PHYSICAL REVIEW D VOLUME 16, NUMBER 2 15 JULY 1977

Localization and causality in relativistic quantum mechanics

J. Fernando Perez*
Instituto de Física-Universidade de São Paulo, São Paulo, Brazil

Ivan F. Wilde†
Department of Physics, Queen Mary College, London, England
(Received 20 September 1976)

It is shown that in relativistic quantum mechanics there is no criterion for the strict localization of a state in a bounded space-time region compatible with causality, translation covariance, and the spectral condition (or positivity of energy together with Lorentz covariance).

I. INTRODUCTION

It is well known that relativity poses some problems concerning the definition of a position operator and the notion of localized states in quantum mechanics. The work of Newton and Wigner[1] shows that for free one-particle systems, a notion of localization is uniquely determined by some natural requirements. A rigorous discussion of these aspects has been made by Wightman.[2]

In the context of quantum field theory the problem of the characterization of localized states has been studied,[3-6] mostly in connection with the consequences of the theorem of Reeh and Schlieder.[7-9]

The present work is motivated by a recent paper by Hegerfeldt[10] in which a notion of strict localization of particles in relativistic quantum mechanics is shown to be incompatible with a causality requirement. The aim of our discussion is to provide a twofold generalization of the statement in Ref. 10 and to stress the fundamental role of the spectral assumption. Namely, we show that in any relativistic quantum theory with positive energy, any (space-time) translationally covariant notion of localization is incompatible with a natural causality requirement. In particular, there is no assumption about the particle structure of the theory.

We also show that Hegerfeldt's notion of localization and his causality requirement are covered by our discussion.

The conclusion of our discussion is that either one is forced to consider essentially localized states in a relativistic quantum theory (as defined, for example, by Haag and Swieca[3] or to admit noncausal behavior, or to allow the energy operator to be unbounded both from above and from below.[1,11,12]

II. THE RESULTS

Our discussion will take place within the following framework.

We suppose we are given a Hilbert space of states, $\mathcal{H}$, together with a strongly continuous unitary representation $U(x)$, $x=(x^0,\vec{x})\in \mathbf{R}^4$, of the group of space-time translations. The joint spectrum of the generators $P=(P^0,\vec{P})$, $U(x)=e^{iP\cdot x}$ ($P\cdot x=P^0x^0-\vec{P}\cdot\vec{x}$), is assumed to be contained in the closed forward light cone $\overline{V}_+=\{p\in\mathbf{R}^4:\ p^2=p\cdot p\geq 0,\ p^0\geq 0\}$. (This property follows if we assume positivity of the energy and relativistic covariance.)

The main ingredient of our discussion is the following result.

Lemma. Let $\psi\in\mathcal{H}$, and let E be a nonempty open set in $\mathbf{R}^4$. If

$$(\psi, U(x)\psi)=0,\quad \forall x\in E\,,$$

then $\psi=0$.

Proof. Because of the assumption on the joint spectrum of P, it follows that the function

$$G(x)=(\psi, U(x)\psi)$$

is the boundary value of a function $F(z)$, $z=x+iy$ analytic for $y\in V_+$, i.e.,

$$G(x)=\lim_{\substack{y\to 0\\ y\in V_+}} F(x+iy)$$

(see for instance Ref. 8). By the edge-of-the-wedge theorem[8] it follows that $F(z)=0$, $\forall z\in \mathbf{C}^4$, $\mathrm{Im}\,z\in V_+$, and so $G(x)=0$, $\forall x\in\mathbf{R}^4$.

In particular, for $x=0$

$$G(0)=(\psi,U(0)\psi)=(\psi,\psi)=0$$

i.e., $\psi=0$. Q.E.D.

If $\mathcal{O}\subset\mathbf{R}^4$ we shall denote its causal complement by $\mathcal{O}'$, i.e., $\mathcal{O}'=\{y\in\mathbf{R}^4:\ (y-x)^2<0,\ \forall x\in\mathcal{O}\}$. If $x\in\mathbf{R}^4$, we denote the x translation of $\mathcal{O}$ by $\mathcal{O}+x=\{y+x,\ y\in\mathcal{O}\}$.

A notion of (strict) localization of a system is a criterion allowing one to tell whether a given state is or is not localized in a given space-time region $\mathcal{O}$. If L is a notion of localization, let $L(\mathcal{O})$ be the set of states in $\mathcal{H}$ localized in $\mathcal{O}$.

The notion L is said to be translationally covariant if

(I) $\psi \in L(\mathcal{O}) \Rightarrow U(a)\psi \in L(\mathcal{O}+a), \quad \forall a \in \mathbb{R}^4, \ \mathcal{O} \subset \mathbb{R}^4.$

The notion L is said to be causal if

(II) $\mathcal{O}_2 \subset \mathcal{O}_1' \Rightarrow (\psi_1, \psi_2) = 0, \quad \forall \psi_1 \in L(\mathcal{O}_1), \ \psi_2 \in L(\mathcal{O}_2).$

Property II is a causality requirement: If a state is in $\mathcal{O}_1$, then it cannot be in $\mathcal{O}_2 \subset \mathcal{O}_1'$. That is, the transition probability between states in $L(\mathcal{O}_1)$ and $L(\mathcal{O}_2)$ should be zero if $\mathcal{O}_2 \subset \mathcal{O}_1'$.

We are now in a position to state our results.

Theorem 1. Let L be a translationally covariant and causal notion of localization. If $\mathcal{O} \subset \mathbb{R}^4$ is bounded, then

$$\psi \in L(\mathcal{O}) \Rightarrow \psi = 0.$$

Proof. Let $\psi \in L(\mathcal{O})$. If $x = (x^0, \vec{x})$ with $|x^0| < r$ and $|\vec{x}| > 4r$, where r is the radius of the base of the smallest diamond containing $\mathcal{O}$, then $(\mathcal{O}+x) \subset \mathcal{O}'$, and so by property I $U(x)\psi \in L(\mathcal{O}')$ and by property II $(\psi, U(x)\psi) = 0$. Since this is true for x lying in an open set in $\mathbb{R}^4$, we have, by the lemma, $\psi = 0$. Q.E.D.

In the course of the proof we did not make full use of the covariance and of the causality assumptions but only of the following very weak causality requirement which is a consequence of properties I and II:

(III) If $\psi \in L(\mathcal{O})$, then there exists $a > 0$ such that $(U(x)\psi, \psi) = 0$ whenever $-x^2 > a$ (a may depend on ψ and $\mathcal{O}$).

Condition III could be called a kind of macroscopic causality condition and it is obviously implied by properties I and II.

So we have proved the following:

Theorem 2. Let L satisfy condition III. If $\mathcal{O} \subset \mathbb{R}^4$ is bounded, then

$$\psi \in L(\mathcal{O}) \Rightarrow \psi = 0.$$

III. DISCUSSION

Let us show that some notions of localization with certain causality requirements are covered by our discussion.

Example 1. Let $\{N_\mathcal{O}, \mathcal{O} \subset \mathbb{R}^4\}$ be a family of self-adjoint operators in x satisfying

$$0 \leqslant N_\mathcal{O} \leqslant 1, \quad \forall \mathcal{O} \subset \mathbb{R}^4 \tag{1a}$$

$$N_{\mathcal{O}+a} = U(a) N_\mathcal{O} U^{-1}(a) \tag{1b}$$

and consider the following notion of localization

$$\psi \in L(\mathcal{O}) \Longleftrightarrow (\psi, N_\mathcal{O}\psi) = (\psi, \psi), \tag{1c}$$

together with the causality requirement

$$\psi \in L(\mathcal{O}_2) \Rightarrow (\psi, N_{\mathcal{O}_1}\psi) = 0 \ \text{ if } \ \mathcal{O}_2 \subset \mathcal{O}_1'. \tag{1d}$$

Then L satisfies conditions I and II. In fact

$$(U(a)\psi, N_{\mathcal{O}+a} U(a)\psi) = (\psi, N_\mathcal{O}\psi),$$

and so condition I is fulfilled.

If $\psi_1 \in L(\mathcal{O}_1)$, $\psi_2 \in L(\mathcal{O}_2)$, with $\mathcal{O}_2 \subset \mathcal{O}_1'$, we have from (1c) and (1d)

$$(\psi_1, N_{\mathcal{O}_1}\psi_1) = (\psi_1, \psi_1) \ \text{ and } \ (\psi_2, N_{\mathcal{O}_1}\psi_2) = 0$$

Thus

$$((1 - N_{\mathcal{O}_1})^{1/2}\psi_1, (1 - N_{\mathcal{O}_1})^{1/2}\psi_1) = 0 \Rightarrow N_{\mathcal{O}_1}\psi_1 = \psi_1$$

and

$$(N_{\mathcal{O}_1}{}^{1/2}\psi_2, N_{\mathcal{O}_1}{}^{1/2}\psi_2) = 0 \Rightarrow N_{\mathcal{O}_1}\psi_2 = 0.$$

From this it follows that $(\psi_1, \psi_2) = 0$ because ψ_1 and ψ_2 are eigenvectors of the self-adjoint operator $N_{\mathcal{O}_1}$ corresponding to different eigenvalues. Condition II is therefore satisfied.

Example 2. If, in example 1, we substitute the conditions (1b) and (1d) by the requirement that if $\psi \in L(\mathcal{O})$, then there exists an $a > 0$ such that for $-x^2 > a$

$$(U(x)\psi, N_\mathcal{O} U(x)\psi) = 0.$$

Then we obtain, as before,

$$(\psi, U(x)\psi) = 0 \ \text{ if } \ -x^2 > a,$$

that is, condition III is fulfilled.

This example corresponds to the notion of localization and the causality requirement of Ref. 10.

ACKNOWLEDGMENTS

The authors wish to express their thanks to Lawrence Landau for many stimulating discussions on this and related questions.

One of us (I.F.W.) would like to thank H. Fleming and J. F. Perez for the hospitality of the Instituto de Fisica, USP, São Paulo, Brazil where this paper originated, and also BNDE for financial support.

*Partial financial support by FAPESP.

†Work supported in part by the Science Research Council.

[1]T. D. Newton and E. P. Wigner, Rev. Mod. Phys. 21, 400 (1949).

[2]A. S. Wightman, Rev. Mod. Phys. 34, 845 (1962).

[3]R. Haag and J. A. Swieca, Commun. Math. Phys. 1, 308 (1965).

[4]S. Schlieder, Commun. Math. Phys. 1, 265 (1965).

[5]J. M. Knight, J. Math. Phys. 2, 459 (1961).

[6]A. L. Licht, J. Math. Phys. 4, 1443 (1963).

[7]H. Reeh and S. Schlieder, Nuovo Cimento 22, 1051

(1961).
[8]R. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964).
[9]R. Jost, *The General Theory of Quantized Fields* (American Mathematical Society, New York, 1963).
[10]G. C. Hegerfeldt, Phys. Rev. D 10, 3320 (1974).
[11]B. Durand, Phys. Rev. D 4, 1554 (1976).
[12]G. N. Fleming, Phys. Rev. 137, B188 (1965).