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Limits to the use of four-vectors in relativistic Newtonian mechanics

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Of all the parity-invariant Poincaré-invariant Newtonian equations of motion for two particles, the only ones that can be written in a recently used four-vector form are shown to have accelerations collinear with the relative velocity.

Consider a classical mechanical system of two particles ($n=1,2$) described by positions $\vec{x}_n$, velocities $\vec{v}_n = d\vec{x}_n/dt$, and Newtonian equations of motion that give accelerations as functions of the positions and velocities at one time,

$$d\vec{v}_n/dt = \vec{f}_n(\vec{x},\vec{v}_1,\vec{v}_2). \tag{1}$$

These equations are made invariant for time translations by letting $\vec{f}_n$ not depend explicitly on time, for space translations by letting $\vec{f}_n$ depend on the positions only through the relative position $\vec{x}=\vec{x}_1-\vec{x}_2$, and for rotations by letting $\vec{f}_n$ be a rotational vector function of $\vec{x},\vec{v}_1,\vec{v}_2$. Then Poincaré invariance, i.e., Lorentz invariance, requires only that the functions $\vec{f}_n$ satisfy the coupled nonlinear equations

$$2v_{nk}f_{nj}+f_{nk}v_{nj}=\sum_{m=1}^{2}\left(\sum_{l=1}^{3}v_{mk}v_{ml}\partial f_{nj}/\partial v_{ml}-\partial f_{nj}/\partial v_{mk}\right)+(x_{n'k}-x_{nk})\sum_{l=1}^{3}(v_{n'l}\partial f_{nj}/\partial x_{n'l}+f_{n'l}\partial f_{nj}/\partial v_{n'l}) \tag{2}$$

for $k,j=1,2,3$ with $n'=2,1$ for $n=1,2$.[1,2]

Four-vector space-time coordinates $x_n=(t_n,\vec{x}_n)$ and velocities $u_n = dx_n/d\tau$ have been used to write equations of motion in the manifestly covariant form

$$du_n/d\tau=\xi_n(x,u_1,u_2) \tag{3}$$

with ξ_n a Lorentz four-vector function of $x=x_1-x_2, u_1, u_2$.[3-8] For parity-invariant Poincaré-invariant equations of motion it is assumed that

$$\xi_n = a_n x + b_{n1}u_1 + b_{n2}u_2 \tag{4}$$

with a_n, b_{n1}, b_{n2} functions of the parity-invariant Lorentz scalars x^2, $x\cdot u_1$, $x\cdot u_2$, $u_1\cdot u_2$.[3-8] (We will not actually require Lorentz scalars; parity-invariant rotational scalars would be all right.) It is also assumed that

$$u_n\cdot\xi_n=0 \tag{5}$$

and

$$\sum_{\mu=0}^{3}(u_{n'\mu}\partial\xi_n/\partial x_{n'\mu}+\xi_{n'\mu}\partial\xi_n/\partial u_{n'\mu})=0, \tag{6}$$

where $n'=2,1$ for $n=1,2$.[3-8]

Here we show that of all the Newtonian equations of motion (1) for two particles that are Poincaré invariant in the sense of Eqs. (2), and also parity invariant, the only ones that can be written in the four-vector form according to Eqs. (3)–(6) have accelerations collinear with the relative velocity, which means the velocities do not change direction in any center-of-mass frame (where the velocities are collinear). In contrast, parity-invariant Poincaré-invariant Newtonian equations of motion for two identical particles are known to exist for practically any reasonable dynamics specified in the center-of-mass frame.[9]

From Eq. (5) it follows that u_n^2 is constant, which means that

$$\left(\frac{dt_n}{d\tau}\right)^2-\left(\frac{d\vec{x}_n}{dt_n}\right)^2\left(\frac{dt_n}{d\tau}\right)^2$$

is constant, so we have

$$d\tau/dt_n \propto [1-(d\vec{x}_n/dt_n)^2]^{1/2}.$$

Thus without loss of generality we can let τ be proper time for each particle. Then

$$u_{n0}=dt_n/d\tau=(1-\vec{v}_n^{\,2})^{-1/2}, \tag{7}$$

$$\vec{u}_n=d\vec{x}_n/d\tau=(1-\vec{v}_n^{\,2})^{-1/2}\vec{v}_n, \tag{8}$$

and

$$u_n^2=\vec{u}_n^{\,2}-u_{n0}^2=-1.$$

From Eqs. (4) and (5) it follows that

$$a_n x\cdot u_n - b_{nn} + b_{nn'}u_1\cdot u_2 = 0,$$

so

$$\xi_n = a_n x + (a_n x\cdot u_n + b_{nn'}u_1\cdot u_2)u_n + b_{nn'}u_{n'}, \tag{9}$$

where again $n'=2,1$ for $n=1,2$.[3,5,8] From Eqs. (7) and (8) we have

$$d\vec{u}_n/d\tau=(1-\vec{v}_n^{\,2})^{-1}d\vec{v}_n/dt_n+\vec{v}_n du_{n0}/d\tau,$$

so[4]

$$d\vec{v}_n/dt_n = (1-\vec{v}_n^{\,2})(d\vec{u}_n/d\tau - \vec{v}_n du_{n0}/d\tau)$$
$$= (1-\vec{v}_n^{\,2})(\vec{\xi}_n - \vec{v}_n \xi_{n0}), \tag{10}$$

which, from Eq. (9), we find to be

$$d\vec{v}_n/dt_n = (1-\vec{v}_n^{\,2})[a_n\vec{x} + (a_n x\cdot u_n + b_{nn'}u_1\cdot u_2)\vec{u}_n + b_{nn'}\vec{u}_{n'} - (a_n x\cdot u_n + b_{nn'}u_1\cdot u_2)u_{n0}\vec{v}_n - b_{nn'}u_{n'0}\vec{v}_n]$$
$$= (1-\vec{v}_n^{\,2})[a_n\vec{x} + b_{nn'}u_{n'0}(\vec{v}_{n'} - \vec{v}_n)] \tag{11}$$

at $t_1 = t_2$.[5,8] We can let $t_1 = t_2$ because Eqs. (6) and (10) imply that ξ_n and $d\vec{v}_n/dt_n$ are not changed if we change the point on the world line at which we take values for the position and velocity of particle n'. Thus, we obtain Newtonian equations of motion (1) with $\vec{f}_n$ of the form

$$\vec{f}_n = A_n\vec{x} + B_n(\vec{v}_n - \vec{v}_{n'}) \tag{12}$$

with $n' = 2, 1$ for $n = 1, 2$.

For parity-invariant Poincaré-invariant Newtonian equations of motion we take A_n, B_n to be functions of the parity-invariant rotational scalars $\vec{x}^2$, $\vec{x}\cdot\vec{v}_1$, $\vec{x}\cdot\vec{v}_2$, $\vec{v}_1^{\,2}$, $\vec{v}_2^{\,2}$, $\vec{v}_1\cdot\vec{v}_2$. (We do not need the more restrictive assumption that $a_n, b_{nn'}$ are functions of the parity-invariant Lorentz scalars.) For $\vec{f}_n$ of the form (12), the Lorentz-invariance equations (2) have terms involving just the nine tensors $x_j x_k$, $x_j v_{1k}$, etc., so from the coefficients of these tensors we get nine separate scalar equations. The key is that there is no δ_{jk} term as there is in general when the coefficients of $\vec{v}_n$ and $\vec{v}_{n'}$ in $\vec{f}_n$ are not the same. Furthermore, Eqs. (6) and (10) imply that

$$\sum_{l=1}^{3}(v_{n'l}\partial f_{nj}/\partial x_{n'l} + f_{n'l}\partial f_{nj}/\partial v_{n'l}) = 0 \tag{13}$$

for $j = 1, 2, 3$ with $n' = 2, 1$ for $n = 1, 2$, so Eqs. (2) reduce to

$$2v_{nk}f_{nj} + f_{nk}v_{nj} = \sum_{m=1}^{2}\left(\sum_{l=1}^{3} v_{mk}v_{ml}\partial f_{nj}/\partial v_{ml} - \partial f_{nj}/\partial v_{mk}\right). \tag{14}$$

From the coefficients of $x_k v_{nj}$ in Eqs. (14) we get

$$A_n = -\partial B_n/\partial\vec{x}\cdot\vec{v}_1 - \partial B_n/\partial\vec{x}\cdot\vec{v}_2$$

and from the coefficients of $x_k v_{n'j}$ we get

$$0 = \partial B_n/\partial\vec{x}\cdot\vec{v}_1 + \partial B_n/\partial\vec{x}\cdot\vec{v}_2 .$$

Therefore $A_n = 0$ and $\vec{f}_n$ is collinear with $\vec{v}_1 - \vec{v}_2$.

It is not necessary to assume that the Lorentz invariance of Newtonian equations of motion is the manifest invariance of four-vector forms. We have shown that writing Newtonian equations of motion in the manifestly invariant form of Eqs. (3)–(6) implies restrictions that are in fact unnecessary and physically unreasonable.

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