Electric and magnetic dipole moments of the bound system of a Dirac particle and a fixed magnetic monopole

Yoichi Kazama*
Fermi National Accelerator Laboratory, Batavia, Illinois 60510 †
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The electric and the magnetic dipole moments are calculated for the bound states of a charged Dirac particle of spin 1/2 with an extra magnetic moment in the field of a fixed magnetic monopole. Unlike ordinary bound systems with P and/or T invariance, this system, lacking both, is found to possess a nonvanishing electric dipole moment. Its magnitude for the loosely bound states with the lowest possible angular momentum increases exponentially in the principal quantum number n. The magnetic moment of the system is found to be, in general, nonvanishing.

I. INTRODUCTION

Recently it was found[1,2] that bound states for a charged Dirac particle, with an extra magnetic moment in the field of a fixed magnetic monopole, exist. The results may be summarized as follows[3]:

(i) For each possible value of the total angular momentum $j=|q|-\frac{1}{2},\ |q|+\frac{1}{2},\ |q|+\frac{3}{2},\ \ldots$, there exists a nondegenerate tightly bound state with $E=0$ for any nonvanishing κ. Moreover, for the lowest angular-momentum state ($j=|q|-\frac{1}{2}$), the $\kappa\to 0$ limit still yields a bound state for this energy.

(ii) For the state with $j=|q|-\frac{1}{2}$, there exists a countably infinite number of bound states if $\kappa|q|>\frac{1}{4}$. The energy spectrum is symmetric about $E=0$ and not bounded either from above or from below. If the above condition is not satisfied, there are no bound states with $E\neq 0$.

As regards the system of a charged particle and a magnetic monopole, it has long been known that it violates the discrete symmetries[4] P and T. This is simply due to the fact that the magnetic field of the monopole, $g\vec{r}/r^3$, has the "wrong" transformation property under P and T, unless one changes the sign of g by hand.

This exceptional feature of the system makes one suspect that the bound states previously found may possess electric dipole moments, which are strictly forbidden for ordinary systems with P and/or T invariance.

In this paper, we shall confirm this intriguing conjecture and give explicit expressions for the electric dipole moments of the bound states. The magnetic-dipole moments, in general nonvanishing, are also computed.

II. TWO TYPES OF BOUND STATES—A SHORT REVIEW

In this section, we shall give a brief review of the two types of bound states found in Ref. 1 and set the notations.

The Hamiltonian of the system is

$$H=\vec{\alpha}\cdot(\vec{p}-e\vec{A})+\beta M-\kappa q\beta\vec{\sigma}\cdot\vec{r}(2mr^3)^{-1}\,, \tag{1}$$

with the usual definitions of $\vec{\alpha}$, β and $\vec{\sigma}$ matrices. To avoid the singularity in the potential $\vec{A}$, the wave function ψ should be considered as a section.[5] The system possesses the conserved total angular momentum $\vec{J}$ given by

$$\vec{J}=\vec{r}\times(\vec{p}-e\vec{A})-q\hat{r}+\tfrac{1}{2}\vec{\sigma}\,, \tag{2}$$

where $\hat{r}$ is the unit vector pointing to the electron from the monopole at the origin. The magnitude j of this angular momentum takes the values $j=|q|-\frac{1}{2},\ |q|+\frac{1}{2},\ |q|+\frac{3}{2},\ldots$. There are[6] two two-component angular eigenfunctions $\xi_{jm}^{(1)}$ and $\xi_{jm}^{(2)}$ if $j>j_{\min}=|q|-\frac{1}{2}$, and only one, η_m, if $j=j_{\min}$. For later purposes, we exhibit their specific forms. $\xi_{jm}^{(1)}$ and $\xi_{jm}^{(2)}$ are obtained from the "familiar" two-component eigenfunctions $\phi_{jm}^{(1)}$ and $\phi_{jm}^{(2)}$ (see below) by a rotation in the space of the two-component spinors as follows:

$$\xi_{jm}^{(1)}=\cos\frac{\alpha}{2}\,\phi_{jm}^{(1)}-\sin\frac{\alpha}{2}\,\phi_{jm}^{(2)}\,,$$
$$\xi_{jm}^{(2)}=\sin\frac{\alpha}{2}\,\phi_{jm}^{(1)}+\cos\frac{\alpha}{2}\,\phi_{jm}^{(2)}\,, \tag{3}$$

where α is given by

$$\sin\alpha=\frac{q}{j+\frac{1}{2}}\,,$$
$$\cos\alpha=\left[1-\frac{q^2}{(j+\frac{1}{2})^2}\right]^{1/2}, \tag{4}$$
$$-\pi/2\leq\alpha\leq\pi/2,$$

and $\phi_{jm}^{(1)}$ and $\phi_{jm}^{(2)}$ are composed of the monopole harmonics[3] $Y_{q,l,m}$ ($q=0$ gives the ordinary spherical harmonics)

$$\phi^{(1)}_{jm}=\begin{bmatrix}\left(\dfrac{j+m}{2j}\right)^{1/2}Y_{q,j-1/2,m-1/2}\\ \left(\dfrac{j-m}{2j}\right)^{1/2}Y_{q,j-1/2,m+1/2}\end{bmatrix},$$

$$\phi^{(2)}_{jm}=\begin{bmatrix}-\left(\dfrac{j-m+1}{2j+2}\right)^{1/2}Y_{q,j+1/2,m-1/2}\\ \left(\dfrac{j+m+1}{2j+2}\right)^{1/2}Y_{q,j+1/2,m+1/2}\end{bmatrix}. \qquad (5)$$

η_m is defined by

$$\eta_m=\phi^{(2)}_{q-1/2,m}=\begin{bmatrix}-\left(\dfrac{|q|+\frac{1}{2}-m}{2|q|+1}\right)^{1/2}Y_{q,|q|,m-1/2}\\ \left(\dfrac{|q|+\frac{1}{2}+m}{2|q|+1}\right)^{1/2}Y_{q,|q|,m+1/2}\end{bmatrix}. \qquad (6)$$

The following property[6] of η_m will prove useful later:

$$(\vec{\sigma}\cdot\hat{r})\eta_m=\frac{q}{|q|}\eta_m. \qquad (7)$$

Using these angular eigenfunctions, we can construct two types of eigenfunctions of J^2, J_z, and H:

type A: $j\geq|q|+\frac{1}{2}$,

$$\psi=\frac{1}{r}\begin{bmatrix}h_1(r)\xi^{(1)}_{jm}+h_2(r)\xi^{(2)}_{jm}\\ -i\dfrac{\kappa q}{|\kappa q|}\left[h_3(r)\xi^{(1)}_{jm}+h_4(r)\xi^{(2)}_{jm}\right]\end{bmatrix}, \qquad (8)$$

type B: $j=|q|-\frac{1}{2}$,

$$\psi=\frac{1}{r}\begin{bmatrix}\dfrac{\kappa q}{|\kappa q|}F(r)\eta_m\\ -i\,G(r)\eta_m\end{bmatrix}. \qquad (9)$$

The radial equation for type A is

$$\begin{bmatrix}-\partial_r+\lambda r^{-1} & 0 & \kappa q(2Mr^2)^{-1} & M+E\\ 0 & -\partial_r-\lambda r^{-1} & M+E & \kappa q(2Mr^2)^{-1}\\ \kappa q(2Mr^2)^{-1} & M-E & -\partial r+\lambda r^{-1} & 0\\ M-E & \kappa q(2Mr^2)^{-1} & 0 & -\partial_r-\lambda r^{-1}\end{bmatrix}\begin{bmatrix}h_1\\ h_2\\ -\dfrac{\kappa q}{|\kappa q|}h_3\\ -\dfrac{\kappa q}{|\kappa q|}h_4\end{bmatrix}=0, \qquad (10)$$

where

$$\lambda\equiv[(j+\tfrac{1}{2})^2-q^2]^{1/2}. \qquad (11)$$

The boundary condition for bound states is that $h_i(r)$ $(i=1,2,3,4)$ vanish at both $r=0$ and $r=\infty$. In general this is a complicated eigenvalue problem and the full spectrum has not been obtained. However, for $E=0$, Eq. (10) decouples into two identical sets of two-component equations, which can be solved exactly to give a bound state with wave functions (up to an overall constant),

$$h_1=-h_3=\left(\frac{r}{2}\right)^{1/2}\exp\left(-\frac{|\kappa q|}{2Mr}\right)K_{\lambda-1/2}(Mr),$$
$$h_2=-h_4=-\frac{q}{|q|}\left(\frac{r}{2}\right)^{1/2}\exp\left(-\frac{|\kappa q|}{2Mr}\right)K_{\lambda+1/2}(Mr). \qquad (12)$$

$K_\nu(x)$ here is the familiar modified Bessel function of order ν. This bound state is nondegenerate. The $\kappa\to 0$ limit does not give a bound state since then the singularity at the origin of the K function makes the solution unnormalizable.

For type-B states the radial equation is

$$\frac{dF}{dr}=\left[\frac{\kappa}{|\kappa|}(E+M)-\frac{|\kappa q|}{2Mr^2}\right]G,$$
$$\frac{dG}{dr}=\left[-\frac{\kappa}{|\kappa|}(E-M)-\frac{|\kappa q|}{2Mr^2}\right]F. \qquad (13)$$

Upon substituting

$$r=\frac{|\kappa q|}{2M}\rho,\quad A=\frac{|\kappa q|}{2},\quad B=\frac{|\kappa q|}{2}\frac{E}{M}, \qquad (14)$$

(13) simplifies to

$$\frac{dF}{d\rho}=\left(A+B-\frac{1}{\rho^2}\right)G,$$
$$\frac{dG}{d\rho}=\left(A-B-\frac{1}{\rho^2}\right)F. \qquad (15)$$

This set of equations, although not soluble in closed form in general, was analyzed in Ref. 1 and countably infinite bound states were found as mentioned in the Introduction. Again $E=0$ [i.e., $B=0$ in (15)] is a special case for which bound-state radial functions may be explicitly exhibited. They are, up to an overall constant,

$$F=-G=\frac{1}{\sqrt{2}}\exp\left(-\frac{|\kappa q|}{2Mr}-Mr\right). \quad (16)$$

III. ELECTRIC DIPOLE MOMENT OF THE BOUND STATES

Having summarized the necessary ingredients, we shall now compute the electric dipole moment of the bound states discussed above.

The electric dipole moment operator for the system is given, as usual, by

$$\vec{d}=e\vec{r}. \quad (17)$$

By angular momentum conservation, the only nonvanishing component of $\vec{d}$ for a state with definite angular momentum is the z component

$$\langle d_z\rangle_{jm}=\int d^3r\,\psi^*_{jm}ez\psi_{jm}\Big/\int d^3r\,\psi^*_{jm}\psi_{jm}. \quad (18)$$

We shall first compute this for the simpler of the two types of states, i.e., for the type-B bound states, somewhat in detail since the calculation for the type-A states as well as the evaluation of the magnetic dipole moment in the following section requires only more involved algebra of the same kind.

A. Type-B bound states

Substituting (9) into (18) we get (omitting the subscript j)

$$\langle d_z\rangle_m=e\int \eta^\dagger_m\eta_m\cos\theta\,d\Omega \times\int_0^\infty r(F^2+G^2)dr\Big/\int_0^\infty (F^2+G^2)dr. \quad (19)$$

The azimuthal-quantum-number dependence may be extracted from the Wigner-Eckart theorem, i.e.,

$$\langle jm\,|\cos\theta|jm\rangle=\langle jj\,|\cos\theta|jj\rangle C(jm\,|10jm)/C(jj\,|10jj) =\frac{m}{j}\langle jj\,|\cos\theta|jj\rangle, \quad (20)$$

where $C(jm\,|j_1m_1j_2m_2)$ is the Clebsch-Gordan coefficients and we have used

$$C(jm\,|10jm)=-m[j(j+1)]^{-1/2}.$$

For $j=m=|q|-\frac{1}{2}$, the two-component eigenfunction η_m takes the form

$$\eta_{|q|-1/2}=\begin{bmatrix}-\left(\frac{1}{2|q|+1}\right)^{1/2}Y_{q,|q|,|q|-1}\\ \left(\frac{2|q|}{2|q|+1}\right)^{1/2}Y_{q,|q|,|q|}\end{bmatrix}, \quad (21)$$

while $\cos\theta$ is conveniently expressed in terms of an ordinary spherical harmonics, which in our notation is

$$\cos\theta=(4\pi/3)^{1/2}Y_{010}. \quad (22)$$

Thus with (21) and (22) the reduced angular integral becomes

$$\int\eta^\dagger_j\eta_j\cos\theta\,d\Omega =(4\pi/3)^{1/2}(2|q|+1)^{-1} \times\int(Y^*_{q,|q|,|q|-1}Y_{010}Y_{q,|q|,|q|-1} +2|q|\,Y^*_{q,|q|,|q|}Y_{010}Y_{q,|q|,|q|})\,d\Omega. \quad (23)$$

The integral of the triple product of monopole harmonics can be computed using the following properties[7] of monopole harmonics:

$$Y^*_{q,l,m}=(-1)^{q+m}Y_{-q,l,-m}, \quad (24)$$

$$\int Y_{q,l,m}Y_{q',l',m'}Y_{q'',l'',m''}d\Omega =\left[\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}\right]^{1/2} \times\begin{pmatrix}l & l' & l''\\ m & m' & m''\end{pmatrix}\begin{pmatrix}l & l' & l''\\ q & q' & q''\end{pmatrix}(-1)^{l+l'+l''}, \quad (25)$$

where the quantity expressed by the large parentheses is the Wigner $3j$ symbol. Thus (23) becomes

$$\int\eta^\dagger_j\eta_j\cos\theta\,d\Omega=(-1)^{3|q|+q}\begin{pmatrix}|q| & |q| & 1\\ q & -q & 0\end{pmatrix} \times\left[\begin{pmatrix}|q| & |q| & 1\\ |q|-1 & 1-|q| & 0\end{pmatrix} -2|q|\begin{pmatrix}|q| & |q| & 1\\ |q| & -|q| & 0\end{pmatrix}\right]. \quad (26)$$

Finally the explicit evaluation of the $3j$ symbols yields

$$\int\eta^\dagger_j\eta_j\cos\theta\,d\Omega=-\frac{q}{|q|}\,\frac{|q|-\frac{1}{2}}{|q|+\frac{1}{2}}. \quad (27)$$

Combining (19), (20) and (27), we obtain

$$\langle d_z\rangle_m=-\frac{q}{|q|}\,\frac{m}{|q|+\frac{1}{2}}e\int_0^\infty(F^2+G^2)r\,dr\Big/\int_0^\infty(F^2+G^2)dr. \quad (28)$$

Before we discuss the radial integrals appearing in (28), it is worth pointing out that the kinematical factor obtained above may be understood by the following simple argument.

Since the only available vector characterizing the system is the total angular momentum $\vec{J}$, the dipole moment $\vec{d}$ must be parallel to $\vec{J}$ with a constant of proportionality γ; γ may not depend on m.

To obtain γ, we form d^2 in two different ways:

$$d^2=\vec{d}\cdot\gamma\vec{J}=\gamma e\vec{r}\cdot\vec{J}, \tag{29a}$$

$$d^2=\gamma^2 J^2. \tag{29b}$$

From this we obtain

$$\vec{d}=\frac{e\vec{J}}{J^2}\vec{r}\cdot\vec{J}. \tag{30}$$

These operator manipulations are valid if we deal with their simultaneous eigenstates, for which they are c numbers. The type-B states are just such states since, from (2),

$$\vec{r}\cdot\vec{J}=r(-q+\tfrac{1}{2}\vec{\sigma}\cdot\hat{r}) \tag{31}$$

and η_m is an eigenstate of $\vec{\sigma}\cdot\hat{r}$ with the eigenvalue $q/|q|$. Thus for such states the kinematical factor for d_z takes the form

$$\frac{em}{j(j+1)}\left(-q+\frac{1}{2}\,\frac{q}{|q|}\right),$$

which, with $j=|q|-\frac{1}{2}$, is precisely

$$-\frac{q}{|q|}\,\frac{m}{|q|+\frac{1}{2}}e.$$

Let us now discuss the radial integrals for the $E=0$ and $E\neq 0$ bound states separately.

Type-B, $E=0$ state. For this case the radial functions are explicitly given by (16). The two integrals to be computed are

$$I_1\equiv\int_0^\infty(F^2+G^2)dr=\int_0^\infty\exp\left(-\frac{|\kappa q|}{2mr}-mr\right)dr, \tag{32}$$

$$I_2\equiv\int_0^\infty(F^2+G^2)r\,dr=\int_0^\infty\exp\left(-\frac{|\kappa q|}{2mr}-mr\right)r\,dr. \tag{33}$$

They are easily evaluated and give the ratio

$$I_2/I_1=\frac{1}{M}\left(\frac{|\kappa q|}{2}\right)^{1/2}K_2(2(2|\kappa q|)^{1/2})/K_1(2(2|\kappa q|)^{1/2}). \tag{34}$$

Thus for this state the complete expression for the electric dipole moment becomes

$$\langle d_z\rangle_m=-\frac{q}{|q|}\,\frac{m}{|q|+\frac{1}{2}}\,\frac{e}{M}\left(\frac{|\kappa q|}{2}\right)^{1/2}\times K_2(2(2|\kappa q|)^{1/2})/K_1(2(2|\kappa q|)^{1/2}). \tag{35}$$

From the asymptotic form of the K functions, large- and small-$|\kappa q|$ limits are easily obtained:

$$\langle d_z\rangle_m\underset{|\kappa q|\to 0}{\sim}-\frac{q}{|q|}\,\frac{m}{|q|+\frac{1}{2}}\,\frac{e}{2M}\times\left(1+4|\kappa q|\ln\frac{1}{|\kappa q|}+\cdots\right), \tag{36}$$

$$\langle d_z\rangle_m\underset{|\kappa q|\to\infty}{\sim}-\frac{q}{|q|}\,\frac{m}{|q|+\frac{1}{2}}\,\frac{e}{M}\left(\frac{|\kappa q|}{2}\right)^{1/2}. \tag{37}$$

In words, as $|\kappa q|$ approaches zero, the relative contribution from around the origin increases owing to the increase of the factor $\exp[-|\kappa q|(2Mr)^{-1}]$ and the dipole moment decreases in magnitude to a nonvanishing value of $-(q|q|)m(|q|+\frac{1}{2})^{-1}e/(2M)$. On the other hand, as $|\kappa q|$ becomes large, the major contribution comes from further and further away from the origin and the moment increases in magnitude indefinitely like $|\kappa q|^{1/2}$.

Type-B, $E\neq 0$ states. For these states, unfortunately, it is difficult to obtain the radial functions even in an approximate manner. In Ref. 1 only the energy levels were obtained for several asymptotic cases.

It turns out, however, that the dominant term can be computed in the limit of very loosely bound states with fixed $|\kappa q|$ without recourse to the full wave functions. This is the most interesting limit as one expects a large moment for such large (in size) bound states.

Since the energy levels are symmetric about $E=0$, we shall concentrate on the positive-energy bound states labeled by a positive integer n such that

$$0<E_1<E_2<\cdots<E_n<\cdots<M. \tag{38}$$

It was found in Ref. 1 that for large n, with $|\kappa q|$ fixed,

$$1-\frac{E_n}{M}\sim\text{const}\times\exp\left[\frac{-4\pi n}{(4|\kappa q|-1)^{1/2}}\right], \tag{39}$$

where the constant in front is of order unity.

Our aim is to compute

$$\langle r\rangle\equiv\int_0^\infty(F^2+G^2)r\,dr\Big/\int_0^\infty(F^2+G^2)dr \tag{40}$$

in the limit $n\to\infty$, i.e., $E_n\to M$. For this purpose, it is convenient to employ the substitution defined in (14). Then we have

$$\langle r\rangle=\frac{|\kappa q|}{2M}\langle\rho\rangle, \tag{41}$$

where

$$\langle\rho\rangle\equiv\int_0^\infty(F^2+G^2)\rho\,d\rho\Big/\int_0^\infty(F^2+G^2)d\rho \tag{42}$$

is dimensionless. In terms of the dimensionless positive parameters A and B in (14), the large-n limit corresponds to the limit $B\to A$.

Let $u\equiv F+G$ and $v\equiv F-G$. Then Eq. (15) becomes

$$\frac{du}{d\rho}=\left(A-\frac{1}{\rho^2}\right)u-Bv\,,$$
$$\frac{dv}{d\rho}=-\left(A-\frac{1}{\rho^2}\right)v+Bu\,. \tag{43}$$

Now fix a number $R\gg A^{-1/2}$ and consider the region where $\rho\geq R$. For such a region, $1/\rho^2$ may be neglected compared with A and we readily find

$$F\sim\frac{1}{2}\left\{1+\frac{A}{B}+\left[\left(\frac{A}{B}\right)^2-1\right]^{1/2}\right\}\exp[-(A^2-B^2)^{1/2}\rho]\;, \tag{44a}$$

$$G\sim\frac{1}{2}\left\{1-\frac{A}{B}-\left[\left(\frac{A}{B}\right)^2-1\right]^{1/2}\right\}\exp[-(A^2-B^2)^{1/2}\rho]. \tag{44b}$$

Thus for large ρ the wave function describes simply an exponential tail, which becomes longer and longer as B approaches A. Moreover, notice that, in this limit, G is negligible compared with F.

We shall now normalize the radial functions such that

$$\int_0^\infty(F^2+G^2)d\rho=1\,, \tag{45}$$

and consider

$$k\equiv\epsilon\int_0^\infty\rho(F^2+G^2)d\rho\,, \tag{46}$$

where

$$\epsilon=2(A^2-B^2)^{1/2}$$
$$=|\kappa q|\left[1-\left(\frac{E_n}{M}\right)^2\right]^{1/2}. \tag{47}$$

k may be split into two terms:

$$k=\epsilon\int_R^\infty\rho(F^2+G^2)d\rho+\epsilon\int_0^R\rho(F^2+G^2)d\rho\,. \tag{48}$$

The second term is, by virtue of (45), at most of order ϵR. For the first term we may use (44) in the form

$$F=fe^{-\epsilon\rho/2}\,,$$
$$G=ge^{-\epsilon\rho/2}\,, \tag{49}$$
$$f,g=\text{const}\,.$$

Then k can be written as

$$k=-\int_R^\infty\rho(f^2+g^2)\frac{d}{d\rho}e^{-\epsilon\rho}d\rho+O(\epsilon R)\,,$$

which upon integration by parts becomes

$$\int_R^\infty(F^2+G^2)d\rho+R(F^2+G^2)(R)+O(\epsilon R)\,. \tag{50}$$

To estimate $(F^2+G^2)(R)$, note that, from (45) and (49),

$$1\geq\int_R^\infty(F^2+G^2)d\rho=(F^2+G^2)(R)e^{\epsilon R}\int_R^\infty e^{-\epsilon\rho}d\rho$$
$$=(F^2+G^2)(R)/\epsilon\,.$$

Thus $R(F^2+G^2)(R)\leq\epsilon R$. Therefore we obtain

$$k=\int_R^\infty(F^2+G^2)d\rho+O(2\epsilon R)\,.$$

Passing to the limit $\epsilon\to0$ (i.e., $B\to A$), we find

$$\lim_{B\to A}k=\lim_{B\to A}\int_R^\infty(F^2+G^2)d\rho\,. \tag{51}$$

This means that in this limit, the contribution is entirely due to the long tail. But in such a case we may explicitly compute $\langle\rho\rangle$, i.e.,

$$\langle\rho\rangle=\int_R^\infty\rho e^{-\epsilon\rho}d\rho\bigg/\int_R^\infty e^{-\epsilon\rho}d\rho=\frac{1}{\epsilon}+R\,. \tag{52}$$

An alternative form is, by multiplying by ϵ,

$$\lim_{B\to A}k=\lim_{B\to A}\int_R^\infty(F^2+G^2)d\rho=1\,. \tag{53}$$

Combining (39), (41), (47), and (51), we finally obtain the leading term for $\langle r\rangle$ in the limit $n\to\infty$ to be

$$\langle r\rangle\underset{n\to\infty}{\sim}\text{const}\times\frac{1}{2M}\exp\left[\frac{2\pi n}{(4|\kappa q|-1)^{1/2}}\right]\,, \tag{54}$$

where the constant is of order 1. It is interesting to compare this expression of the mean radius with that for a hydrogen atom. For the latter, neglecting relativistic corrections, $\langle r\rangle$ is given by

$$\langle r\rangle=\tfrac{1}{2}[3n^2-l(l+1)]\;.$$

Thus there exists a sharp contrast between these two types of binding.

B. Type-A bound states

The evaluation proceeds in a similar manner. Upon substituting the form (8) into (18) with the definitions of $\xi_{jm}^{(1)}$ and $\xi_{jm}^{(2)}$ given by (3), (4), and (5), one obtains, after some algebra, the following expression for the electric-dipole moment:

$$\langle d_z\rangle_{jm}=-\frac{mq}{j(j+1)}\,e\left(\alpha+\frac{1}{2|q|}\beta\right),$$

with

$$\alpha=\frac{1}{N}\int_0^\infty dr\,r(|h_1|^2+|h_2|^2+|h_3|^2+|h_4|^2)\,,$$
$$\beta=\frac{1}{N}\int_0^\infty dr\,r(h_1^*h_2+h_2^*h_1+h_3^*h_4+h_4^*h_3)\,, \tag{55}$$

where

$$N=\int_0^\infty dr(|h_1|^2+|h_2|^2+|h_3|^2+|h_4|^2)\,.$$

For the only known type-A bound state, with $E=0$, the radial integrals take the form [see Eq. (12)]

$$\alpha=\frac{1}{N}\int_0^\infty dr\, r\exp\left(-\frac{|\kappa q|}{Mr}\right)\{[K_{\lambda+1/2}(Mr)]^2+[K_{\lambda-1/2}(Mr)]^2\},\tag{56a}$$

$$\beta=\frac{2}{N}\int_0^\infty dr\, r\exp\left(-\frac{|\kappa q|}{Mr}\right)K_{\lambda+1/2}(Mr)K_{\lambda-1/2}(Mr),\tag{56b}$$

with

$$N=\int_0^\infty dr\exp\left(-\frac{|\kappa q|}{Mr}\right)\{[K_{\lambda+1/2}(Mr)]^2+[K_{\lambda-1/2}(Mr)]^2\}.\tag{56c}$$

When λ [defined in (11)] takes an integer value, for example for $(\lambda,j,q)=(2,2,3/2)$, $(4,8,15/2)$, etc., these integrals can be explicitly evaluated in closed forms. One obtains

$$\alpha=\frac{1}{D}\frac{1}{M}\left\{\sum_{l=0}^{2\lambda}[f_\lambda(l)+f_{\lambda-1}(l)]Z^{1-l}K_{l-1}(Z)\right\},\tag{57a}$$

$$\beta=\frac{1}{D}\frac{1}{M}\left\{\sum_{l=0}^{2\lambda-2}[f_\lambda(l)+f_{\lambda-1}(l)]Z^{1-l}K_{l-1}(Z),+2[(2\lambda-1)!!]^2Z^{2-2\lambda}K_{2\lambda-2}(Z)\right\},\tag{57b}$$

with

$$D\equiv\sum_{l=0}^{2\lambda}[f_\lambda(l)+f_{\lambda-1}(l)]Z^{-l}K_l(Z),$$

$$Z\equiv\left(\frac{|\kappa q|}{2}\right)^{1/2},\tag{57c}$$

and $f_\lambda(l)$ is a number defined by

$$f_\lambda(l)\equiv l!\,2^{-l}\sum_{k=0}^{l}\binom{l}{k}\binom{\lambda+k}{l}\binom{\lambda-k+l}{l}.$$

In the asymptotic region of small and large $|\kappa q|$, α and β behave like

$$\alpha\underset{|\kappa q|\to 0}{\sim}\frac{1}{2M}\frac{|\kappa q|}{\lambda-1}\quad(\lambda\geq 2),\tag{58}$$

$$\beta\underset{|\kappa q|\to 0}{\sim}\begin{cases}\dfrac{3}{4M}\dfrac{1}{\ln(1/|\kappa q|)} & (\lambda=2),\\[2mm] \dfrac{1}{2M}\dfrac{|\kappa q|}{\lambda-3/2} & (\lambda\geq 3),\end{cases}\tag{59}$$

$$\left.\begin{aligned}\alpha&\underset{|\kappa q|\to\infty}{\sim}\frac{1}{M}\left(\frac{|\kappa q|}{2}\right)^{1/2}\\ \beta&\underset{|\kappa q|\to\infty}{\sim}\frac{1}{M}\left(\frac{|\kappa q|}{2}\right)^{1/2}\end{aligned}\right\}\ \text{independent of }\lambda.\tag{60}$$

Notice that the moment vanishes as $|\kappa q|\to 0$ in contrast with the type-B bound state with $E=0$ as described before. This is due to the divergent character of the K function at the origin. As $|\kappa q|$ becomes small, the factor $\exp[-|\kappa q|(2Mr)^{-1}]$ in the wave function which suppresses the divergence becomes weaker and the probability density becomes highly concentrated around the origin. Hence the vanishing of the moment in this limit.

On the other hand, for large $|\kappa q|$ the same factor wipes out the order dependent part of $K_\nu(x)$, leaving only the exponential tail, common to all orders, as the sole contributor to the moment.

IV. MAGNETIC DIPOLE MOMENT OF THE BOUND STATES

The magnetic-moment operator of the system is given by

$$\vec{\mu}=\vec{\mu}_0+\vec{\mu}_1,\quad \vec{\mu}_0=\frac{e}{2}\vec{r}\times\vec{\alpha},\quad \vec{\mu}_1=\frac{\kappa e}{2M}\beta\vec{\sigma}.\tag{61}$$

One can easily verify that $\vec{\mu}_0$ and $\vec{\mu}_1$ are both axial-vector operators even under the presence of the monopole. Thus again μ_z is the only nonvanishing component.

The evaluations of μ_{0z} and μ_{1z} are similar to those of the electric-dipole moment. We shall quote only the results and discuss their salient features.

Type-B states:

$$\langle\mu_{0z}\rangle_m=0,\tag{62}$$

$$\langle\mu_{1z}\rangle_m=-\frac{\kappa e}{2M}\frac{m}{|q|+\frac{1}{2}}\int_0^\infty(F^2-G^2)dr\bigg/\int_0^\infty(F^2+G^2)dr.\tag{63}$$

Type-A states:

$$\langle\mu_{0z}\rangle_{jm}=\frac{\kappa e\lambda}{2|\kappa|}\frac{m}{j(j+1)}\int_0^\infty r\,dr(h_1^*h_4-h_2^*h_3)\bigg/N,\tag{64}$$

$$\langle\mu_{1z}\rangle_{jm}=\frac{\kappa e}{2M}\frac{m}{j(j+1)}\left\{\int_0^\infty dr[(\tfrac{1}{2}+\lambda)(|h_1|^2-|h_3|^2)+(\tfrac{1}{2}-\lambda)(|h_2|^2-|h_4|^2)]+2|q|\int_0^\infty dr(h_1^*h_2-h_3^*h_4)\right\}\bigg/N,\tag{65}$$

where N is defined in (55).

(1) First, notice that for $E=0$ bound states (both type A and type B), the magnetic dipole moment vanishes identically since the integrands all vanish

[see (12) and (16)]. This is due to the property of these states under the "charge-conjugation" *operation* defined by

$$C=\gamma_2 K,$$

where K is the complex conjugation operator. This unitary operation is *not* a symmetry of the Hamiltonian. Indeed one finds

$$CHC^\dagger = -H.$$

By direct calculation, the magnetic-moment operator is found to change sign under this operation, i.e.,

$$C\vec{\mu}C^\dagger = -\vec{\mu}.$$

Now observe that if ψ is a state with $E=0$, $C\psi$ is also an eigenstate with $E=0$ since

$$HC\psi = -C^\dagger H\psi = 0.$$

Nondegeneracy of the $E=0$ bound state for each j then dictates that $C\psi$ and ψ differ at most by a constant phase factor ξ. Thus

$$\begin{aligned}\langle\psi|\vec{\mu}|\psi\rangle &= \langle C\psi|C\vec{\mu}C^\dagger|C\psi\rangle\\ &= -\xi\xi^*\langle\psi|\vec{\mu}|\psi\rangle\\ &= -\langle\psi|\vec{\mu}|\psi\rangle\\ &= 0.\end{aligned}$$

(2) Next, the vanishing of $\vec{\mu}_0$ for the type-B states (62) may be understood as follows. Since there exists only one two-component angular eigensection η_m for $j=|q|-\frac{1}{2}$, the two cross terms produced by the matrix $\vec{\alpha}$ in $\vec{\mu}_0$ have the identical angular structure, i.e., $\eta_m^\dagger(\hat{r}\times\vec{\sigma})\eta_m$. Recall that η_m is an eigenfunction of $\vec{\sigma}\cdot\hat{r}$ with the eigenvalue $q/|q|$. It is easily checked that $\vec{\sigma}\cdot\hat{r}$ anticommutes with $\hat{r}\times\vec{\sigma}$. Therefore $\vec{\mu}_0$ must vanish for the type-B bound states.

(3) Can we estimate μ_{1z} for weakly bound type-B states? As was discussed in Sec. III, in the limit of large n the contribution to the normalization integral is dominated entirely by the exponential tail of the wave functions, for which $\lim_{n\to\infty}G^2/F^2=0$. Therefore the ratio of the integrals in (63) approaches 1 in this limit and one obtains the following result: In the limit of very loosely bound states with $j=|q|-\frac{1}{2}$, the magnetic moment of the system approaches a value proportional to the extra magnetic moment of the charged particle

$$\lim_{n\to\infty}\langle\mu_z\rangle = -\frac{m}{|q|+\frac{1}{2}}\frac{\kappa e}{2M}. \tag{66}$$

V. SUMMARY

The violation of both parity and time-reversal invariance for the system of a charged Dirac particle, with an extra magnetic moment $\kappa e/2M$, bound in the field of a fixed magnetic monopole allows it to have a nonvanishing electric dipole moment.

For the bound states with $E=0$, the moment was evaluated using the explicit wave functions available for these states. It was found that for all these states the moment increases like $|\kappa q|^{1/2}$ as $|\kappa q|$ tends to infinity, while in the small $|\kappa q|$ limit the moment vanishes. An exception occurs for the state with the lowest angular momentum, for which the $\kappa\to 0$ limit still yields a bound state. For this state the moment approaches a finite value as $|\kappa q|$ tends to zero.

For the infinite number of bound states with $j=j_{\min}$, $E=E_n\neq 0$, for which the wave functions are not known, the leading term for the electric-dipole moment was obtained in the weak-binding (i.e., the large-n) limit by relating it to the corresponding asymptotic energy spectrum of the system. The moment is found to increase in its magnitude exponentially in the principal quantum number n.

The magnetic dipole moment was evaluated for these bound states in a similar manner. Owing to the special property under the charge-conjugation operation, the magnetic moment vanishes identically for the states with $E=0$. For the case of loosely bound states with $j=j_{\min}$ mentioned above, the magnetic moment approaches a finite limiting value proportional to the extra magnetic moment of the charged particle as $n\to\infty$.

The results summarized above would undoubtedly be subject to quantum-electrodynamical corrections. Lacking a consistent theory of QED with monopoles, however, the assessment of the nature and the degree of modification is difficult at the moment.

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† Present address: Department of Physics, University of Michigan, Ann Arbor, Michigan 48109.

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[3]We choose units so that $c=\hbar=1$. e stands for the charge of the Dirac particle, g for the monopole strength. The product $q\equiv eg$ is quantized according to Dirac and takes integer or half-integer values. Extra magnetic moment of the charged particle is denoted by $\kappa e/2M$, where M is the mass of the particle.

[4]Charge-conjugation *symmetry*, in the absence of the second-quantized theory of electromagnetism with monopoles, may not be properly defined. Nevertheless one may define an *operation* which may be called a charge-conjugation operation (see Sec. IV).

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