Comments and Addenda

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Note on the spacetimes of Szekeres

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We find that the "quasispherical" spacetimes of Szekeres have conformally flat space slices in comoving coordinates. These spacetimes can represent pressureless nonspherical gravitational collapse with no gravitational radiation. We give a restriction on general-relativistic exact solutions for matter with pressure which admit flat comoving slices.

I. INTRODUCTION

Recently, Szekeres discovered[1] a remarkable special class of exact solutions to Einstein's equations for a source consisting of pressureless matter or "dust." Subsequently he discussed[2] the interpretation of the "quasispherical" subclass of these solutions in terms of nonspherical gravitational collapse. Here we shall remark on some further properties of these solutions.

The spacetime metric of these solutions is of the form

$$ds^2 = -dt^2 + X^2 dr^2 + Y^2 d\zeta d\bar{\zeta} \ , \tag{1a}$$

where t and r are two real coordinates, ζ is a complex coordinate standing for two real ones, $\zeta \equiv x + iy$, and

$$Y = \phi(r,t)/P(r,\zeta,\bar{\zeta}) \ , \tag{1b}$$

$$X = P(r,\zeta,\bar{\zeta}) Y_{,r}/W(r) \ ; \tag{1c}$$

here

$$P = a(r)\zeta\bar{\zeta} + B(r)\zeta + \bar{B}(r)\bar{\zeta} + c(r) \tag{1d}$$

with the quasispherical subclass being determined by the further restriction

$$ac - B\bar{B} = \tfrac{1}{4} \ . \tag{2}$$

The equation of motion for $\phi(r,t)$ is

$$(\phi_{,t})^2 = W^2 - 1 + 2\phi^{-1} \int W M_{,r} dr \ , \tag{3}$$

where $M(r)$ is the conserved total proper mass inside coordinate radius r; most of our computations will not need Eq. (3).

Although these spacetimes do not in general admit any Killing vectors[3] (or homothetic Killing vectors), they do admit a preferred 2-parameter family of 2-surfaces of constant curvature, which in the quasispherical case, Eq. (2), are spheres of radius ϕ. These preferred 2-spheres are, however, not nested inside one another in a spherically symmetric way; rather, one sphere is offset by an infinitesimal translation in an arbitrary direction from the next. These Szekeres solutions can therefore be regarded as certain asymmetric generalizations of the familiar Tolman-Bondi[4,5] solutions for dust, which are strictly spherically symmetric.

The mathematical nature of these solutions has been considerably clarified by Wainwright,[6] who discovered that they are algebraically special of the Petrov type D, and gave a detailed characterization of them among type-D dust solutions. The Weyl tensor of the spacetime given by Eqs. (1) is

$$\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0 \ , \tag{4a}$$

$$\psi_2 = (\phi/6WX)\{[(\phi_{,t})^2 - W^2 + 1]/\phi^2\}_{,r} \ , \tag{4b}$$

in the Newman-Penrose tetrad[7]

$$\vec{l} = 2^{-1/2}(\partial_t + X^{-1}\partial_r) \ , \quad \vec{n} = 2^{-1/2}(\partial_t - X^{-1}\partial_r) \ , \tag{5a}$$

$$\vec{m} = 2^{1/2} Y^{-1} \partial_\zeta \ . \tag{5b}$$

In Sec. II we point out that these spacetimes have the further curious property that the comoving space slices t = const are conformally flat [or flat if $W \equiv 1$ and Eq. (2) holds]. We discuss the existence of further exact solutions with conformally flat or flat comoving space slices in Sec. III. In Sec. IV we remark that these spacetimes contain no gravitational radiation according to usual criteria, to follow up an observation by Bonnor.[3,8]

II. SPACE SLICES ARE CONFORMALLY FLAT

We demonstrate that each comoving space slice t = const of a spacetime given by Eqs. (1) is conformally flat; this result follows from the vanishing of York's dual[9,10] of the standard 3-dimensional Weyl tensor,

$$C_{ijk} = R_{ij|k} - R_{ik|j} - \tfrac{1}{4} g_{ij} R_{|k} + \tfrac{1}{4} g_{ik} R_{|j} \ , \tag{6}$$

where g_{ij} is the 3-metric, R_{ij} is the 3-Ricci-tensor, i,j,k run over the spatial coordinates $x^i = (r, \zeta, \bar{\zeta})$, and the subscript $|k$ denotes covariant differentiation according to g_{ij}. A short calculation shows $C_{ijk} = 0$.

In the "marginally bound, quasispherical" case defined by Eq. (2) and $W \equiv 1$, each comoving space slice t = const is in fact flat, i.e., Euclidean, because R_{ij} itself vanishes. Each such space slice turns out to be a particular example of the class of solutions to the initial-value problem recently given by one of us (D.E.[11]), of the form

$$g_{ij} \text{ flat}, \tag{7a}$$

$$K_{ij} \equiv -\tfrac{1}{2}\partial_t g_{ij} = \kappa_{|ij} \tag{7b}$$

for a scalar "velocity potential" κ. For the Szekeres spacetimes it turns out that

$$\kappa = (\lambda \zeta \bar{\zeta} + \mu \zeta + \bar{\mu} \bar{\zeta} + \sigma)/P \ , \tag{8a}$$

where the real functions $\lambda(r,t)$ and $\sigma(r,t)$, and the complex function $\mu(r,t)$, are solutions of the system of ordinary differential equations in r:

$$(\lambda/\phi)_{,r} = (a/\phi)_{,r} f \ , \tag{8b}$$

$$(\mu/\phi)_{,r} = (B/\phi)_{,r} f \ , \tag{8c}$$

$$(\sigma/\phi)_{,r} = (c/\phi)_{,r} f \ , \tag{8d}$$

with

$$f \equiv 2(\lambda c + \sigma a - \mu \bar{B} - \bar{\mu} B) - \phi \phi_{,t} \ . \tag{8e}$$

The generic solution of the flat initial-value problem of the form given by Eqs. (7) has the property that the comoving space slices become curved during time evolution. The marginally bound, quasispherical Szekeres solutions have the remarkable property that the flatness of g_{ij} is conserved by the evolution equations.

Since the marginally bound, quasispherical solutions have flat slices, the "velocity-dominated approximation" ($R_{ij} \approx 0$), first given in the well-known work of Lifshitz and Khalatnikov on singularities,[12] and used here in the particular sense of Eardley, Liang, and Sachs,[13] is in fact exact for these solutions.

III. RESTRICTIONS ON NEW PERFECT-FLUID SOLUTIONS

One wonders how many exact solutions for irrotational perfect-fluid matter with pressure might be found with this curious property of flat or conformally flat comoving space slices. We shall give one negative partial result along these lines, and refer to some further examples in the literature.

Proposition. Given Einstein's equations with zero cosmological constant, a perfect-fluid source, and the three assumptions (1) spherical symmetry, including a spatial origin of spherical symmetry, (2) an equation of state with pressure a function of energy density alone, and with speed of sound nonzero, and (3) flat comoving space slices, then the only solution is the $k=0$ Robertson-Walker (RW) spacetime.

The proof follows by construction from equations given by Cahill and Taub[14] for the line element

$$ds^2 = -e^{2\phi(r,t)}dt^2 + e^{2\psi(r,t)}dr^2 + R^2(r,t)d\Omega^2 \ . \tag{9}$$

Since the assumption of flat comoving space slices is equivalent to setting $e^{\psi} = R_{,r}$, the field equation

$$R_{,rt} - R_{,t}\phi_{,r} - R_{,r}\psi_{,t} = 0 \tag{10}$$

immediately implies that the product $R_{,t}\phi_{,r}$ vanishes. The vanishing of $R_{,t}$ leads directly to a special case of the $k=0$ RW spacetime, namely empty space with mass function $m(r,t) \equiv 0$. The vanishing of $\phi_{,r}$, in which case one can set $\phi = 0$, implies from the Bianchi identities that the spatial pressure gradient $p_{,r}$ and the four-acceleration both vanish. Assumption (2) allows us to conclude that $\rho_{,r} = 0$, where ρ is the total energy density. Using the continuity equation and the Gauss-Codazzi relation for the spatial curvature scalar in irrotation flow,[15]

$$\rho_{,t} + (\rho + p)\theta = 0 \tag{11}$$

and

$$^{(3)}R = 2(\rho - \tfrac{1}{3}\theta^2 + \sigma^2) \ , \tag{12}$$

where θ is the volume expansion rate and σ is the shear scalar, we find that $p = p(t)$ and $\rho = \rho(t)$ implies $\theta = \theta(t)$ and that $^{(3)}R = 0$ implies $\sigma = \sigma(t)$. Since σ vanishes at the spatial origin by spherical symmetry, therefore $\sigma \equiv 0$, which leads again to the homogeneous isotropic $k=0$ RW spacetime.

The cosmological import of this result is as follows. For epochs when pressure is important, there are well-known difficulties in constructing models for lumps. Any lump possesses by definition a spatial density gradient; when assumption (2) is satisfied there must also exist a spatial pressure gradient, therefore nonzero four-acceleration, and therefore deviation from the geodesic flow that obtains in all dust cosmologies. By the result above, an additional complication is the necessary presence of spatial curvature in any spherical lump, since spatially flat spherical inhomogeneities cannot exist where assumption (2) on the pressure is satisfied.

If we give up assumption (2), the proposition is false. In fact, exact solutions with pressure a function of time alone, $p=p(t)$, were indicated by Bondi[5] in the spherical case, were derived by Szafron[16] in the quasispherical case, and were derived by Szafron and Wainwright[17] for a related set of Szekeres solutions. All of these solutions have conformally flat comoving space slices, and some of them have all flat slices.

A class of exact static solutions for stiff matter, $p=\rho$, found by Melnick and Tabensky,[18] has nonvanishing pressure and pressure gradients, and admits conformally flat but not flat space slices. These spacetimes are also "quasispherical."

IV. LACK OF GRAVITATIONAL RADIATION

Bonnor has shown[3,8] that any piece of a quasispherical Szekeres spacetime, interior to some coordinate radius r_0, can be smoothly matched onto the Schwarzschild exterior solution. He thereby concludes that quasispherical spacetimes are nonradiative. Here we remark that these spacetimes are in fact nonradiative according to the usual definition of gravitational radiation in a spacetime which is asymptotically flat at future null infinity, due to Bondi *et al.*,[19] Sachs,[20] and Penrose.[21]

As a reasonably general criterion for asymptotic flatness in a quasispherical spacetime, we assume as $r\to\infty$ that

$$\phi=O(R)\ , \tag{13a}$$

$$\int WdM \to \text{const} \equiv \text{total mass,} \tag{13b}$$

and

$$a\to \text{const},\quad B\to\text{const},\quad c\to\text{const}, \tag{13c}$$

where R is a luminosity distance. Then from Eqs. (4), the Weyl tensor obeys

$$\psi_2=(\phi/3WX)\left(\int WdM/\phi^3\right)_{,r} \tag{14a}$$

$$=O(R^{-3})\ , \tag{14b}$$

$$\psi_0=\psi_1=\psi_3=\psi_4=0\ . \tag{14c}$$

Now at future null infinity, gravitational radiation is the $O(R^{-1})$ part of the Weyl tensor, which is purely ψ_4 in a suitable null tetrad. This part vanishes in Eqs. (14); it follows that the quasispherical Szekeres spacetimes are nonradiative whenever they are asymptotically flat.

It is extremely surprising that there should exist these strongly nonspherical collapsing configurations of dust which nevertheless emit no gravitational radiation. A generic dust collapse will certainly emit some radiation, although exactly how much is not clear.

Various workers have given local definitions of "gravitational radiation" in terms of the geometry of space slices, notably Arnowitt, Deser, and Misner[22] and York.[10] One understands that these definitions are necessarily slicing-dependent, but they may have considerable theoretic, heuristic, and calculational value for suitably chosen slices. In particular, York[10] has suggested conformal curvature as "radiation amplitude." Since each comoving slice of a Szekeres spacetime is conformally flat, we can say additionally that these spacetimes are "nonradiative" according to York's definition. It is then quite clear that the lack of radiation in these spacetimes is a nongeneric property in all dust collapses. For instance, dust collapses defined by the initial-value solution of Eqs. (7) have vanishing initial amplitude and time derivative of "gravitational radiation"; however, a generic solution of this form will generate conformal curvature of the comoving space slices at order $O(\Delta t^2)$ when time evolution is turned on; that is, it will generate "radiation" later.

One can think of a quasispherical spacetime as having nonzero monopole and dipole moment contributions from each shell of dust, but all higher moments vanish. The remarkable feature is that no higher moments are generated during time evolution of the system.

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