an important dynamical mechanism leading to single-particle inclusive distributions decreasing like inverse powers of p_T (Refs. 1, 4, and 5) in accord with well-established experimental facts.

(b) In association with a large-p_T trigger, the same mechanism leads to hadron multiplicities increasing with the trigger's p_T (Refs. 4 and 5), again in accord with experiments.

((c) With regard to hadron correlations, together with a large-p_T trigger, bremsstrahlung predicts a strong jet of hadrons on the opposite side to the trigger as well as another jet (somewhat weaker) on the same side.[1,6] This feature is one of the most interesting recent discoveries of experimental hadron physics at large p_T.[7]

On the other hand, it has been known for quite some time that hadronic bremsstrahlung accounts well for the basic features of elastic scattering of hadrons at large p_T,[8,9] for the asymptotic behavior of electromagnetic form factors[8] and for some of the features of deep-inelastic electron-proton scattering.[10,11]

There are two basic processes occurring in scattering described by hadronic bremsstrahlung. An elastic scattering of off-mass-shell hadrons (protons for example) is accompanied by the emission of neutral vector mesons from the external lines. The rapid drop of the elastic pp differential cross section with increasing p_T (Refs. 8 and 9) is partially compensated by the larger phase space available owing to increased vector-meson (V) emission.

The basic structure of these models is described in Refs. 2 and 3. There the amplitude for the process $pp \to pp+nV$ is taken to be

$$M_n(p_1,p_2,p_3,p_4)=g^n M_0(p_1,p_2,p_3,p_4) \times \prod_{i=1}^{n} V_\mu(k_i)\epsilon^\mu(k_i,\lambda_i) , \tag{1.1}$$

where ϵ^μ is the vector-meson polarization vector, M_0 is the elastic scattering amplitude, and

$$V_\mu(k)=\sum_{j=1}^{4}\frac{\eta_j p_{j\mu}}{p_j\cdot k} , \quad \eta_j=\begin{cases}1 & j=1,2\\ -1 & j=3,4\end{cases} . \tag{1.2}$$

The differential cross section for $pp\to pp+nV$, averaged over initial and summed over final polarizations, is

$$d\sigma_n=\frac{|M_0|^2}{4\pi^2}\frac{m^4}{E_1E_2v_{12}}\,\delta^4\Big(p_1+p_2-p_3-p_4-\sum_i k_i\Big)\prod_{i=1}^{n}\left[-\frac{g^2}{(2\pi)^3}V^2(k_i)\frac{d^3k_i}{2\omega_i}\right]\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4} . \tag{1.3}$$

If none of the vector mesons is observed then we perform the k integrations and insert a $1/n!$ symmetrization factor. Using the integral representation for the δ function we have

$$d\sigma_n=\frac{|M_0|^2}{4\pi^2}\frac{m^4}{E_1E_2v_{12}}\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}\int\frac{d^4x}{(2\pi)^4}\frac{1}{n!}K^n(x)\exp[ix(p_1+p_2-p_3-p_4)] , \tag{1.4}$$

where

$$K(x) = -\frac{g^2}{(2\pi)^3}\int \frac{d^3k}{2\omega} e^{-ikx} V^2(k) \ . \tag{1.5}$$

We now sum (1.4) over all n to obtain

$$d\sigma = \frac{|M_0|^2}{4\pi^2}\frac{m^4}{E_1E_2v_{12}}\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}\Delta(p_1+p_2-p_3-p_4) \ , \tag{1.6}$$

where the "grand partition function" is

$$\Delta(P) = \int \frac{d^4x}{(2\pi)^4}\exp[iPx + K(x)] \ . \tag{1.7}$$

Following the same procedure, we find that $d\sigma$ for the emission of n observed vector mesons and an arbitrary number of unobserved vector mesons is obtained from (1.6) by including a factor $[-g^2/(2\pi)^3] \times (d^3k/2\omega)V^2(k)$ for each observed V and adding $-k$ to the argument of Δ for each observed V.

In addition to cross sections, bremsstrahlung models give definite predictions for associated V multiplicities. With (1.4) we find

$$\sum_n n\, d\sigma_n = \frac{|m_0|^2}{4\pi^2}\frac{m^4}{E_1E_2v_{12}}\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}\int\frac{d^4x}{(2\pi)^4}K(x)\exp[ix(p_1+p_2-p_3-p_4)+K(x)] \ . \tag{1.8}$$

Since $K(x)$ is linear in g^2 we can obtain the integral in (1.8) by differentiating $\Delta(P)$ with respect to g^2 and multiplying by g^2. The multiplicity of neutral vector mesons associated with finding the final protons in the momentum-space elements d^3p_3 and d^3p_4 is

$$\langle n\rangle = \frac{\sum_n n\, d\sigma_n}{\sum_n d\sigma_n} = g^2\frac{d}{dg^2}\ln[\Delta(p_1+p_2-p_3-p_4)] \ . \tag{1.9}$$

If there is a strong correlation between the final-state protons then (1.9) will remain valid for the single-proton distribution. Notice that the associated multiplicity is not dependent upon M_0.

In the usual treatment the grand partition function is estimated by expanding the function $K(x)$ in (1.7) around $x=0$. If only the first term in this expansion is kept then (1.7) becomes

$$\Delta(P) = e^{K(0)}\delta^4(P) \ . \tag{1.10}$$

This approximation clearly does not conserve energy or momentum. The latter is conserved, however, if one makes the assumption that the unobserved vector mesons carry off, on the average, no momentum. Energy conservation can be restored by postulating a probability distribution for energy loss to unobserved vector mesons. In the case with no observed vector mesons, as in (1.6), the choice made is[2-6]

$$\Delta(p_1+p_2-p_3-p_4) = \delta^3(\vec{p}_3+\vec{p}_4)\int_0^1 d\eta\,\delta(E_3-\eta E_1)P(\eta)e^{K(0)} \tag{1.11}$$

in the c.m. frame.

An immediate difficulty arises because $K(x)$ diverges at $x=0$. This is avoided by the use of a cutoff. We shall remove this problem by expanding $K(x)$ around the saddle point of the x_0 integral rather than $x_0=0$. This approach also avoids the introduction of the unknown function $P(\eta)$. In Sec. II we develop a first approximation to the grand partition function in which momentum conservation is imposed as discussed above. In Sec. III we obtain a more accurate form which displays both energy and momentum distributions.

Another difficulty with the usual approach concerns the momenta of the protons participating in the elastic scattering subprocess described by M_0. For near elastic scattering, in which little energy is lost to vector mesons, it is reasonable to let the internal proton momenta be the same as the external momenta. If a significant amount of energy is lost, however, then this approximation is certainly not accurate. This problem is discussed in Sec. IV.

There are two appendixes included. In Appendix A we present a detailed calculation of the function $K(x)$ which is valid for small x and $p_i \cdot p_j \gg m^2$. In Appendix B we discuss the properties of functions which are frequently used in this paper.

II. FIRST APPROXIMATION TO $\Delta(P)$

In this section we shall examine our simplest and most readily calculated approximation to $\Delta(P)$. We shall assume that the unobserved vector mesons are constrained so that their three-momenta satisfy

$$\sum_i \vec{k}_i = \vec{\kappa} \ , \tag{2.1}$$

where $\vec{\kappa}$ is a function of the momenta p_i and the momenta of the observed vector mesons. For example, if we observe no vector mesons and if we work in the c.m. system then it is reasonable to take $\vec{\kappa}=0$.

With the constraint (1.1) we find

$$\Delta(P)=\delta^3(\vec{P}-\vec{\kappa})\tilde{\Delta}(P^0)\ , \tag{2.2}$$

where $P=(P^0, \vec{P})$ and

$$\tilde{\Delta}(P^0)=\int \frac{dy}{2\pi}\exp[iP^0y+K(y)]\ . \tag{2.3}$$

$K(y)$ in (2.3) is now an abbreviation for $K(y,0,0,0)$.

The function $K(y)$ in approximated in Appendix A. Substituting (A15) and (A35) into (A2) we find

$$K(-iw/\mu)\approx -\frac{g^2}{4\pi^2}\sum_i\left[\mathrm{Ci}^2\left(w\frac{E_i}{m}\right)+\mathrm{si}^2\left(w\frac{E_i}{m}\right)+g\left(w\frac{E_i}{m}\right)-\frac{\pi^2}{4}\right] + \frac{g^2}{4\pi^2}(\gamma+\ln w)\sum_{i\neq j}\eta_i\eta_j\ln[2(1-z_{ij})]-\frac{g^2}{16\pi^2}\sum_{i\neq j}\eta_i\eta_j\{\ln^2[2(1-z_{ij})]-2G(z_{ij})\}\ . \tag{2.4}$$

z_{ij} is the cosine of the angle between $\vec{p}_i$ and $\vec{p}_j$, $G(z)$ is defined in (A34) and approximated in (A36), $g(x)$ is a well-known auxiliary function defined by

$$g(x)=-\mathrm{Ci}(x)\cos x-\mathrm{si}(x)\sin x\ , \tag{2.5}$$

and Ci and si are defined in Appendix B. The approximation is good for w small, $E_i\gg m$, and $z_{ij}\neq 1$.

We may simplify (2.4) somewhat if we assume that the final-state protons are emitted back to back in the c.m. system:

$$\vec{p}_1+\vec{p}_2=0,\quad \hat{p}_3=-\hat{p}_4\ . \tag{2.6}$$

This is certainly the case if $\vec{\kappa}=0$ and also if $\vec{\kappa}$ is in the direction of $\vec{p}_3$ or $\vec{p}_4$. In this case

$$z_{12}=z_{34}=-1\ ,\quad z_{13}=z_{24}=\cos\theta\equiv z\ ,\quad z_{14}=z_{23}=-\cos\theta\equiv -z\ , \tag{2.7}$$

where θ is the angle between $\vec{p}_1$ and $\vec{p}_3$. With these substitutions we easily find

$$\sum_{i\neq j}\eta_i\eta_j\ln[2(1-z_{ij})]=-8\ln\sin\theta \tag{2.8}$$

and

$$\sum_{i\neq j}\eta_i\eta_j\{\ln^2[2(1-z_{ij})]-2G(z_{ij})\}=4\{\ln^2 4-\ln^2[2(1-z)]-\ln^2[2(1+z)]-2G(-1)+2G(z)+2G(-z)\}\ . \tag{2.9}$$

with (A36) we see that $G(-1)=\pi^2/3$ and

$$G(z)+G(-z)=\frac{\pi^2}{2}+\ln\left(\frac{1-z}{2}\right)\ln\left(\frac{1+z}{2}\right)\ . \tag{2.10}$$

Upon substitution into (2.9) we have

$$\sum_{i\neq j}\eta_i\eta_j\{\ln^2[2(1-z_{ij})]-2G(z_{ij})\}=16\left[\frac{\pi^2}{12}-\ln^2\sin\theta+\ln\left(\frac{1-\cos\theta}{2}\right)\ln\left(\frac{1+\cos\theta}{2}\right)\right]\ . \tag{2.11}$$

Equation (2.4) therefore becomes

$$K(-iw/\mu)=-\frac{g^2}{4\pi^2}\sum_i\left[\mathrm{Ci}^2\left(w\frac{E_i}{m}\right)+\mathrm{si}^2\left(w\frac{E_i}{m}\right)+g\left(w\frac{E_i}{m}\right)\right] + \frac{g^2}{\pi^2}\left[(\gamma+\ln w-\ln\sin\theta)^2-\ln\left(\frac{1-z}{2}\right)\ln\left(\frac{1+z}{2}\right)+\frac{\pi^2}{6}\right]\ . \tag{2.12}$$

In the region of validity of our approximation $K(0)$ is positive-infinite and $K(y)$ is real along the negative imaginary y axis. For $P^0>0$ the function iP^0y is real along the negative imaginary y axis and becomes positive-infinite as $y\to -i\infty$. The exponent in the integrand of (2.3) is therefore real along the negative imaginary y axis and has a minimum on that ray. With Cauchy's equations we can easily see that the minimum is a saddle point.

Let us suppose that the minimum is at $y=-i\xi/\mu$. Then changing variables in (2.3) to $x=y\mu+i\xi$ and expanding the exponent about $x=0$, we have

$$\tilde{\Delta}(P^0)\approx e^{(P^0/\mu)\xi+K(-i\xi/\mu)}\int_{-\infty}^{\infty}\frac{dx}{2\pi\mu}e^{-(x^2/2)(d^2/d\xi^2)K(-i\xi/\mu)}$$
$$=\frac{1}{\mu\sqrt{2\pi}}\left[\frac{d^2}{d\xi^2}K\left(-\frac{i\xi}{\mu}\right)\right]^{-1/2}e^{(P^0/\mu)\xi+K(-i\xi/\mu)}\ . \qquad (2.13)$$

The value of ξ is determined from

$$P^0/\mu+(d/d\xi)K(-i\xi/\mu)=0\ . \qquad (2.14)$$

With (2.12) this condition becomes

$$\frac{P^0}{\mu}\xi+\frac{g^2}{\pi^2}\left[1+2\gamma+2\ln\xi-2\ln\sin\theta-\sum_i B_1(x_i)\right]=0\ , \qquad (2.15)$$

where $x_i=E_i\xi/m$ and

$$B_1(x)=\tfrac{1}{4}[\mathrm{Ci}(x)(x\sin x+2\cos x)-\mathrm{si}(x)(x\cos x-2\sin x)]$$
$$=\tfrac{1}{4}[xf(x)-2g(x)]\ . \qquad (2.16)$$

The auxiliary function $g(x)$ is defined in (2.5) while[12]

$$f(x)=\mathrm{Ci}(x)\sin x-\mathrm{si}(x)\cos x\ . \qquad (2.17)$$

The first form for B_1 is useful for small x while the second is useful for large x.

If we write (2.13) in the form

$$\tilde{\Delta}(P^0)=\frac{1}{\mu\sqrt{2\pi}}F_3^{-1/2}e^{F_2}\ , \qquad (2.18)$$

then we find

$$F_2=\frac{P^0\xi}{\mu}+\frac{g^2}{\pi^2}\left[(\gamma+\ln\xi-\ln\sin\theta)^2+\frac{\pi^2}{6}-\ln\left(\frac{1-\cos\theta}{2}\right)\ln\left(\frac{1+\cos\theta}{2}\right)-\sum_i B_2(x_i)\right]\ , \qquad (2.19)$$

where

$$B_2(x)=\tfrac{1}{4}[\mathrm{Ci}^2(x)+\mathrm{si}^2(x)-\mathrm{Ci}(x)\cos x-\mathrm{si}(x)\sin x]$$
$$=\tfrac{1}{4}[g^2(x)+f^2(x)+g(x)] \qquad (2.20)$$

and

$$F_3=\frac{g^2}{\pi^2\xi^2}\left[\sum_i B_3(x_i)-(1+2\gamma+2\ln\xi-2\ln\sin\theta)\right], \qquad (2.21)$$

where

$$B_3(x)=\tfrac{1}{4}\{\mathrm{Ci}(x)[2x\sin x+(2-x^2)\cos x]-\mathrm{si}(x)[2x\cos x-(2-x^2)\sin x]\}$$
$$=\tfrac{1}{4}[2xf(x)+(x^2-2)g(x)]\ . \qquad (2.22)$$

The procedure for evaluation of $\tilde{\Delta}$ is therefore a two-step one. First, we solve (2.15) to determine ξ. Second, we use this result in (2.19) and (2.21) to determine F_2 and F_3 and then, with (2.18), $\tilde{\Delta}$.

The procedure outlined above is rather tedious. Unfortunately, it seems to be the only way to get a good approximation to $\tilde{\Delta}$ in most regions. There are, however, two cases in which simpler forms are available.

Consider first the case in which $P^0\pi^2/\mu g^2\equiv R$ is small while the energies of the external protons remain large. In this case ξ is small but the x_i are not and we can use the asymptotic expansions for f and g to simplify the expressions. Equation (2.15) becomes

$$R\xi+2(\gamma+\ln\xi-\ln\sin\theta)\approx 0\ , \qquad (2.23)$$

while (2.19) and (2.21) are

$$F_2\approx\frac{g^2}{\pi^2}\left[R\xi+\tfrac{1}{4}(R\xi)^2+\frac{\pi^2}{6}-\ln\left(\frac{1-\cos\theta}{2}\right)\ln\left(\frac{1+\cos\theta}{2}\right)\right] \qquad (2.24)$$

and

$$F_3\approx\frac{g^2}{\pi^2\xi^2}(R\xi+2)\ . \qquad (2.25)$$

Although the procedure for determining F_2 and F_3 is still a two-step one, the functions to be calculated are much simpler. Notice that for g^2/π^2 and θ fixed $\tilde{\Delta}(P^0)$ in this approximation is a function only of P^0, the energy available for the production of neutral vector mesons.

Next, consider the case in which R is large. This will occur, for example, if the coupling is small. In this case ξ will be very small so that the x_i will be small as well. We can therefore use the small-argument expansions for Ci, si, cos, and sin. Equation (2.15) becomes

$$\xi=\frac{2\sum_i\ln(E_i/m)+8\ln\sin\theta-4}{4R+(\pi/2)\sum_i E_i/m}\ , \qquad (2.26)$$

while (2.19) and (2.21) give

$$F_2 \approx -\frac{g^2}{4\pi^2}\left\{\sum_i \left[\ln^2\left(\frac{E_i}{m}\right) - \ln\left(\frac{E_i}{m}\right)\right] + \frac{\pi^2}{3} + 4\ln\left(\frac{1-\cos\theta}{2}\right)\ln\left(\frac{1+\cos\theta}{2}\right) - 4\ln^2\sin\theta + (\gamma + \ln\xi - 1)\left[2\sum_i \ln\left(\frac{E_i}{m}\right) + 8\ln\sin\theta - 4\right]\right\} \tag{2.27}$$

and

$$F_3 \approx \frac{g^2}{4\pi^2}\,\frac{1}{\xi}\left(4R + \frac{\pi}{2}\sum_i \frac{E_i}{m}\right) . \tag{2.28}$$

In this case we have a one-step procedure since (2.26) gives a value for ξ directly. Upon substitution we obtain

$$\tilde{\Delta}(P^0) \approx \frac{e^{-B-\gamma A}\{(g^2/4\pi^2)[4R + (\pi/2)\sum_i E_i/m]\}^{A-1}}{\mu[(2\pi/A)^{1/2}e^A A^{-A}]} , \tag{2.29}$$

where

$$A = \frac{g^2}{4\pi^2}\left[2\sum_i \ln\left(\frac{E_i}{m}\right) + 8\ln\sin\theta - 4\right] \tag{2.30}$$

and

$$B = \frac{g^2}{4\pi^2}\left\{\sum_i \left[\ln^2\left(\frac{E_i}{m}\right) - \ln\left(\frac{E_i}{m}\right)\right] + \frac{\pi^2}{3} + 4\ln\left(\frac{1+\cos\theta}{2}\right)\ln\left(\frac{1-\cos\theta}{2}\right) - 4\ln^2\sin\theta\right\} . \tag{2.31}$$

We have written (2.29) in that particular form because the bracket in the denominator is just the first term in the asymptotic expansion for $\Gamma(A)$. If we substitute the small-y expansion for $K(y)$ directly into (2.3) then we do, in fact, obtain

$$\tilde{\Delta}(P^0) \approx \frac{e^{-B-\gamma A}}{\mu\Gamma(A)}\left(\frac{P^0}{\mu} + \frac{g^2}{8\pi^2}\sum_i \frac{E_i}{m}\right)^{A-1} . \tag{2.32}$$

In Fig. 1 we plot $\tilde{\Delta}(P^0)$ using all three methods outlined in this section. We present the result for the process $pp \to p + X$ so that $P^0 = E_1 + E_2 - E_3 - E_4$. The c.m. energy is fixed at $E_1 = E_2 = \epsilon m$ with $\epsilon = 100$, while the c.m. scattering angle is $\theta = 90°$. We take μ to be the ρ mass and $g^2/\pi^2 = 1$. The function is plotted against the variable $\eta = E_3/E_1 = E_4/E_1$. Note that η is close to $x_T = 2p_T/\sqrt{s}$.

With Fig. 1 we see that the small-R method is quite accurate for $\eta - 1$ while the large-R approximation is good for small η. Although the approximations used in obtaining $\tilde{\Delta}(P^0)$ are not valid at $\eta = 0$ or 1, the shape of Fig. 1 and similar calculations carried out at other energies indicate that

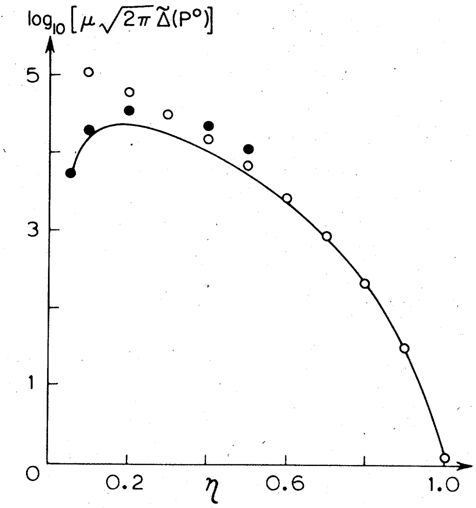

FIG. 1. $\log_{10}[\mu\sqrt{2\pi}\,\tilde{\Delta}(P^0)]$ vs η for $E_1 = E_2 = \epsilon m$, $E_3 = E_4 = \epsilon\eta m$ with $\epsilon = 100$, $\theta = 90°$, and $g^2/\pi^2 = 1$. The solid line represents the exact saddle-point method. Open circles represent the small-R approximation. Solid points represent the large-R approximation.

$\tilde{\Delta}(P^0)$ can be parametrized by

$$\tilde{\Delta}(P^0) \approx (1 - x_T)^\beta p_T^\alpha . \tag{2.33}$$

This is the form realized by the more primitive forms of the bremsstrahlung model and leads [with (1.6)] to a single-particle inclusive distribution similar to those of the parton model. As in some simpler bremsstrahlung formulations,[4] α in (2.33) increases slowly with energy.

To conclude this section we give the expression for the associated vector-meson multiplicity as determined from (1.9). Because of the constraint (2.1) the final protons are strongly correlated and the expression is valid for the single-proton distribution. Using $\tilde{\Delta}$ as determined from (2.15), (2.19), and (2.21) we obtain

$$\langle n\rangle = -\tfrac{1}{2} + F_2 - (P^0/\mu)\xi + \frac{(P^0/\mu)}{F_3\xi} + \frac{(P^0/\mu)}{F_3^{\,2}\xi^3}\,\frac{g^2}{\pi^2}\sum_i B_4(x_i) , \tag{2.34}$$

where

$$B_4(x) = \frac{x}{8}\{1 - x[\mathrm{Ci}(x)\sin x - \mathrm{si}(x)\cos x]\} = \frac{x}{8}[1 - xf(x)] . \tag{2.35}$$

III. EVALUATION OF $\Delta(P)$

In Sec. II we obtained an approximation to $\Delta(P)$ based on the assumptions that the final-state protons were strongly correlated and that they were emitted back to back in the center-of-mass frame. In this section we shall drop the first assumption and weaken the second. In particular, we shall assume that the proton lines are coplanar and that there exists a frame in which all protons are relativistic and $\hat{p}_1=-\hat{p}_2$, $\hat{p}_3=-\hat{p}_4$. This is a reasonable assumption since we expect each vector meson to be emitted roughly in the direction of motion of the emitting proton. These assumptions can be written

$$\frac{\vec{p}_1}{E_1}=-\frac{\vec{p}_2}{E_2}=\hat{a}\ ,\qquad \frac{\vec{p}_3}{E_3}=-\frac{\vec{p}_4}{E_4}=\hat{b}\ , \tag{3.1}$$

where these relations hold in the "collinear frame."

The procedure for evaluating (1.7) is basically the same as in Sec. II. We change variables to $y=\mu x+i\xi$ in that equation to obtain

$$\begin{aligned}\Delta(P)&=\frac{1}{(2\pi\mu)^4}\int d^4y\exp\left[i(y-i\xi)\frac{P}{\mu}+K((y-i\xi)/\mu)\right]\\ &\approx\frac{e^{(p/\mu)\xi+K(-i\xi/\mu)}}{(2\pi\mu)^4}\int d^4y\exp\left\{iy_\alpha\left[\frac{P^\alpha}{\mu}+\partial^\alpha K(-i\xi/\mu)\right]-y_\alpha y_\beta\tfrac{1}{2}\partial^\alpha\partial^\beta K(-i\xi/\mu)\right\}\ ,\end{aligned} \tag{3.2}$$

where $\partial^\alpha=\partial/\partial\xi_\alpha$. K is now a function of the four-vector ξ_α and is defined in Appendix A by (A2) with (A38) and (A39).

In evaluating the derivative of K we shall use the fact that $G(z)$ is slowly varying and therefore we shall ignore its derivatives. We shall also anticipate ourselves by assuming that ξ_α is dominated by ξ_0 so that the argument of G is $z_{ij}=\cos\theta_{ij}$ and not a complicated function of ξ. Then working in the collinear frame with the scattering angle θ we find

$$\begin{aligned}K(-i\xi/\mu)=&\frac{g^i}{4\pi^2}\sum_i\left\{\left[\gamma+\ln\left(\frac{p_i\cdot\xi}{E_i}\right)-\ln\sin\theta\right]^2+\frac{\pi^2}{6}-\ln\left(\frac{1+\cos\theta}{2}\right)\ln\left(\frac{1-\cos\theta}{2}\right)-4B_2\left(\frac{p_i\cdot\xi}{m}\right)\right\}\\ &+\frac{g^2}{16\pi^2}\sum_{i\neq j}\eta_i\eta_j\ln^2\left(\frac{p_i\cdot\xi}{p_j\cdot\xi}\,\frac{E_j}{E_i}\right)\ ,\end{aligned} \tag{3.3}$$

$$\partial^\alpha K(-i\xi/\mu)=\frac{g^2}{4\pi^2}\sum_i\frac{p_i^\alpha}{p_i\cdot\xi}\left[1+2\gamma-2\ln\sin\theta+2\ln\left(\frac{p_i\cdot\xi}{E_i}\right)-\eta_i\sum_i\eta_j\ln\left(\frac{p_j\cdot\xi}{E_i}\right)-4B_1\left(\frac{p_i\cdot\xi}{m}\right)\right]\ , \tag{3.4}$$

and

$$\begin{aligned}\partial^\alpha\partial^\beta K(-i\xi/\mu)=&\frac{g^2}{4\pi^2}\sum_i\frac{p_i^\alpha p_i^\beta}{(p_i\cdot\xi)^2}\left[4B_3\left(\frac{p_i\cdot\xi}{m}\right)-2\gamma-2\ln\left(\frac{p_i\cdot\xi}{E_i}\right)+2\ln\sin\theta-1+\eta_i\sum_j\eta_j\ln\left(\frac{p_j\cdot\xi}{E_j}\right)\right]\\ &-\frac{g^2}{4\pi^2}\sum_{i,j}\eta_i\eta_j\frac{p_i^\alpha}{p_i\cdot\xi}\,\frac{p_j^\beta}{p_j\cdot\xi}\ .\end{aligned} \tag{3.5}$$

The functions B_N are defined in (2.16), (2.20), and (2.22).

The simplest choice for the expansion point is

$$\xi^\alpha=(\xi,0,0,0)\ . \tag{3.6}$$

With this choice we have

$$K(-i\xi/\mu)=\frac{g^2}{\pi^2}\left[(\gamma+\ln\xi-\ln\sin\theta)^2+\frac{\pi^2}{6}-\ln\left(\frac{1-\cos\theta}{2}\right)\ln\left(\frac{1+\cos\theta}{2}\right)-\sum_i B_2\left(\frac{E_i\xi}{m}\right)\right]\ , \tag{3.7}$$

which is the same as the result of Sec. II. We also have

$$\partial^\alpha K(-i\xi/\mu)=\frac{g^2}{4\pi^2\xi}\sum_i\frac{p_i^\alpha}{E_i}\left[1+2\gamma+2\ln\xi-2\ln\sin\theta-4B_1\left(\frac{E_i\xi}{m}\right)\right] \tag{3.8}$$

and

$$\partial^\alpha\partial^\beta K(-i\xi/\mu)=\frac{g^2}{4\pi^2\xi^2}\sum_i\frac{p_i^\alpha}{E_i}\,\frac{p_i^\beta}{E_i}\left[4B_3\left(\frac{E_i\xi}{m}\right)-2\gamma-2\ln\xi+2\ln\sin\theta-1\right]-\frac{g^2}{4\pi^2\xi^2}\sum_{i,j}\eta_i\eta_j\frac{p_i^\alpha}{E_i}\,\frac{p_j^\beta}{E_j}\ . \tag{3.9}$$

With (3.1) the spatial derivatives in (3.8) are

$$\partial^k K(-i\xi/\mu) = \frac{g^2}{\pi^2\xi}\left\{\hat{a}^k\left[B_1\left(\frac{E_2\xi}{m}\right) - B_1\left(\frac{E_1\xi}{m}\right)\right] + \hat{b}^k\left[B_1\left(\frac{E_4\xi}{m}\right) - B_1\left(\frac{E_3\xi}{m}\right)\right]\right\}, \tag{3.10}$$

while the time derivative is the same as in Sec. II:

$$\partial^0 K(-i\xi/\mu) = \frac{g^2}{\pi^2\xi}\left[1 + 2\gamma + 2\ln\xi - 2\ln\sin\theta - \sum_i B_1\left(\frac{E_i\xi}{m}\right)\right]. \tag{3.11}$$

The second derivatives are

$$\partial^0\partial^0 K(-i\xi/\mu) = \frac{g^2}{\pi^2\xi^2}\left[\sum_i B_3\left(\frac{E_i\xi}{m}\right) - (1 + 2\gamma + 2\ln\xi - 2\ln\sin\theta)\right]. \tag{3.12}$$

As in Sec. II,

$$\partial^0\partial^k K(-i\xi/\mu) = \frac{g^2}{\pi^2\xi^2}\left\{\hat{a}^k\left[B_3\left(\frac{E_1\xi}{m}\right) - B_3\left(\frac{E_2\xi}{m}\right)\right] + \hat{b}^k\left[B_3\left(\frac{E_3\xi}{m}\right) - B_3\left(\frac{E_4\xi}{m}\right)\right]\right\} \tag{3.13}$$

and

$$\partial^k\partial^l K(-i\xi/\mu) = \frac{\Lambda_a}{2}\hat{a}^k\hat{a}^l + \frac{\Lambda_b}{2}\hat{b}^k\hat{b}^l, \tag{3.14}$$

where

$$\Lambda_a = \frac{g^2}{\pi^2\xi^2}\left[2B_3\left(\frac{E_1\xi}{m}\right) + 2B_3\left(\frac{E_2\xi}{m}\right) - (1 + 2\gamma + 2\ln\xi - 2\ln\sin\theta)\right],$$
$$\Lambda_b = \frac{g^2}{\pi^2\xi^2}\left[2B_3\left(\frac{E_3\xi}{m}\right) + 2B_3\left(\frac{E_4\xi}{m}\right) - (1 + 2\gamma + 2\ln\xi - 2\ln\sin\theta)\right]. \tag{3.15}$$

The advantage of the choice (3.6) for ξ^α is now clear. If $E_1 \approx E_2$ or $E_3 \approx E_4$ then (3.13) is very small. Even if these conditions are not satisfied we still expect (3.13) to be much smaller than (3.12) and (3.14). We therefore ignore this term and write (3.2) in the separable form

$$\Delta(P) \approx \frac{\tilde{\Delta}(P^0)}{(2\pi\mu)^3}\int d^3y \exp\left\{-i\vec{y}\cdot\left[\frac{\vec{P}}{\mu} + \vec{\nabla}K(-i\xi/\mu)\right] - \tfrac{1}{2}(\vec{y}\cdot\vec{\nabla})^2 K(-i\xi/\mu)\right\}, \tag{3.16}$$

where $\tilde{\Delta}$ is the function discussed in Sec. II. We now fix ξ to be the saddle point of Sec. II.

The integral in the direction normal to $\hat{a}$ and $\hat{b}$ gives a trivial δ function and we find

$$\Delta(P) \approx \frac{\tilde{\Delta}(P^0)\delta(P_\perp)e^{-(1/2)r\cdot A^{-1}\cdot r}}{2\pi\mu^2(\det A)^{1/2}} \tag{3.17}$$

when A is the two-dimensional matrix (3.14), $P_\perp$ is the component of $\vec{P}$ normal to $\hat{a}$ and $\hat{b}$, and r is the projection of the vector $\vec{P}/\mu + \vec{\nabla}K(-i\xi/\mu)$ in the $\hat{a}-\hat{b}$ plane. We easily find

$$\det A = \tfrac{1}{4}\Lambda_a\Lambda_b\sin^2\theta \tag{3.18}$$

and

$$\tfrac{1}{2}rA^{-1}r = \frac{r_a^2}{\Lambda_a} + \frac{r_b^2}{\Lambda_b}, \tag{3.19}$$

where

$$\vec{r} = \hat{a}r_a + \hat{b}r_b. \tag{3.20}$$

We therefore have

$$\Delta(P) \approx \frac{\tilde{\Delta}(P^0)\delta(P_\perp)}{\mu^2\sin\theta}\frac{e^{-r_a^2/\Lambda_a}}{(\pi\Lambda_a)^{1/2}}\frac{e^{-r_b^2/\Lambda_b}}{(\pi\Lambda_b)^{1/2}} \tag{3.21}$$

in the collinear frame. This approximation is expected to be very good for $r_a^2 \ll \Lambda_a$, $r_b^2 \ll \Lambda_b$ since in this case we are quite close to a saddle point in the spatial integration. Notice that we can use (2.21) to write the Λ as

$$\Lambda_a = F_3 + \frac{g^2}{\pi^2\xi^2}\sum_i \eta_i B_3(x_i),$$
$$\Lambda_b = F_3 - \frac{g^2}{\pi^2\xi^2}\sum_i \eta_i B(x_i). \tag{3.22}$$

We can easily see the physical significance of the last two factors of (3.21). For both the large- and small-R approximations of Sec. II we see that if R is not too small

$$\xi \sim \frac{2\ln\epsilon}{R} = \frac{2\mu g^2}{\pi^2}\frac{\ln\epsilon}{P^0}, \tag{3.23}$$

where ϵ is some characteristic energy. Λ_a and

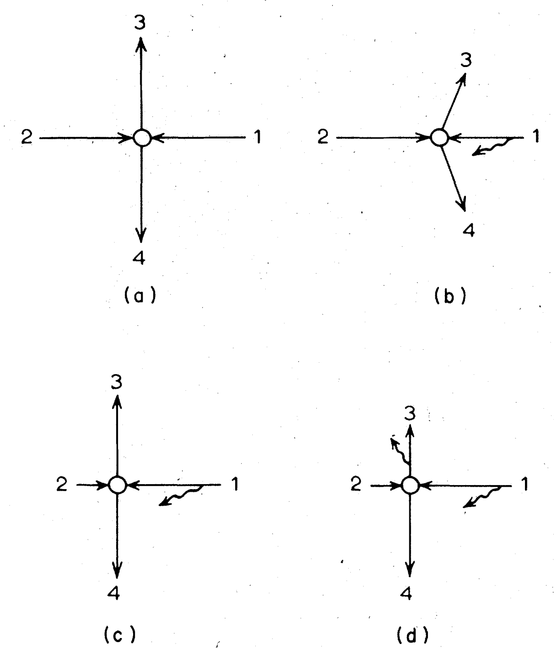

FIG. 2. (a) Elastic scattering at 90° in the c.m. frame. (b) Scattering in the c.m. frame with vector meson emitted off line 1. (c) Transformation to collinear system. This restores 90° scattering but gives $|\vec{p}_1|>|\vec{p}_2|$. (d) Emission of vector meson off line 3 in $\hat{p}_3$ direction preserves angles but gives $|\vec{p}_3|<|\vec{p}_4|$.

Λ_b grow slightly less rapidly than $(P^0)^2$, the square of the energy available for vector-meson production, and fall inversely with g^2. Consider elastic pp scattering at 90° in the c.m. frame. In this case there is no energy available for vector-meson production. As the process becomes inelastic, releasing energy for vector-meson production, a vector meson may be produced off p_1 for example as in Fig. 2. This will be produced near the direction $\hat{p}_1$. In order to conserve momentum, p_3 and p_4 are no longer emitted at 90° but have components in the $\hat{p}_2$ direction. In the collinear system p_3 and p_4 will again be emitted at 90° but $\vec{p}_1+\vec{p}_2$, and hence r_a, will be nonzero. The width of the r_a distribution, which is proportional to $\sqrt{\Lambda_a}$, will therefore grow with energy. The increase is slightly less than linear owing to multiple emission of vector mesons. As g^2 increases the probability of multiple emission increases and the width of the distribution tends to decrease.

If vector mesons are emitted from p_3 and p_4 then a similar broadening of the r_b distribution will result. In that case the scattering angle remains unchanged, so that p_3 and p_4 will still be collinear, but $\vec{p}_3+\vec{p}_4$ will become nonzero.

It is not difficult to see that if the vector mesons are produced exactly in the direction of the emitting proton then the collinear system is the c.m. system for the elastic scattering process described by M_0. The collinear scattering angle θ is just the c.m. scattering angle for that subprocess. In the next section we shall obtain an estimate for the momenta of the protons participating in the subprocess.

The expression (3.21) for the grand partition function is useful if we wish to describe two-proton distributions. In most cases, however, we shall want to integrate over one or both of the final proton momenta. Our expression is not well suited to such applications owing to its complexity. Fortunately, we can make some simplifications which can aid in these integrations.

Suppose first that any observed vector mesons are found at large angles. In this case they are predominantly associated with the outgoing proton lines, and we can safely write

$$r_a \approx \frac{1}{\mu}(E_1-E_2) - \frac{g^2}{\pi^2\xi}\left[B_1\left(\frac{\xi E_1}{m}\right) - B_1\left(\frac{\xi E_2}{m}\right)\right], \quad (3.24)$$

so that the second factor of (3.21) peaks at $E_1=E_2$. With the estimate (3.23) we find

$$\frac{E_i\xi}{m} \approx \frac{2\mu}{m}\frac{g^2}{\pi^2}\frac{E_i}{P^0}\ln\epsilon . \quad (3.25)$$

Unless the collinear frame is very far from the c.m. frame of the incoming protons, E_1 and E_2 are comparable to or larger than $P^0/2$. If $g^2/\pi^2\sim 1$, a reasonable phenomenological value, then (3.25) is not small for the incoming protons $(i=1,2)$. This means that the function $B_N(E_1\xi/m)$ and $B_N(E_2\xi/m)$ which occur throughout the derivations of ξ and $\tilde{\Delta}(P^0)$ are slowly varying. If we fix the c.m. energy of the incoming protons then any changes of E_1 and E_2 must be in opposite directions since $E_1E_2 \approx E^2$. Since the B_N functions are slowly varying for the incoming variables we expect ξ, $\tilde{\Delta}(P^0)$, Λ_a, and Λ_b to be nearly constant as functions of E_1-E_2.

On both experimental and theoretical grounds we expect M_0 to be a slowly varying function of its c.m. energy but rapidly varying with respect to momentum transfer. These internal variables are discussed in the next section, but here we need only mention that the momentum transfer depends only weakly on E_1 and E_2. We therefore expect a weak dependence on E_1-E_2 for M_0.

For fixed c.m. energy we therefore expect most of the E_1-E_2 dependence of (1.6) to be concentrated in the second factor of (3.21). This factor is a Gaussian of unit area in the variable r_a. Since we intend to integrate we can replace this factor by $\delta(r_a)$. Changing the variable in the δ function to $(E_1-E_2)\sin\theta$, the component of $\vec{P}$ in the $\hat{a}\hat{b}$ plane normal to $\hat{b}$, we find

$$\delta(r_a)=\frac{\mu\sin\theta\delta((E_1-E_2)\sin\theta)}{1-(g^2/4\pi^2x_1)[2-x_1^2g(x_1)-x_1f(x_1)]}\ , \quad (3.26)$$

where $x_1=E_1\xi/m$. With the estimate (3.25) we can see that the second term in the denominator is much smaller than the first. We therefore neglect it and write

$$\Delta(P)\approx\tilde{\Delta}(P^0)\delta^2(P_\perp)\frac{e^{-r_b{}^2/\Lambda_b}}{\mu(\pi\Lambda_b)^{1/2}}\ , \quad (3.27)$$

where $P_\perp$ is normal to $\hat{b}$. Remember that this expression is suitable only for integration.

At this point we have almost returned to the formalism of Sec. II. If the Gaussian peak in (3.27) could be replaced by a δ function then we could use (2.2) for $\Delta(P)$ and identify $\vec{\kappa}$ from $r_b=0$. However, there are two things which stand in the way. First, in some cases of interest E_3 and E_4 are not comparable to P^0 so that the B_N functions are not necessarily slowly varying. Second, M_0 does vary rapidly as a function of momentum transfer and therefore as a function of E_3 and E_4. In short, we cannot expect to isolate the r_b dependence as a Gaussian with unit area as we did with the r_a dependence.

IV. INTERNAL MOMENTA

So far we have been concerned with the function Δ appearing in (1.6). We shall now turn to the factor $|M_0|^2$ of that equation. As we mentioned in Sec. I, the simple form of the bremsstrahlung model takes the internal momenta, the momenta of the protons in the scattering described by M_0, to be the same as the external momenta. This is seen in (1.1). Suppose we change our definition of V from (1.2) to

$$\vec{V}_\mu(k)=\sum_{j=1}^{4}\frac{\eta_j p_{j\mu}}{p_j\cdot k}\,e^{-\eta_j k\cdot\vec{\partial}_j}\ , \quad (4.1)$$

where $\vec{\partial}_j^\mu=\partial/\partial p_{j\mu}$, acting to the right. The exponential clearly has the property that

$$e^{-\eta_j k\cdot\vec{\partial}_j}F(p_j)=F(p_j-\eta_j k)e^{-\eta_j k\cdot\vec{\partial}_j}\ . \quad (4.2)$$

The correct expression for M_0 is

$$M_n(p_1,p_2,p_3,p_4)=g^n\Big[\prod_{i=1}^{n}\epsilon^\mu(k_i,\lambda)\vec{V}_\mu(k_i)\Big]_s \times M_0(p_1,p_2,p_3,p_4)\ . \quad (4.3)$$

Since $\vec{V}$ depends upon the proton momenta we have symmetrized to take into account all orderings of vector mesons.

This new expression is very difficult to deal with. We can simplify things if we ignore changes in the p_i appearing in V. In other words, we allow $\vec{\partial}$ to act only on M_0. Then the only change in the formalism appearing in Appendix A is the replacement

$$K_{ij}(-iw^\alpha/\mu)\rightarrow K_{ij}\left(-\frac{i}{\mu}(w^\alpha+\mu\eta_i\vec{\partial}_i^\alpha+\mu\eta_j\overleftarrow{\partial}_j^\alpha)\right)\ , \quad (4.4)$$

where $\vec{\partial}$ acts on M_0 and $\overleftarrow{\partial}$ on $M_0^\dagger$. Expanding (4.4) in a Taylor series we have

$$K_{ij}(-iw/\mu)\rightarrow K_{ij}(-iw/\mu)+\mu(\eta_i\vec{\partial}_i^\alpha+\eta_j\overleftarrow{\partial}_j^\alpha)\partial_\alpha K_{ij}(-iw/\mu)+\cdots\ . \quad (4.5)$$

In carrying out the saddle-point procedure for approximating $\Delta(P)$ we will find, evaluating the second term of (4.5) at the saddle point,

$$\Delta(P)\rightarrow\Delta(P)\exp\sum_{i,j}[\partial_\alpha K_{ij}(-i\xi/\mu)\mu(\eta_i\vec{\partial}_i^\alpha+\eta_j\overleftarrow{\partial}_j^\alpha)]\ . \quad (4.6)$$

With (4.2) this means that the internal momenta p_i are, approximately,

$$\tilde{p}_i^\alpha\approx p_i^\alpha-\eta_i k_i^\alpha+\mu\sum_j\eta_j\partial^\alpha K_{ij}(-i\xi/\mu)\ , \quad (4.7)$$

where k_i^α is the total four-momentum of observed vector mesons emitted from the ith line.

Suppose that we demand momentum conservation for the elastic scattering described by M_0. This condition is

$$\sum_i\eta_i\tilde{p}_i=0\ . \quad (4.8)$$

Substituting (4.7) we find

$$P^\alpha+\mu\partial^\alpha K(-i\xi/\mu)=0\ . \quad (4.9)$$

The time component of this is just the condition we used to determine ξ. The spatial components on the left-hand side are just $\mu\vec{r}$, where the vector $\vec{r}$ is defined in Sec. III. In that section, however, we did not demand that $\vec{r}$ vanish. The condition (4.9) is an approximation obtained by neglecting, among other things, second derivations of K. Had we done the same in Sec. III then the last two factors of (3.21) would have been $\delta(r_a)\delta(r_b)$, which is consistent with (4.9). This suggests that a more accurate estimate of $\tilde{p}_i$ is

$$\tilde{p}_i^\alpha=p_i^\alpha-\eta_i k_i^\alpha-\mu\eta_i\rho_i^\alpha+\mu\sum_j\eta_j\partial^\alpha K_{ij}(-i\xi/\mu)\ , \quad (4.10)$$

where $\rho_i^0=0$ and

$$\vec{\rho}_1=\vec{\rho}_2=\tfrac{1}{2}r_a\hat{a}\ ,\qquad \vec{\rho}_3=\vec{\rho}_4=\tfrac{1}{2}r_b\hat{b}\ . \quad (4.11)$$

We are working in the collinear system with $\hat{a}$ and $\hat{b}$ defined by (3.1). The choice (4.10) conserves energy and momentum in the elastic scattering subprocess.

If we neglect derivatives of $G(z)$ as we did in Sec. III and make the choice (3.6) for ξ then we find

$$\tilde{p}_i^\alpha \approx p_i^\alpha - \eta_i k_i^\alpha - \mu\eta_i \rho_i^\alpha + \frac{\mu g^2}{4\pi^2\xi}\frac{\eta_i p_i^\alpha}{E_i}\left[1+2\gamma+2\ln\xi - 2\ln\sin\theta - 4B_1\left(\frac{E_i\xi}{m}\right)\right]$$
$$-\frac{\mu g^2}{4\pi^2\xi}\sum_j \frac{\eta_j p_i^\alpha}{E_i}\left[\gamma+\ln\xi+g\left(\frac{E_i\xi}{m}\right)\right]+\frac{\mu g^2}{8\pi^2\xi}\sum_{j\neq i}\frac{\eta_j p_i^\alpha}{E_j}\ln[2(1-z_{ij})] \;, \qquad (4.12)$$

where θ is the collinear scattering angle and the z_{ij} are given by (2.7).

V. DISCUSSION

In Sec. II we have developed a simple approximation to the grand partition function $\Delta(P)$. In later sections this approximation is refined so that formulas on several levels of accuracy are available. Naturally, the more elementary forms of Sec. II are more easily applied. As discussed in Secs. III and IV, however, the elementary forms are expected to be only roughly accurate. Nevertheless, it is encouraging that these forms yield results in accord with both the parton model and the simpler bremsstrahlung formalisms.

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APPENDIX A

In this appendix we evaluate the function

$$K(x) = -\frac{g^2}{(2\pi)^3}\int \frac{d^3k}{2\omega} e^{-ikx} V^2(k) \;. \qquad (A1)$$

With (1.2) we have

$$K(x) = \sum_{i,j=1}^{4} \eta_i\eta_j K_{ij}(x) \;, \qquad (A2)$$

where

$$K_{ij}(x) = -\frac{g^2}{(2\pi)^3}\int\frac{d^3k}{2\omega} e^{-ikx}\frac{p_i\cdot p_j}{(p_i\cdot k)(p_j\cdot k)} \;. \qquad (A3)$$

We may rewrite this as

$$K_{ij}(x) = -\frac{2g^2}{(2\pi)^3}\int_{-1}^{1} d\rho \int\frac{d^3k}{2\omega} e^{-ikx}\frac{p_i\cdot p_j}{[K\cdot P_{ij}(\rho)]^2} \;, \qquad (A4)$$

where

$$P_{ij}(\rho) = p_i(1-\rho) + p_j(1+\rho)$$
$$\equiv (E_{ij}(\rho), \vec{Q}_{ij}(\rho)) \;. \qquad (A5)$$

Let us now restrict ourselves to the case

$$x = (z/\mu, 0, 0, 0) \;. \qquad (A6)$$

We do this for two reasons. First, in Sec. II we are concerned only with this case. Second, K_{ij} is a Lorentz-invariant quantity and is a function of the Lorentz scalars x^2, $p_i\cdot x$, $p_j\cdot x$, and $p_i\cdot p_j$. If we choose the frame in which (6) is true then the resulting K_{ij} will be a function of z^2/μ^2, $E_i z/\mu$, $E_j z/\mu$, and $p_i\cdot p_j$. If we can determine K_{ij} in this special frame then the generalization to arbitrary x should be trivial.

With x given by (A6) the angular part of the K integration in (A4) is trivial. After a change of variables to $\lambda = \omega/\mu$ we have

$$K_{ij}(z/\mu) = -\frac{g^2}{2\pi^2}\int_{-1}^{1} d\rho \frac{p_i\cdot p_j}{P_{ij}^{\;2}}\int_1^\infty \frac{d\lambda\, e^{-i\lambda z}(\lambda^2-1)^{1/2}}{\lambda^2-1+E_{ij}^{\;2}/P_{ij}^{\;2}} \;. \qquad (A7)$$

Remember that P_{ij} and E_{ij} are functions of ρ.

Our first project will be the evaluation of the λ integral in (A7). This integral has the form

$$I(A,w) = \int_1^\infty d\lambda \; \frac{e^{-w\lambda}(\lambda^2-1)^{1/2}}{\lambda^2-1+A^2} \;. \qquad (A8)$$

In evaluating $\Delta(P)$ we shall only need to know $K(x)$ for small x. We shall therefore evaluate (A8) for $|w|$ small and arbitrary A^2. Notice that for $w=0$ the integral diverges, the divergence coming from the large-λ region of integration. It is therefore useful to separate I into

$$I(A,w) = I_1(A,w) + I_2(A,w) \;, \qquad (A9)$$

where

$$I_1(A,w) = \int_0^\infty d\lambda\, \frac{\lambda e^{-w\lambda}}{\lambda^2 + A^2} \qquad (A10)$$

contains all of the divergent behavior. The integral (A10) is well known and we have

$$I_1(A,w) = g(wA)$$
$$= -\mathrm{Ci}(wA)\cos wA - \mathrm{si}(wA)\sin wA, \qquad (A11)$$

where $\mathrm{Re}A > 0$, $\mathrm{Re}w > 0$.

It is easy to check that for $\mathrm{Re}\,w > 0$, $A^2 > 0$, $I_2(A,w)$ is bounded by $I_2(A,0)$. Furthermore, the expansion of $I_2(A,w)$ about $w=0$ contains no term linear in w. Finally, $I_2(A,0)$ vanishes as $A^2 \to \infty$. We shall therefore make the approximation

$$I_2(A,w) \approx I_2(A,0) = \ln 2A - \frac{1}{2}\left(\frac{A^2}{A^2-1}\right)^{1/2} \times \ln\left[\frac{(A^2)^{1/2}+(A^2-1)^{1/2}}{(A^2)^{1/2}-(A^2-1)^{1/2}}\right]. \tag{A12}$$

Remember that this is a small-w approximation.

To summarize, the function K_{ij} is given by

$$K_{ij}(-iw/\mu) = -\frac{g^2}{2\pi^2}\int_{-1}^{1} d\rho\, \frac{p_i\cdot p_j}{P_{ij}^{\ 2}}\, I\left(\left(\frac{E_{ij}^{\ 2}}{P_{ij}^{\ 2}}\right), w\right), \tag{A13}$$

where $I(A,w)$ is well approximated by (A11) and (A12) for small w.

The diagonal functions K_{ii} are easily evaluated since

$$P_{ii}^{\ 2} = 4p_i^{\ 2} = 4m^2, \quad E_{ii}^{\ 2} = 4E_i^{\ 2} \tag{A14}$$

are independent of ρ. For $E_i \gg m^2$, $I_2(E_i/m, w)$ is small and we can write

$$K_{ii}(-iw/\mu) \approx -\frac{g^2}{4\pi^2}\, g\left(w\,\frac{E_i}{m}\right). \tag{A15}$$

The neglected terms are

$$O\left(\frac{m^2}{E_i^{\ 2}}\ln\left(\frac{E_i}{m}\right)\right).$$

In order to evaluate the nondiagonal functions K_{ij} it is convenient to write

$$K_{ij} = \tilde{K}_{ij} + \tilde{K}_{ji}, \tag{A16}$$

where

$$\tilde{K}_{ij}(-iw/\mu) = -\frac{g^2}{2\pi^2}\int_0^1 d\rho\, \frac{p_i\cdot p_j}{P_{ij}^{\ 2}}\, I\left(\left(\frac{E_{ij}^{\ 2}}{P_{ij}^{\ 2}}\right)^{1/2}, w\right). \tag{A17}$$

The leading high-energy behavior of (A17) comes from the $\rho \sim 1$ region of integration where $P_{ij}^{\ 2}$ is small compared to $p_i\cdot p_j$. Only I_1 contributes to this behavior since $(E_{ij}^{\ 2}/P_{ij}^{\ 2})^{1/2}$ is large. We can extract this contribution by replacing $P_{ij}^{\ 2}$ and $E_{ij}^{\ 2}$ by their $\rho \sim 1$ expansions to obtain

$$-\frac{g^2}{2\pi^2}\int_0^1 d\rho\, \frac{p_i\cdot p_j}{m^2+p_i\cdot p_j(1-\rho)}\, g\left(w\left(\frac{E_i^{\ 2}}{m^2+p_i\cdot p_j(1-\rho)}\right)^{1/2}\right). \tag{A18}$$

This can be integrated immediately to give

$$\frac{g^2}{8\pi^2}\left[\mathrm{Ci}^2\left(w\,\frac{E_i}{m}\right) + \mathrm{si}^2\left(w\,\frac{E_i}{m}\right) - \mathrm{Ci}^2\left(w\left(\frac{E_i^{\ 2}}{p_i\cdot p_j}\right)^{1/2}\right) - \mathrm{si}^1\left(w\left(\frac{E_i^{\ 2}}{p_i\cdot p_j}\right)^{1/2}\right)\right]. \tag{A19}$$

Since w is assumed to be small and with the assumption that $E_j^{\ 2}/p_i\cdot p_j$ is not large we may use the small-argument expansions for Ci and si to rewrite (A19) as

$$\frac{g^2}{8\pi^2}\left\{\mathrm{Ci}^2\left(w\,\frac{E_i}{m}\right) + \mathrm{si}^2\left(w\,\frac{E_i}{m}\right) - \frac{\pi^2}{4} - \left[\gamma + \ln w + \tfrac{1}{2}\ln\left(\frac{E_i}{E_i}\right) - \tfrac{1}{2}\ln\left(\frac{p_i\cdot p_j}{E_iE_j}\right)\right]^2\right\}. \tag{A20}$$

For high energies

$$\frac{p_i\cdot p_j}{E_iE_j} \approx 1 - \cos\theta_{ij} \equiv 1 - z_{ij}, \tag{A21}$$

where θ_{ij} is the angle between $\vec{p}_i$ and $\vec{p}_j$. For reasons which will become clear shortly, we rewrite (A20) as

$$\frac{g^2}{8\pi^2}\left(\mathrm{Ci}^2\left(w\,\frac{E_i}{m}\right) + \mathrm{si}^2\left(w\,\frac{E_i}{m}\right) - \frac{\pi^2}{4} - \{\gamma + \ln w - \tfrac{1}{2}\ln[2(1-z_{ij})]\}^2\right) - \frac{g^2}{8\pi^2}\left\{(\gamma+\ln w)\ln 2 + \tfrac{1}{4}\ln^2 2 - \tfrac{1}{2}\ln 2 - \tfrac{1}{2}\ln 2\ln[2(1-z_{ij})] + \tfrac{1}{4}\ln^2\left(\frac{E_i}{E_j}\right)\right\}. \tag{A22}$$

In obtaining this we have dropped terms linear in $\ln(E_i/E_j)$ since these vanish in the sum (A16).

The expression (A22) contains the leading behavior of $\tilde{K}_{ij}(-iw/\mu)$. The remaining contribution is given by (A17) and (A18). Since the $\rho \sim 1$ region of integration is expected to give only a small contribution we can use the small argument expansion for the function g and set $m^2 = 0$. The remaining contribution is approximately

$$\frac{g^2}{8\pi^2}\int_0^1 d\rho\, \frac{1}{1-\rho}\left\{\frac{2}{1+\rho}\left[\gamma + \ln w - \ln 2 + \frac{1}{2}\left(\frac{A^2}{A^2-1}\right)^{1/2}\ln\left(\frac{(A^2)^{1/2}+(A^2-1)^{1/2}}{(A^2)^{1/2}-(A^2-1)^{1/2}}\right)\right.\right. - \gamma - \ln w + \tfrac{1}{2}\ln\left[(1-\rho)(1-z_{ij})\frac{E_i}{E_j}\right]\Bigg\}, \tag{A23}$$

where $A^2 = E_{ij}{}^2/P_{ij}{}^2$ with $P_{ij}{}^2$ evaluated with $m^2=0$. The terms proportional to $\gamma + \ln w$ are easily seen to be

$$\frac{g^2}{8\pi^2}(\gamma+\ln w)\int_0^1 d\rho\,\frac{1}{\rho+1}=\frac{g^2}{8\pi^2}(\gamma+\ln w)\ln 2. \tag{A24}$$

This cancels the first term in the last curly bracket of (A22). The remaining terms of (A23) may be written

$$\frac{g^2}{8\pi^2}\int_0^1 d\rho\,\frac{1}{1-\rho}\left(\tfrac{1}{2}\ln\left[(1-\rho)(1-z_{ij})\frac{E_i}{E_j}\right]+\frac{2}{1+\rho}\left\{\left(\frac{A^2}{A^2-1}\right)^{1/2}\ln[(A^2)^{1/2}+(A^2-1)^{1/2}]-\ln 2\right\}\right). \tag{A25}$$

Let us define

$$x_{ij}=E_i/E_j=1/x_{ji}\ . \tag{A26}$$

We may write

$$A^2=(\lambda^2x_{ij}+2\lambda+x_{ji})\frac{1}{2\lambda(1-z_{ij})}\quad, \tag{A27}$$

where

$$\lambda=(1-\rho)/(1+\rho)\ . \tag{A28}$$

Making the changes of variables (A28) for the last two terms of (A25) and $1-\rho=2\lambda$ for the first term we find

$$\frac{g^2}{8\pi^2}\int_0^1\frac{d\lambda}{\lambda}\left(\tfrac{1}{2}\ln[2\lambda(1-z_{ij})x_{ij}]-\ln 2+\left(\frac{\lambda^2x_{ij}+2\lambda+x_{ji}}{\lambda^2x_{ij}+2\lambda z_{ij}+x_{ji}}\right)^{1/2}\ln\left\{\frac{(\lambda^2x_{ij}+2\lambda+x_{ji})^{1/2}+(\lambda^2x_{ij}+2\lambda z_{ij}+x_{ji})^{1/2}}{[2\lambda(1-z_{ij})]^{1/2}}\right\}\right)$$
$$-\frac{g^2}{8\pi^2}\int_{1/2}^1\frac{d\lambda}{\lambda}\,\tfrac{1}{2}\ln[2\lambda(1-z_{ij})x_{ij}]\,. \tag{A29}$$

The second integral in (A29) is easily seen to cancel the second and third terms in the last curly bracket of (A22).

The first integral in (A29) can be rewritten, with the change of variables $\beta=\lambda x_{ij}$, as

$$\frac{g^2}{8\pi^2}\int_0^{x_{ij}}\frac{d\beta}{\beta}\,[\tfrac{1}{2}\ln\beta+S(\beta+\beta^{-1})]\ , \tag{A30}$$

where

$$S(\beta+\beta^{-1})=\tfrac{1}{2}\ln[2(1-z_{ij})]-\ln 2+\left(\frac{\beta+2+\beta^{-1}}{\beta+2z_{ij}+\beta^{-1}}\right)^{1/2}\ln\left\{\frac{(\beta+2+\beta^{-1})^{1/2}+(\beta+2z_{ij}+\beta^{-1})^{1/2}}{[2(1-z_{ij})]^{1/2}}\right\}\ . \tag{A31}$$

The first term of (A30) can be written

$$\frac{g^2}{8\pi^2}\left(\tfrac{1}{2}\int_0^1\frac{d\beta}{\beta}\ln\beta+\tfrac{1}{2}\int_1^{x_{ij}}\frac{d\beta}{\beta}\ln\beta\right)\ . \tag{A32}$$

The second integral cancels the last term in the last curly bracket of (A22). Notice that all terms in that bracket have now been canceled. The second term of (A30) can be written

$$\frac{g^2}{8\pi^2}\left[\int_0^1\frac{d\beta}{\beta}S(\beta+\beta^{-1})+\tfrac{1}{2}\int_1^{x_{ij}}\frac{d\beta}{\beta}S(\beta+\beta^{-1})+\tfrac{1}{2}\int_1^{x_{ji}}\frac{d\beta}{\beta}S(\beta+\beta^{-1})\right.$$
$$\left.+\tfrac{1}{2}\int_1^{x_{ij}}\frac{d\beta}{\beta}S(\beta+\beta^{-1})-\tfrac{1}{2}\int_1^{x_{ji}}\frac{d\beta}{\beta}S(\beta+\beta^{-1})\right]\ . \tag{A33}$$

The last two terms give no contribution upon substitution into (A16). If we change variables to $\beta\to1/\beta$ in the second integral we find that it cancels the third integral. We are therefore left with the first terms of (A32) and (A33). We write these terms as $(g^2/16\pi^2)G(z_{ij})$, where

$$G(z)=\int_0^1\frac{d\beta}{\beta}\left\{\frac{\beta+1}{(\beta^2+2\beta z+1)^{1/2}}\ln\left[\frac{\beta+1+(\beta^2+2\beta z+1)^{1/2}}{\beta+1-(\beta^2+2\beta z+1)^{1/2}}\right]+\ln\left[\frac{\beta(1-z)}{2}\right]\right\}\ . \tag{A34}$$

Adding this contribution to the surviving terms of (A22) and using (A16), we obtain our final estimate:

$$K_{ij}(-iw/\mu) \approx \frac{g^2}{4\pi^2}\left(\frac{1}{2}\left[\mathrm{Ci}^2\left(w\,\frac{E_i}{m}\right)+\mathrm{Ci}^2\left(w\,\frac{E_j}{m}\right)\right]+\frac{1}{2}\left[\mathrm{si}^2\left(w\,\frac{E_i}{m}\right)+\mathrm{si}^2\left(w\,\frac{E_j}{m}\right)\right]\right.$$
$$\left.-\{\gamma+\ln w-\tfrac{1}{2}\ln[2(1-z_{ij})]\}^2+\tfrac{1}{2}G(z_{ij})-\tfrac{1}{4}\pi^2\right)\,. \qquad \text{(A35)}$$

This is expected to be valid for w small, $E \gg m$, and $z_{ij}\neq 1$. Recall that the diagonal term K_{ii} is given by (A15).

We are not yet finished with the evaluation of K_{ij} since the integral (A34) has not been evaluated. Since we were not clever enough to do this analytically we have resorted to numerical methods. In Fig. 3 we plot $G(z)$ in the allowed region $|z|\leq 1$. The numerical results strongly suggest that

$$G(z)=\frac{\pi^2}{4}+\tfrac{1}{2}\ln\left(\frac{1-z}{2}\right)\ln\left(\frac{1+z}{2}\right)-\frac{\pi^2}{12}z+R(z)\,, \qquad \text{(A36)}$$

where R is an odd function of z plotted in Fig. 4 for $0\leq z\leq 1$. $R(z)$ is close to, but not exactly equal to, $z(1-z^2)^{1/2}8=\sin 2\theta/16$. Note also that $R(0)=R(1)=0$. In practice, we shall need only the even part of $G(z)$ given by the first two terms of (A36).

As mentioned at the beginning of this Appendix, the generalization from the special case (A6) to arbitrary x is trivial. With (A21) we see that the necessary substitutions for going from $x=(-iw/\mu,0,0,0)$ to $x_\alpha=-iw_\alpha/\mu$ are

$$wE_i/m \to p_i\cdot w/m)$$
$$\frac{w^2}{2(1-z_{ij})}\to\frac{(p_i\cdot w)(p_j\cdot w)}{2p_i\cdot p_j}\,, \qquad \text{(A37)}$$
$$z_{ij}\to\frac{p_{i\alpha}p_{j\beta}}{(p_i\cdot w)(p_j\cdot w)}\,(-w^2g^{\alpha\beta}+w^\alpha w^\beta)\,.$$

With these substitutions we find

$$K_{ii}(-iw^\alpha/\mu)\approx-\frac{g^2}{4\pi^2}\,g(p_i\cdot w/m)\,, \qquad \text{(A38)}$$

$$K_{ij}(-iw^\alpha/\mu)\approx\frac{g^2}{4\pi^2}\left(\frac{1}{2}\left[\mathrm{Ci}^2\left(\frac{p_i\cdot w}{m}\right)+\mathrm{Ci}^2\left(\frac{p_j\cdot w}{m}\right)\right]+\frac{1}{2}\left[\mathrm{si}^2\left(\frac{p_i\cdot w}{m}\right)+\mathrm{si}^2\left(\frac{p_j\cdot w}{m}\right)\right]\right.$$
$$\left.+\tfrac{1}{2}G(z_{ij})-\left\{\gamma+\tfrac{1}{2}\ln\left[\frac{(p_i\cdot w)(p_j\cdot w)}{2p_i\cdot p_j}\right]\right\}^2-\frac{\pi^2}{4}\right)\,, \qquad \text{(A39)}$$

where z_{ij} is given by the substitution of (A37).

Before concluding this Appendix we should mention that if a more accurate form is used for (1.2), namely the replacement of $p_j\cdot k$ by $p_j\cdot k-\eta_j\mu^2$ in that expression, then the only change in $K(x)$ at high energies is the addition of the term

$$\delta K(x)=\frac{g^2}{\pi^2}\,[(\tan^{-1}\beta)^2-\beta\tan^{-1}\beta]\,, \qquad \text{(A40)}$$

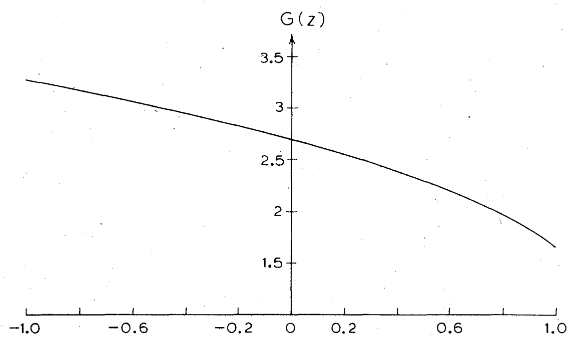

FIG. 3. The function $G(z)$.

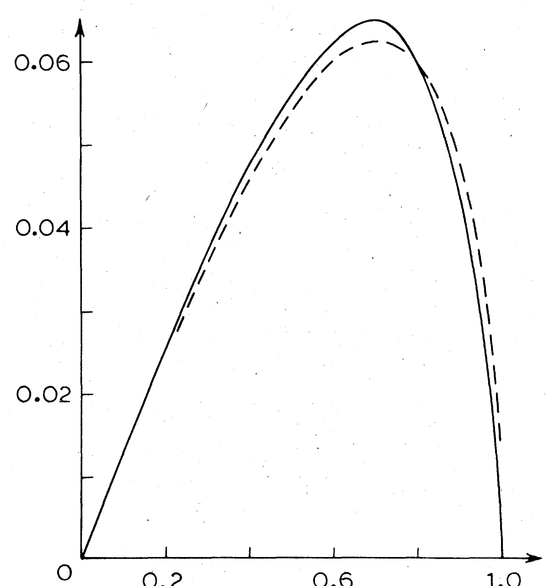

FIG. 4. The solid line represents $R(z)$, and the dashed line represents $z(1-z^2)^{1/2}/8$.

where $\beta=\mu(4m^2-\mu^2)^{-1/2}$. Using the proton and ρ masses we find

$$\delta K(x)=-0.0115g^2/\pi^2 . \tag{A41}$$

The calculation leading to this addition is extremely tedious and we do not present it here. In all work we shall ignore δK since it changes the cross sections only by a multiplicative factor close to unity and adds a small constant term to the multiplicities.

APPENDIX B

In this Appendix we shall review the properties of some of the functions used in the text.

1. Ci(x) and si(x)

The functions Ci(x) and si(x) are defined by

$$\mathrm{Ci}(x)=-\int_x^\infty dt\,\frac{\cos t}{t} , \tag{B1}$$

$$\mathrm{si}(x)=-\int_x^\infty dt\,\frac{\sin t}{t} . \tag{B2}$$

With these definitions we see that

$$\frac{d}{dx}\mathrm{Ci}(x)=\frac{\cos x}{x};\quad \frac{d}{dx}\mathrm{si}(x)=\frac{\sin x}{x} . \tag{B3}$$

The series expansions are

$$\mathrm{si}(x)=-\frac{\pi}{2}+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \tag{B4}$$

and

$$\mathrm{Ci}(x)=\gamma+\ln x+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2n(2n)!} . \tag{B5}$$

2. f(x) and g(x)

The auxiliary functions $f(x)$ and $g(x)$ have the integral representations[12]

$$f(x)=\int_0^\infty dt\,\frac{e^{-xt}}{t^2+1} , \tag{B6}$$

$$g(x)=\int_0^\infty dt\,\frac{t e^{-xt}}{t^2+1} \tag{B7}$$

for Re$x>0$. In terms of the Ci and si functions we have

$$f(x)=\mathrm{Ci}(x)\sin x-\mathrm{si}(x)\cos x \tag{B8}$$

and

$$g(x)=-\mathrm{Ci}(x)\cos x-\mathrm{si}(x)\sin x . \tag{B9}$$

From (B3) we find

$$\frac{d}{dx}f(x)=-g(x) \tag{B10}$$

and

$$\frac{d}{dx}g(x)=-\frac{1}{x}+f(x) . \tag{B11}$$

For small x we can determine f and g by substituting (B4) and (B5) into (B8) and (B9). Notice that for very small x

$$f(x)\approx\frac{\pi}{2}+x(\gamma+\ln x-1) \tag{B12}$$

and

$$g(x)\approx-(\gamma+\ln x)+\frac{\pi}{2}x . \tag{B13}$$

For large x we may use the following asymptotic expansions:

$$f(x)\sim\frac{1}{x}\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\frac{6!}{x^6}+\cdots\right) \tag{B14}$$

and

$$g(x)\sim\frac{1}{x^2}\left(1-\frac{3!}{x^2}+\frac{5!}{x^4}-\frac{7!}{x^6}+\cdots\right) . \tag{B15}$$

More accurate approximations to f and g are found in Ref. 12.

3. The $B_N(x)$

The function $B_2(x)$ is defined in the text as

$$B_2(x)=\tfrac{1}{4}[g^2(x)+f^2(x)+g(x)] . \tag{B16}$$

With (B12) and (B13) we see that for very small x

$$B_2(x)\approx\tfrac{1}{4}[(\gamma+\ln x)^2-(\gamma+\ln x)+\pi^2/4] , \tag{B17}$$

while with (B14) and (B15) we find the asymptotic expansion

$$B_2(x)\sim\frac{1}{2}\frac{1}{x^2}-\frac{9}{4}\frac{1}{x^4}+40\frac{1}{x^6}-\cdots . \tag{B18}$$

The function $B_1(x)$ is defined by

$$B_1(x)=\tfrac{1}{4}[xf(x)-2g(x)] . \tag{B19}$$

For small x we have

$$B_1(x)\approx\tfrac{1}{4}[2(\gamma+\ln x)-(\pi/2)x] , \tag{B20}$$

while the asymptotic expansion is

$$B_1(x)\sim\tfrac{1}{4}-\frac{1}{x^2}+\frac{9}{x^4}-\cdots . \tag{B21}$$

$B_1(x)$ is related to B_2 by

$$B_1(x)=\tfrac{1}{4}+x\frac{x}{dx}B_2(x) . \tag{B22}$$

The function $B_3(x)$ is defined by

$$B_3(x)=\tfrac{1}{4}[2xf(x)+(x^2-2)g(x)] . \tag{B23}$$

The small-x behavior is

$$B_3(x)\approx\tfrac{1}{2}(\gamma+\ln x)+O(x^2) , \tag{B24}$$

while the asymptotic behavior is

$$B_3(x) \sim \tfrac{3}{4} - 3/x^2 + 45/x^4 - \cdots \; . \qquad \text{(B25)}$$

In terms of the other B_i functions we have

$$B_3(x) = \tfrac{1}{2} - x^2 \frac{d^2}{dx^2} B_2(x)$$

$$= \tfrac{1}{2} - x^2 \frac{d}{dx}\left[\frac{1}{x} B_1(x)\right] \; . \qquad \text{(B26)}$$

Finally, the function $B_u(x)$ which is used for calculating multiplicities is defined by

$$B_u(x) = \frac{x}{8}[1 - xf(x)] \, . \qquad \text{(B27)}$$

For small x this becomes

$$B_4(x) \approx \frac{x}{8} - x^2 \frac{\pi}{16} \, , \qquad \text{(B28)}$$

while its asymptotic expansion is

$$B_4(x) \sim \frac{1}{4x} - \frac{3}{x^3} + \frac{90}{x^5} - \cdots \; . \qquad \text{(B29)}$$

In terms of $B_3(x)$ we have

$$B_4(x) = \frac{1}{4x} - \frac{1}{2}\frac{d}{dx} B_3(x) \, . \qquad \text{(B30)}$$

The errors in the asymptotic expansions for B_i are of the order of the first neglected terms. If we keep only the first two terms in those expansions then (B18), (B21), and (B25) are accurate to better than 10% for $x \gtrsim 5$, while (B30) has the same accuracy for $x \gtrsim 10$. The small-x expansions are better than 10% for the following values of x: (B17) $x \lesssim 0.25$, (B20) $x \lesssim 0.5$, (B24) $x \lesssim 0.4$, (B28) $x \lesssim 0.2$.

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