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Spectra of fermions in monopole fields—Exactly soluble models*

Constantine J. Callias

Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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We investigate the structure of the energy spectrum of an isospin-1/2 Dirac particle in the field of the SU(2) magnetic monopole of 't Hooft and Polyakov. We show that aside from the zero-energy mode, which is always present, there are at most a finite number of bound states. To clarify the interaction of the fermion with the various components of the monopole field, we consider two different extrapolations of the background field to limiting forms. The corresponding Dirac equations turn out to be exactly soluble. In the first limiting model, only the Higgs field is retained, and the Dirac equation is found to be equivalent to the nonrelativistic Coulomb problem. The second model is just the point monopole, and our problem is equivalent to a doublet of massive Dirac particles interacting with an Abelian magnetic monopole. This classical problem admits a simple treatment in the context of non-Abelian gauge theories; we present its solution in this formulation; we point out the hitherto unnoticed fact that the Hamiltonian is not self-adjoint on the customary domain of nonsingular wave functions and we study its self-adjoint extensions and bound states.

I. INTRODUCTION

Dirac equations in the background field of classical solutions to non-Abelian gauge theories have been investigated in a number of different examples.[1] They exhibit an interesting mathematical property with intriguing physical consequences. Whenever the background field has "interesting topological characteristics" a zero-energy mode has been found to be invariably imposed on the corresponding Dirac equation. If, moreover, this zero-energy mode is nondegenerate, the coupling of the background non-Abelian gauge fields to the fermionic field leads to the assignment of a fermion number of $\frac{1}{2}$ to their classical solutions, suitably interpreted in the quantum theory.

In this paper we shall be concerned with isospinor fermion fields in the presence of an SU(2) magnetic monopole. The zero-energy modes of the associated Dirac equation have been investigated by Jackiw and Rebbi.[2] The finite-mass monopole background field of 't Hooft[3] and Polyakov[4] leads to a nondegenerate mode. A more in-depth analysis of this phenomenon, however, or the possibility of use of these models in calculations of other effects, necessitates an investigation of the entire spectrum and not only of the zero-energy solutions. We would like to know, in particular, whether the rest of the bound-state spectrum is discrete or whether it exhibits any pathological features.[5] Questions of this sort we undertake to answer in this work.

A suitable formalism, including a partial-wave analysis, for our problem has been given by Jackiw and Rebbi.[2] It is parametrized by two functions $A(r)$ and $\Phi(r)$, giving the strength of the gauge and the Higgs fields, respectively. These functions can be obtained by numerical integration of ordinary differential equations deduced from the SU(2) gauge field equations.[3] However, they are not known in closed form. At least partly for this reason, the problem with $A(r)$ and $\Phi(r)$ corresponding to a finite-mass monopole cannot be attacked directly. Our approach is rather to deform these functions to concrete expressions, which, it will turn out, lead to exactly soluble Dirac equations. These limiting cases are then of interest in themselves, quite aside from the fact that they give us information about how fermions interact with the various components of the non-Abelian monopole. The qualitative conclusions that we draw from this information we subsequently substantiate by a direct rigorous study of the original problem.

In the first limit, only the Higgs field part is retained in the potential. This choice is the simplest one containing all the topological structure of the monopole solutions. In this case we find that the problem is exactly soluble in terms of the nonrelativistic Coulomb problem. The Hamiltonian does not exhibit any unusual features and the zero-energy mode is nondegenerate as usual. We discuss this model in Sec. III.

The other limit we consider is actually another monopole solution of any SU(2) gauge theory coupled to a triplet of Higgs fields, with only one additional δ-function source for the gauge fields. This limiting case is also equivalent to the ordinary Abelian monopole solution of the Maxwell theory for a point magnetic charge, which has been studied extensively in the literature. As a solution of a non-Abelian gauge field theory it has a certain model-independent character, in that it depends only on a certain combination of the parameters characterizing the Higgs field (mass, self-coupling). The Dirac Hamiltonian in this field has the singular feature that it is not essentially self-adjoint on the customary domain of infinitely differentiable functions of compact support. Only one self-adjoint extension will be found to possess a zero-energy mode, and, moreover, we get the interesting result that the latter is doubly degenerate, unlike the general case. The analysis of these phenomena is the main subject of Sec. IV.

In the last section we comment on what we learn from these examples about the more general case and then proceed to derive upper bounds on the number of bound states for each partial wave. We conclude that the total number of bound states is finite and that therefore the bound-state spectrum is discrete.

As a preparation for the discussion we summarize, in Sec. II, the monopole solutions relevant to our calculations and we clarify their relation to our model potentials for the Dirac equation. We review briefly the Jackiw-Rebbi formalism, which will be used extensively throughout the rest of this paper, and we discuss the zero-energy solution found by these authors.

II. REVIEW OF MONOPOLE SOLUTIONS AND FORMALISM

The class of models we shall be concerned with is described by a Lagrangian $\mathcal{L}=\mathcal{L}_B+\mathcal{L}_F$, consisting of a bosonic term and a fermionic term. The former term describes a spontaneously broken SU(2) gauge-invariant system of a triplet (isovector) of scalar fields coupled minimally to a triplet of vector gauge fields, in Minkowski space of three space and one time dimensions[3,4]:

$$\mathcal{L}_B=-\tfrac{1}{4}F_a^{\mu\nu}F_{a\mu\nu}+\tfrac{1}{2}(D_\mu\Phi)_a(D^\mu\Phi)_a -\frac{1}{g^2}U(g^2|\Phi|^2)\,, \tag{2.1}$$

where

$$F_a^{\mu\nu}=\partial^\mu A_a^\nu-\partial^\nu A_a^\mu+g\epsilon_{abc}A_b^\mu A_c^\nu\,, \tag{2.2}$$

$$(D^\mu\Phi)_a=\partial^\mu\Phi_a+g\epsilon_{abc}A_b^\mu\Phi_c\,. \tag{2.3}$$

$U(|\Phi|^2)$ is required to have a minimum at a nonzero value of its argument, say F^2. For the fermionic part we pick a model already considered by Jackiw and Rebbi[2] as one of their examples: a doublet (isospinor) of Dirac fields gauge invariantly coupled to A_a^μ and Φ_a:

$$\mathcal{L}_F=i\bar\psi_n\gamma^\mu(D_\mu\psi)_n-Gg\bar\psi_n\tau_{nm}^a\psi_m\Phi_a\,, \tag{2.4}$$

where

$$(D^\mu\psi)_n=\partial^\mu\psi_n-ig\tau_{nm}^aA_a^\mu\psi_m \tag{2.5}$$

and $\tau^a=\frac{1}{2}\sigma^a$, where σ^a are the Pauli matrices. G is a dimensionless constant. The τ^a obviously play the role of the generators of global gauge transformations on the Fermi fields.

The fact that the Fermi fields appear only quadratically in the Lagrangian implies that their effects in classical solutions come into play in an order in g higher than the lowest. Our approach will be, therefore, to first find solutions of the problem $\mathcal{L}=\mathcal{L}_B$ and then consider the equations governing the fermions in the background field of these solutions, ignoring the feedback. It will then turn out that the static (zero-energy) solution to the Dirac equation is such that it gives no contribution as a source of the gauge and Higgs fields, and therefore it forms, together with the background fields, an exact solution of the coupled system $\mathcal{L}=\mathcal{L}_B+\mathcal{L}_F$. This, however, does not occur in the case of nonzero-energy solutions to the Dirac equations.

We now outline the well-known results on classical solutions of the systems (1), and in the light of these solutions we describe the models that we study in detail in the following sections. 't Hooft[3] and Polyakov[4] have deduced the existence of solutions with magnetic charge by making the spherically symmetric ansatz

$$A_a^0=0\,,\qquad A_a^i=\epsilon^{aij}\hat r_j\frac{A(r)}{g}\,, \tag{2.6}$$

$$\Phi_a=\hat r_a\frac{\Phi(r)}{g}\,. \tag{2.7}$$

In terms of the two functions $A(r)$ and $\Phi(r)$ the Lagrangian (1) becomes

$$\mathcal{L}_B=4\pi\int_0^\infty r^2dr\left[-\left(\frac{dA}{dr}\right)^2-\frac{2}{r}A\frac{dA}{dr}-\frac{3}{r^2}A^2-\frac{2g}{r}A^3 -\tfrac{1}{2}g^2A^4-\frac{1}{2}\left(\frac{d\Phi}{dr}\right)^2-\frac{1}{r^2}\Phi^2-\frac{2g}{r}A\Phi^2 -g^2A^2\Phi^2-\frac{1}{g^2}U(g^2|\Phi|^2)\right] \tag{2.8}$$

which leads to the equations of motion

$$\frac{2}{r^2}\frac{d}{dr}\left(r^2\frac{dA}{dr}\right)-\frac{2}{r}\frac{dA}{dr}+\frac{2}{r^2}\frac{d}{dr}(rA)-\frac{6}{r^2}A-\frac{6g}{r}A^2 -2g^2A^3-\Phi\left(\frac{2g}{r}+2g^2A\right)=0\,, \tag{2.9}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right)-\frac{2}{r^2}\Phi-\frac{4g}{r}A\Phi -2g^2A^2\Phi-2U'(g^2|\Phi|^2)\Phi=0\,, \tag{2.10}$$

and boundary conditions

$$\left(r^2\frac{dA}{dr}+2rA\right)_{r=0}=0, \tag{2.11}$$

$$\left(r^2 \frac{d\Phi}{dr}\right)_{r=0} = 0\,. \tag{2.12}$$

Single-valuedness of Φ_a and A_a^i at the origin further requires that $\Phi(0) = A(0) = 0$. The system (9) and (10) has a unique smooth solution satisfying these conditions. At infinity this solution has the behavior

$$A(r) = -\frac{1}{r}\,, \tag{2.13}$$

$$\Phi(r) = F\,. \tag{2.14}$$

In Sec. IV we study the Dirac equation with the exact forms (13) and (14) used for $A(r)$ and $\Phi(r)$. Note that (13) and (14) satisfy Eqs. (9) and (10) exactly. However, they violate the boundary conditions (11) and (12). In fact, they correspond to a δ-function source for the gauge fields at the origin.

Next we note that (13) and (14) become exact in a certain limit for the parameters defining the potential $U(|\Phi|^2)$. This limit is given by $F \to \infty$. It is deduced quite easily from (9), which tells us that (13) becomes approximately correct at a distance $F^{-1/2}$ from the origin. Thus, the Dirac equation we will be considering in Sec. IV can be viewed as a certain kind of limit of equations with meaningful potentials as $F \to \infty$ while $\mu = GF/2$ is kept constant.

The model that we study in Sec. III has no interpretation as a limit of realistic models. It corresponds to a particular case of the ansatz (6)–(7): that with $A(r) \equiv 0$ and $\Phi(r) \equiv$ constant. The asymptotic behavior of the Φ field, however, gives all the topological information[6] contained in the fields, and the latter seems to be the cause of the interesting phenomena associated with the Dirac equation. Clearly, this model is the simplest extrapolation of the asymptotic behavior of Φ_a to all space. The abnormally rotated vacuum, which was used by 't Hooft[3] and Polyakov only as a boundary condition at infinity, is now introduced at all points of space, and one may expect high symmetry in phenomena associated with such a configuration. We will in fact see that in the case of the Dirac equation we do obtain nontrivial symmetry, as much as is associated with the Coulomb problem.

As a preparation for the analysis of Secs. III and IV we now outline the formalism of Jackiw and Rebbi[2] for dealing with the problem (4). The Dirac equation deduced from (4) after substitution of (6) and (7) for the potential is

$$[-i\vec{\alpha}\cdot\vec{\nabla}\delta_{nm} + \tfrac{1}{2}A(r)\sigma^a_{nm}(\vec{\alpha}\times\hat{r})_a + \mu\sigma^a_{nm}\hat{r}_a\beta]\psi_m = E\psi_n\,. \tag{2.15}$$

The indices n, m correspond to isospin and take the values 1, 2. σ^a are the Pauli matrices. The γ matrices are given by $\gamma_i = \beta\alpha_i$ where

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = -i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Finally $\mu = G\Phi/2$ and will be a constant in Secs. III and IV.

The Hamiltonian in (15) commutes with the sum of ordinary angular momentum (orbital and spin) and isospin. To take advantage of this conservation law one proceeds with the following transformation. Separate ψ into upper and lower components, as is usually done with the Dirac equation:

$$\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}.$$

Each of the $\psi^\pm$ has four components, $\psi^\pm_{im}$, where $i, m = 1, 2$, corresponding to spin and isospin, respectively. Now we can define two scalar and two vector fields, uniquely related to $\psi^\pm$ by the linear relation

$$\psi^\pm_{im} = (g^\pm\delta_{im} + \vec{g}^\pm\cdot\vec{\sigma}_{in})\sigma_{nm}^{\;2}\,. \tag{2.16}$$

The possibility of expressing the problem in terms of scalar and vector functions is a manifestation of the fact that the true spin of our Dirac particle is not $\frac{1}{2}$ but rather 0 or 1: The generator of rotations, which commutes with the Hamiltonian (15), is not $\vec{L} + \vec{S}$ but rather $\vec{L} + \vec{S} + \vec{T}$, the sum of orbital and spin angular momenta *and* isospin; $\vec{S} + \vec{T}$ then plays the role of the true spin.[7] Substitutions of (16) in (15) now leads to the following system of equations:

$$(\vec{D}^\pm - A\hat{r})g^\pm + i\vec{D}^\mp \times \vec{g}^\pm = iE\vec{g}^\mp\,, \tag{2.17a}$$

$$(\vec{D}^\pm + A\hat{r})\cdot\vec{g}^\pm = iEg^\mp\,, \tag{2.17b}$$

where $\vec{D}^\pm = \vec{\nabla} \pm \mu\hat{r}$. We will be using both this form of our Dirac equation and a partial-wave analysis of it. The latter is performed by expanding $g^\pm$ and $\vec{g}^\pm$ into scalar and vector spherical harmonics:

$$g^\pm(\vec{r}) = \sum_{J, J_3} iG^\pm_{JJ_3}(r)Y_{JJ_3}(\Omega)\,, \tag{2.18a}$$

$$\vec{g}^\pm(\vec{r}) = \sum_{J, J_3}\left[P^\pm_{JJ_3}(r)\hat{r}Y_{JJ_3} + B^\pm_{JJ_3}(r)\frac{1}{j}r\vec{\nabla}Y_{JJ_3} + iC^\pm_{JJ_3}(r)\frac{1}{j}\vec{L}Y_{JJ_3}\right], \tag{2.18b}$$

where $j = [J(J+1)]^{1/2}$, $\vec{L} = -i\vec{r}\times\vec{\nabla}$, the ordinary orbital angular momentum operator, and $Y_{JJ_3}(\Omega)$ are the ordinary spherical harmonics. Substitution of (18) in (17) gives the following system of equations:

$$(D^{\pm}-\sigma)G^{\pm}_J-\frac{j}{r}C^{\pm}_J=EP^{\mp}_J,\quad \text{all } J \tag{2.19a}$$

$$(D^{\pm}+\sigma)P^{\pm}_J-\frac{j}{r}B^{\pm}_J=-EG^{\mp}_J,\quad \text{all } J \tag{2.19b}$$

$$D^{\mp}B^{\pm}_J-\frac{j}{r}P^{\pm}_J=EC^{\mp}_J,\quad J\geq 1 \tag{2.19c}$$

$$D^{\mp}C^{\pm}_J-\frac{j}{r}G^{\pm}_J=-EB^{\mp}_J,\quad J\geq 1 \tag{2.19d}$$

where $D^{\pm}=d/dr+1/r\pm\mu$, and $\sigma=1/r+A(r)$.

Finally we remark that the the transformations (16) and (18) on the field variables are unitary. The inner product of two spinors $\psi^{(1)}$ and $\psi^{(2)}$ in the different representations is given by

$$\begin{aligned}(\psi^{(1)},\psi^{(2)})&=\int d^3r\sum_{\pm}(\psi^{(1)\pm}_{im})^*(\psi^{(2)\pm}_{im})\\&=2\int d^3r\sum_{\pm}\left((g^{(1)\pm})^*(g^{(2)\pm})+(\vec{g}^{(1)\pm})^*\cdot(\vec{g}^{(2)\pm})\right)\\&=2\int_0^\infty r^2dr\sum_{J,J_3}^{\pm}[(P^{(1)\pm}_{JJ_3})^*(P^{(2)\pm}_{JJ_3})+(G^{(1)\pm}_{JJ_3})^*(G^{(2)\pm}_{JJ_3})\\&\qquad+(C^{(1)\pm}_{JJ_3})^*(C^{(2)\pm}_{JJ_3})+(B^{(1)\pm}_{JJ_3})^*(B^{(2)\pm}_{JJ_3})].\end{aligned}$$

Yet another representation to be used in Secs. IV and V is an algebraic combination of the harmonic components, defined by

$$\begin{aligned}&\Pi^{\pm}_{JJ_3+}=P^{\pm}_{JJ_3}+B^{\pm}_{JJ_3},\quad \Pi^{\pm}_{JJ_3-}=P^{\pm}_{JJ_3}-B^{\pm}_{JJ_3},\\&\Gamma^{\pm}_{JJ_3+}=G^{\pm}_{JJ_3}+C^{\pm}_{JJ_3},\quad \Gamma^{\pm}_{JJ_3-}=G^{\pm}_{JJ_3}-C^{\pm}_{JJ_3}.\end{aligned}\tag{2.20}$$

The inner product in this representation is given by

$$\begin{aligned}(\psi^{(1)},\psi^{(2)})&=\int_0^\infty r^2dr\sum_{J,J_3}^{\pm}[(\Pi^{(1)\pm}_{JJ_3+})^*(\Pi^{(2)\pm}_{JJ_3+})+(\Pi^{(1)\pm}_{JJ_3-})^*(\Pi^{(2)\pm}_{JJ_3-})\\&\qquad+(\Gamma^{(1)\pm}_{JJ_3+})^*(\Gamma^{(2)\pm}_{JJ_3+})+(\Gamma^{(1)\pm}_{JJ_3-})^*(\Gamma^{(2)}_{JJ_3-})].\end{aligned}$$

This representation diagonalizes the model of Sec. IV. It is also the correct one for deriving upper bounds on the number of bound states of each partial wave—a task which is undertaken in Sec. V.

The zero-energy mode of the Dirac equation. We first exhibit the form of the unique zero-energy eigenstate of the Dirac equation.[2] In terms of the harmonic analysis it is given by

$$P^{\pm}_{JJ_3}=G^{\pm}_{JJ_3}=C^{\pm}_{JJ_3}=B^{\pm}_{JJ_3}=0,\quad \text{for } J>0$$

$$G^{+}_{00}(r)=c\times\exp\left\{\int_0^\infty dr'[A(r')-\tfrac{1}{2}G\Phi(r')]\right\},\tag{2.21}$$

$$G^{-}_{00}(r)=P^{\pm}_{00}=C^{\pm}_{00}=B^{\pm}_{00}=0.$$

In terms of $g^{\pm}(\vec{r})$ and $\vec{g}^{\pm}(\vec{r})$,

$$\begin{aligned}&g^{-}(\vec{r})=0,\quad \vec{g}^{\pm}(\vec{r})=0,\\&g^{+}(\vec{r})=c\times\exp\left\{\int_0^r dr'[A(r')-\tfrac{1}{2}G\Phi(r')]\right\}.\end{aligned}\tag{2.22}$$

Finally, in terms of the original spinors,

$$\begin{aligned}&\psi^{-}_{im}=0,\\&\psi^{+}_{im}=N\left(\exp\left\{\int_0^r dr'[A(r')-\tfrac{1}{2}G\Phi(r')]\right\}\right)\\&\qquad\times(S^{+}_iS^{-}_m-S^{-}_iS^{+}_m),\end{aligned}\tag{2.23}$$

where S^+ (S^-) is the positive- (negative-) eigenvalue eigenvector of σ^3.

From (23) we can easily check the fact mentioned earlier in this section, that although the Dirac equation was solved in a background-field approach, it happens to form, together with the given monopole solution of the pure bosonic part of the system, an exact solution of the coupled boson-fermion problem. In fact, the fermionic sources in the field equations for the gauge and Higgs fields respectively are given by

$$g\bar{\psi}_n\gamma^\mu\tau^a_{nm}\psi_m$$

and

$$-Gg\bar{\psi}_n\tau^a_{nm}\psi_m.$$

Both vanish when the zero-energy eigenstate given above is substituted for ψ, merely on account of the fact that only the upper component of ψ is nonvanishing.

It is easy to see that nonzero-energy bound states have both upper and lower components and do not satisfy the coupled equations in the self-consistent way the zero-energy mode does. Thus, the latter is interpreted as forming part of the soliton, and the Dirac equation (15) describes the fermionic part of its collective excitations. This soliton will retain its meaning and identity as a particle when an external field is introduced only if the spectrum of excitations is discrete in its neighborhood. In the last section we will show that this is indeed the case.

III. AN INTERESTING EQUIVALENCE

In this section we analyze completely the first of our two models, the simplest configuration of background fields with nonvanishing Kronecker index.[6] In (2.6) and (2.7) we choose

$$\begin{aligned}&A(r)=0,\\&\Phi(r)=F,\end{aligned}\tag{3.1}$$

where F is a constant such that $U'(F^2)=0$. We show that the Dirac equation (2.15) is reduced completely to solving two very familiar problems: the nonrelativistic Schrödinger equations for a spinless particle in a Coulomb field, and the nonrelativistic free Schrödinger equation. The bound states are labeled exactly like the bound states of the Coulomb problem; the spectrum is

$$E=\pm\mu\left(1-\frac{1}{n^2}\right)^{1/2},\quad n=1,2,\ldots \tag{3.2}$$

and each eigenvalue has the familiar n^2 degeneracy. Our question about the discreteness of the bound-state spectrum is obviously answered in the affirmative in this model.

Using the formalism of the preceding section, we proceed to give the equations that lead to (3.2), as well as useful forms for the components of the Dirac spinor for both the bound-state and the continuous spectrum. The form (2.17) of our Dirac equation reduces to

$$\vec{D}^{\pm}g^{\pm}+i\vec{D}^{\mp}\times\vec{g}^{\pm}=iE\vec{g}^{\mp}, \tag{3.3a}$$

$$\vec{D}^{\pm}\cdot\vec{g}^{\pm}=iEg^{\mp}, \tag{3.3b}$$

while the system of Eqs. (19) becomes

$$\left(D^{\pm}-\frac{1}{r}\right)G^{\pm}-\frac{j}{r}C^{\pm}=EP^{\mp}, \tag{3.4a}$$

$$\left(D^{\pm}+\frac{1}{r}\right)P^{\pm}-\frac{j}{r}B^{\pm}=-EG^{\mp}, \tag{3.4b}$$

$$D^{\mp}B^{\pm}-\frac{j}{r}P^{\pm}=EC^{\mp}, \tag{3.4c}$$

$$D^{\mp}C^{\pm}-\frac{j}{r}G^{\pm}=-EB^{\mp}. \tag{3.4d}$$

We have dropped the subscript J, since we will be working with single partial waves. We first make some manipulations on (3.3) to get our simplest results. Dotting (3.3a) with $\vec{D}^{\mp}$ and using (3.3b) and the identity

$$-\vec{D}^{\pm}\cdot\vec{D}^{\mp}=-\nabla^2\pm\frac{2\mu}{r}+\mu^2, \tag{3.5}$$

we get

$$\left(-\nabla^2\mp\frac{2\mu}{r}\right)g^{\pm}=(E^2-\mu^2)g^{\pm}. \tag{3.6}$$

We recognize this as the familiar Coulomb problem. Equation (3.6) tells us that for a bound state $g^-=0$ (assuming that $\mu>0$) and gives the spectrum (3.2). We remark that the latter includes the zero-energy mode ($n=1$).

Of course, we do not know whether the entire set of values (3.2) is included in the spectrum or whether each eigenvalue has the familiar degeneracy of the Coulomb problem until we have found what equations the remaining functions $\vec{g}^{\pm}$, satisfy. It is easy to obtain them. Taking the cross product of $\vec{D}^{\pm}$ with (3.3b) and using successively the identities

$$[\vec{D}^{\pm}\times(\vec{D}^{\mp}\times\vec{g})]_i=-(\vec{D}^{\pm}\cdot\vec{D}^{\mp})g_i+\vec{D}^{\pm}\cdot(D_i^{\mp}\vec{g}), \tag{3.7}$$

$$D_j^{\pm}D_i^{\mp}-D_i^{\mp}D_j^{\pm}=\mp\frac{2\mu}{r}(\delta_{ij}-\hat{r}_i\hat{r}_j), \tag{3.8}$$

and (3.5), we obtain

$$-\nabla^2\vec{g}^{\pm}\pm\frac{2\mu}{r}\hat{r}(\hat{r}\cdot\vec{g}^{\pm})=(E^2-\mu^2)\vec{g}^{\pm}. \tag{3.9}$$

Rather than working with (3.4), we can obtain a useful harmonic analysis of the problem directly from (3.6) and (3.9). Using the decomposition (2.18) in these equations we obtain

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dG^{\pm}}{dr}\right)+\frac{j^2}{r^2}G^{\pm}\mp\frac{2\mu}{r}G^{\pm}=(E^2-\mu^2)G^{\pm}, \tag{3.10a}$$

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dP^{\pm}}{dr}\right)+\frac{j^2+2}{r^2}P^{\pm}-\frac{2j}{r^2}B^{\pm}\pm\frac{2\mu}{r}P^{\pm}=(E^2-\mu^2)P^{\pm}, \tag{3.10b}$$

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dB^{\pm}}{dr}\right)+\frac{j^2}{r^2}B^{\pm}-\frac{2j}{r^2}P^{\pm}=(E^2-\mu^2)B^{\pm}, \tag{3.10c}$$

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dC^{\pm}}{dr}\right)+\frac{j^2}{r^2}C^{\pm}=(E^2-\mu^2)C^{\pm}. \tag{3.10d}$$

Equations (3.10b) and (3.10c) are mutually coupled, but the fact that (3.10a) and (3.10b) are completely decoupled will suffice in order to solve for all eight fields $P^{\pm}$, $G^{\pm}$, $C^{\pm}$, $B^{\pm}$ in terms of standard functions. First, we examine the bound states, to prove the statement about the spectrum (3.2). Equation (3.10d) implies that $C_J^{\pm}=0$ for a bound state, while from (3.10a) we see that $G_J^-=0$. The system (3.4) is then satisfied consistently if

$$B_J^+=P_J^+=0,$$

$$P_J^-=\frac{1}{E}\left(D^+-\frac{1}{r}\right)G_J^+,\quad \text{all } J$$

$$B_J^-=\frac{j}{E}\left(\frac{G_J^+}{r}\right).$$

Next we point out how, not only the bound-state wave functions, but the entire spectral representation of the problem as well can be determined in terms of our ample knowledge about the Schrödinger equation for a Coulomb potential and for a free particle. Very similarly to the case of bound states, $G_J^{\pm}$ and $C_J^{\pm}$ are determined from (3.10a) and (3.10d). Equations (3.4a) and (3.4d) then give $P_J^{\pm}$ and $B_J^{\pm}$ in terms of $G_J^{\pm}$ and $C_J^{\pm}$:

$$P^{\pm}_J = \frac{1}{E}\left[\left(D^{\mp} - \frac{1}{r}\right)G^{\mp}_J - \frac{j}{r}C^{\mp}_J\right],$$

$$B^{\pm}_J = -\frac{1}{E}\left(D^{\pm}C^{\mp}_J - \frac{j}{r}G^{\mp}_J\right).$$

IV. ANOTHER LIMIT: THE ABELIAN MAGNETIC MONOPOLE IN THE SPHERICALLY SYMMETRIC GAUGE

We now turn to our second example, the one for which the potentials in (2.15) are given by

$$A(r) = -\frac{1}{r}, \qquad \Phi(r) = F. \tag{4.1}$$

The Dirac equation for this potential, when expressed in the unitary gauge, corresponds simply to the problem of an isospin doublet of fermions of mass $\mu/2$ and charge $\pm e$ in the field of an ordinary Abelian monopole of charge g, where $eg = \frac{1}{2}$.[8] The latter is a classical problem which was studied a long time ago, independently by Banderet and Harish-Chandra.[9] Their result is that there are no bound states if the wave function is restricted to be nonsingular everywhere.

We shall show, however, that for $J=0$ the eigenvalue problem is meaningless until we have relaxed the requirement that the wave function be nonsingular at $r=0$. This can be done in a one-parameter family of ways. For certain ranges of the parameters there will be a bound state. In the language of unbounded operator theory, the Hamiltonian is symmetric but not self-adjoint on the customary domain, and it has a one-parameter family of self-adjoint extensions.[10]

In subsection (a) below we state the solution of the eigenvalue problem for $J>0$. In (b) we analyze the questions of self-adjointness pertaining to the $J=0$ partial wave.

The separation of the problem into radial and angular parts was given in Sec. II. We rewrite Eqs. (2.19) in the form that they take in the particular case (4.1):

$$D^{\pm}G^{\pm} - \frac{j}{r}C^{\pm} = EP^{\mp}, \quad J \geq 0 \tag{4.2a}$$

$$D^{\pm}P^{\pm} - \frac{j}{r}B^{\pm} = -EG^{\mp}, \quad J \geq 0 \tag{4.2b}$$

$$D^{\mp}B^{\pm} - \frac{j}{r}P^{\pm} = EC^{\mp}, \quad J \geq 1 \tag{4.2c}$$

$$D^{\mp}C^{\pm} - \frac{j}{r}G^{\pm} = -EB^{\mp}, \quad J \geq 1. \tag{4.2d}$$

We immediately see a symmetry of the eigenvalue problem under the transformation S:

$$G^{\pm} \to P^{\pm}, \quad B^{\pm} \to -C^{\pm}, \qquad P^{\pm} \to -G^{\pm}, \quad C^{\pm} \to B^{\pm}. \tag{4.3}$$

This is the charge symmetry.[11] It will be used in subsection (b) below to restrict the self-adjoint extensions of the Hamiltonian.

(*a*) *$J>0$ partial waves.* This case is straightforward and was solved some 30 years ago.[9] Questions of self-adjointness do not arise. There are no bound states, and if E is in the spectrum for $J>0$, $|E| > |\mu|$. We give the explicit solution in our formalism. In the representation (2.20)

$$\begin{pmatrix}\Pi^{\pm}_{JJ_3+}\\ \Gamma^{\pm}_{JJ_3+}\end{pmatrix} = \begin{pmatrix}\alpha^{\pm}_{JJ_3+}\\ \beta^{\pm}_{JJ_3+}\end{pmatrix} r^{-1/2}J_{j-1/2}((E^2-\mu^2)^{1/2}r),$$
$$\begin{pmatrix}\Pi^{\pm}_{JJ_3-}\\ \Gamma^{\pm}_{JJ_3-}\end{pmatrix} = \begin{pmatrix}\alpha^{\pm}_{JJ_3-}\\ \beta^{\pm}_{JJ_3-}\end{pmatrix} r^{-1/2}J_{j+1/2}((E^2-\mu^2)^{1/2}r). \tag{4.4}$$

The coefficients of $\alpha^{\pm}_{\pm}$, $\beta^{\pm}_{\pm}$ satisfy the linear relations

$$-(E^2-\mu^2)^{1/2}\gamma^{\pm}_{+} \pm \mu\gamma^{\pm}_{-} = iE\gamma^{\mp}_{-},$$
$$(E^2-\mu^2)^{1/2}\gamma^{\pm}_{-} \pm \mu\gamma^{\pm}_{+} = iE\gamma^{\mp}_{+}, \tag{4.5}$$

where $\gamma = \alpha$ or β.

(*b*) *S-wave bound states.* Now consider Eqs. (4.2) for $J=0$. They simplify to the simple-looking problem

$$D^{\pm}G^{\pm} = EP^{\mp}, \qquad D^{\pm}P^{\pm} = -EG^{\mp}. \tag{4.6}$$

Define $H^{\pm} = rG^{\pm}$, $Q^{\pm} = rP^{\pm}$. Then (4.6) gives simply

$$\left(\frac{d}{dr} \pm \mu\right)H^{\pm} = EQ^{\mp}, \qquad -\left(\frac{d}{dr} \pm \mu\right)Q^{\pm} = EH^{\mp} \tag{4.7}$$

on the Hilbert space defined by the norm

$$\int_0^{\infty} dr \sum_{\pm} (|H^{\pm}|^2 + |Q^{\pm}|^2). \tag{4.8}$$

The peculiar problems alluded to before are in relation to the choice of a suitable domain on which the operator appearing in (4.7) acts. Call the latter operator A, acting on vectors

$$\begin{pmatrix}H^{+}\\ Q^{-}\\ H^{-}\\ Q^{+}\end{pmatrix},$$

and note first that A can be written as a direct sum of two operators, acting on isomorphic invariant subspaces,

$$A = \begin{pmatrix}K_1 & 0\\ 0 & K_2\end{pmatrix},$$

where both K_1 and K_2 are given by the 2×2 matrix differential operator

$$K=\begin{pmatrix} 0 & \frac{d}{dr}+\mu \\ -\frac{d}{dr}+\mu & 0 \end{pmatrix}, \qquad (4.9)$$

but may differ in the domain on which they act. To preserve the charge symmetry (4.3), we are forced to pick identical domains for K_1 and K_2. It suffices then to study the operator K. A will have the same eigenvalues as K, with twice the multiplicity. We proceed to discuss the domain $D(K)$ of K.

From the definition of $H^\pm$ and $Q^\pm$ we see that continuity of $G^\pm$ and $P^\pm$ demands the boundary conditions at the origin:

$$H^\pm(0)=Q^\pm(0)=0.$$

Thus, the natural domain of definition for K is the set of vectors $\psi=\binom{\psi_1}{\psi_2}$, where the components are absolutely continuous square-integrable functions with absolutely continuous first derivatives, and satisfy $\psi_1(0)=\psi_2(0)=0$. It is easy to see that K, with this domain, is not self-adjoint, although a simple integration by parts would lead one to believe that it is. The reason is that the adjoint K^*, although it is defined by the same differential operator (4.9), acts on a domain $D(K^*)$ which is different, in fact larger, than $D(K)$: $D(K^*)$ consists of the same kind of vectors $\psi=\binom{\psi_1}{\psi_2}$ as $D(K)$, but with no boundary condition at the origin. To see this recall that $D(K^*)$ is defined as the set of all vectors ϕ such that there exists a vector $K^*\phi$ with $(\phi,K\psi)=(K^*\phi,\psi)$ for all $\psi\in D(K)$, and the statements above satisfy this definition since, by simple integration by parts,

$$(\phi,K\psi)-(K^*\phi,\psi)=\bar\phi_1(0)\psi_2(0)-\bar\phi_2(0)\psi_1(0)\,. \qquad (4.10)$$

This is 0, without any conditions on $\phi_1(0),\phi_2(0)$, because the conditions on $\psi_1(0)$ and $\psi_2(0)$ are very strong. Thus, it is clear that in order to make K self-adjoint we should extend its domain, by relaxing the latter conditions on ψ so that (4.10) imposes *the same* conditions on ϕ. These modifications are already dictated to us by (4.10). Fixing ϕ, we find that, of necessity, for all $\psi\in D(K^*)$,

$$\psi_1(0)+a\psi_2(0)=0\,, \qquad (4.11)$$

where a is a constant. Picking $\phi=\psi$ we find that a is allowed to be real or ∞. Equation (4.10) then implies the same conditions on arbitrary ϕ.

Thus, the operator K_a, defined as the differential operator (4.9) acting on the domain specified by (4.11), is self-adjoint. The necessity of picking a domain on which the Hamiltonian is self-adjoint has long been understood in quantum mechanics, and so we concentrate on the class of domains given by (4.11). We are then faced with the dilemma of picking the most physical one out of an infinite range of possibilities. An analysis of the associated spectra, which we shall give momentarily, will point to the answer to this question.

All of the operators K_a exhibit a continuous spectrum for $|E|>|\mu|$. It is also trivial to compute the bound-state energies. We state the results: If $\mu>0$, there exists a nondegenerate bound state for $|a|<1$ with energy

$$E_a=-\frac{2a\mu}{a^2+1}\,. \qquad (4.12)$$

In particular, for $a=0$, and only for that value of a, we get a nondegenerate zero-energy mode for the corresponding self-adjoint extension of (4.9). If $\mu<0$, we get results equivalent to (4.12); we have in this case $|a|>1$, but (4.12) is invariant under $\mu\to-\mu$, $a\to-1/a$.

Thus, there appears to be a most natural choice of domain out of (4.11). It is that which gives zero as the unique bound-state energy. For a monopole ($\mu>0$) the appropriate extension is K_0, corresponding to the boundary condition $\psi_1(0)=0$. For the antimonopole, we are forced to pick out K_∞; the boundary condition is $\psi_2(0)=0$ in this case. We note in passing that these boundary conditions imply that $\psi_1(r)=0$ and $\psi_2(r)=0$ everywhere respectively. But there is an important remark to be made. Although we picked out these particular self-adjoint extensions just in order to preserve the existence of a zero-energy mode shared by the family of Dirac equations of which (4.2) is a limiting case, there appears a discontinuous behavior in passing to the limit: The zero-energy mode acquires a degeneracy. This is due to the fact that the eigenvalues of (4.6) have twice the degeneracy of the eigenvalues of K.[12]

This intuitive, although no less rigorous, treatment of the self-adjoint extensions of K can be understood in the more abstract context of the theory of deficiency subspaces.[13] The fact that $D(K)$ is so small makes $D(K^*)$ so large that the entire complex plane is included in the spectrum of K^*. The dimensions of the deficiency subspaces $\mathrm{Ker}(K^*\pm i)$ measure how large $D(K^*)$ is or, alternatively, how small $D(K)$ is. (Ker$A$ denotes the kernel of the operator A, the set of vectors ψ such that $A\psi=0$.) In our case, $\mathrm{Ker}(K^*\pm i)$ is generated by a single vector:

$$\psi_\pm=\begin{pmatrix}1\\ \pm i(1+\sqrt2)\end{pmatrix}e^{-\sqrt2|\mu|r}\,.$$

The self-adjoint extensions of K are well known to be in one-to-one correspondence with the isometries of $\mathrm{Ker}(K^*-i)$ onto $\mathrm{Ker}(K^*+i)$. In our case such an isometry is given by

$$u_\alpha:\ \psi_- \mapsto e^{i\alpha}\psi_+,\quad \alpha\in \mathbb{R}\ .$$

The corresponding self-adjoint extension K_α of K is described as follows:

$$D(K_\alpha)=\{\psi+\beta(\psi_-+u_\alpha\psi_-)\,|\,\psi\in D(K),\quad \beta\in \mathbb{C}\},$$

$$K_\alpha(\psi+\beta(\psi_-+u_\alpha\psi_-))=K\psi+i\beta\psi_- - i\beta u_\alpha\psi_-\ .$$

It can be checked that K_α as defined here is equivalent to our previous definition of K_a where a and α are related by

$$a=-\frac{1}{\sqrt{2}-1}\cot\tfrac{1}{2}\alpha\ .$$

Thus, the theory of deficiency subspaces explains why we found a one-parameter family of self-adjoint extensions.

We give the resulting spectral analysis of the problem (4.6) under the condition of self-adjointness of the Hamiltonian as analyzed above. As with the $J>0$ partial waves, we express the result in the representation (2.20). For the S waves, $\Pi^\pm_{00,+}=\Pi^\pm_{00-}$, $\Gamma^\pm_{00+}=\Gamma^\pm_{00-}$. There is a one-parameter family of possibilities corresponding to the $r=0$ boundary conditions

$$[r\Gamma^\pm(r)]_{r=0}+a[r\Pi^\mp(r)]_{r=0}=0\ ,$$

where a is an arbitrary real number or ∞. The bound-state energy is given by (4.12) and the corresponding two-dimensional eigenspace is

$$\Gamma^\pm(r)=\alpha_\pm\frac{1}{r}\,e^{-(\mu^2-E^2)^{1/2}r},$$

$$\Pi^\pm(r)=\alpha_\pm\frac{1}{rE}\,[\mu-(\mu^2-E^2)^{1/2}]e^{-(\mu^2-E^2)^{1/2}r}.$$

There is a continuous spectrum for $|E|\geqslant|\mu|$. The corresponding wave functions are

$$\Gamma^\pm(r)=\frac{1}{r}\,(\alpha_\pm e^{i(E^2-\mu^2)^{1/2}r}+\beta_\pm e^{-i(E^2-\mu^2)^{1/2}r}),$$

$$\Pi^\pm(r)=\frac{1}{rE}\,\{[\mu+i(E^2-\mu^2)^{1/2}]\,\alpha_\pm e^{i(E^2-\mu^2)^{1/2}r}+[\mu-i(E^2-\mu^2)^{1/2}]\,\beta_\pm e^{-i(E^2-\mu^2)^{1/2}r}\},$$

where $\alpha_\pm$, $\beta_\pm$ satisfy

$$\alpha_\pm+\beta_\pm+\frac{a}{E}\,\{[\mu+i(E^2-\mu^2)^{1/2}]\,\alpha_\mp+[\mu-i(E^2-\mu^2)^{1/2}]\,\beta_\mp\}=0\ .$$

Lastly, we remark that had we not required that the self-adjoint extensions preserve the charge symmetry we would have obtained a larger family of them, parametrized by SU(2) rather than U(1). For if A represents the operator in (4.7) with the boundary condition of vanishing wave function at the origin, it can easily be seen that $\mathrm{Ker}(A^*\pm i)$ are two-dimensional; then according to the theory of deficiency subspaces summarized above the self-adjoint extensions are in one-to-one correspondence with SU(2), the set of isometries of $\mathrm{Ker}(A^*-i)$ onto $\mathrm{Ker}(A^*+i)$.

V. THE FINITE-MASS MONOPOLE BACKGROUND FIELD

We already remarked that it is impossible to solve the general Dirac equation (2.15) exactly, but we can study the qualitative features of the spectrum. This is our task in this section. The models that we studied in the preceding sections provide us with sufficient information to be able to make a reasonable guess even before going into the precise arguments to be given below. We have seen that the divergence of the spherically symmetric Higgs field gave rise to an infinite number of bound states, and the introduction of the $A(r)=-1/r$ potential wiped them all out completely, leaving only the zero-energy eigenvalue. When now the Higgs field is modified near the origin to smoothly approach the value 0 at $r=0$, we expect that the spectrum will remain qualitatively the same, and only the position of the eigenvalues will be altered slightly. In fact, in terms of the correspondence proved in Sec. III, such a modification of the Higgs field would be equivalent to replacing the point center of the Coulomb field by a finite charge distribution. As to the effect of the $A(r)$ field, we expect that, if $A(r)=-1/r$ only outside a finite radius, while it goes to zero smoothly at the origin, all but a finite number of the bound states created by the Higgs field will be wiped out. We can support this assertion by proving that it holds in the case of the S waves. For $J=0$, Eqs. (2.19) can be reduced to a single Schrödinger equation for G^+:

$$\left[-\frac{d^2}{dr^2}+(\sigma'+\sigma^2-2\mu\sigma)\right](rG^+)=(E^2-\mu^2)(rG^+)\ . \quad (5.1)$$

σ vanishes exponentially outside a finite radius a. Using the well-known upper bound on the number of bound states of the lth partial wave of a Schrödinger equation in a potential V,[14]

$$n_l<\frac{1}{2l+1}\int_{V(r)<0} r\,dr\,|V(r)| \quad (5.2)$$

we easily see that the number of bound states of (5.1) is finite, and bounded above by a number of the order of μa.

We now show how bounds of this sort can also be derived for the higher partial waves of our Dirac equation. Squaring the operator on the left-hand side of (2.19) and transforming to the representation (2.20) we get the following system of equations:

$$\begin{bmatrix} -\frac{d^2}{dr^2}+\frac{j(j-1-s)+1}{r^2}+v_1^\pm & \frac{1}{r^2}f_{12}^\pm & 0 & 0 \\ \frac{1}{r^2}f_{12}^\pm & -\frac{d^2}{dr^2}+\frac{j(j+1+s)+1}{r^2}+v_2^\pm & 0 & 0 \\ 0 & 0 & -\frac{d^2}{dr^2}+\frac{j(j-1+s)}{r^2}+w_1^\pm & g_{12}^\pm \\ 0 & 0 & g_{12}^\pm & -\frac{d^2}{dr^2}+\frac{j(j+1-s)}{r^2}+w_2^\pm \end{bmatrix}$$

$$\times\begin{bmatrix} r\Pi^\pm_{JJ_3^+} \\ r\Pi^\pm_{JJ_3^-} \\ r\Pi^\pm_{JJ_3^+} \\ r\Gamma^\pm_{JJ_3^-} \end{bmatrix} = (E^2-\mu^2)\begin{bmatrix} r\Pi^\pm_{JJ_3^+} \\ r\Pi^\pm_{JJ_3^-} \\ r\Gamma^\pm_{JJ_3^+} \\ r\Gamma^\pm_{JJ_3^-} \end{bmatrix}, \qquad (5.3)$$

where $s(r)=r(1/r+A(r))$ and the remaining functions $v_i^\pm(r), w_i^\pm(r), f_{12}^\pm(r), g_{12}^\pm(r)$ are given by expressions containing $A(r)$ and $\Phi(r)$. The functions $s(r), f_{12}^\pm(r)$ share the properties given below for $s(r)$:

(1) $s(0)=1$.

(2) $s(r)\to 0$ exponentially as $r\to\infty$. The exponential decay sets in outside the radius of the monopole, roughly at $r=F^{-1/2}$.

The remaining functions, $v_i^\pm(r), w_i^\pm(r), g_{12}^\pm(r)$ satisfy (2), while at the origin they are less singular than $1/r$.

In the operator in (5.3) we separate a positive term [an operator A is positive if $(\psi, A\psi)>0$ for all ψ in the Hilbert space]

$$\begin{bmatrix} \frac{1}{r^2}(3-2\sqrt{2})\lvert f_{12}^\pm(r)\rvert & \frac{1}{r^2}f_{12}^\pm(r) & 0 & 0 \\ \frac{1}{r^2}f_{12}^\pm(r) & \frac{1}{r^2}(3+2\sqrt{2})\lvert f_{12}^\pm(r)\rvert & 0 & 0 \\ 0 & 0 & \lvert g_{12}^\pm(r)\rvert & g_{12}^\pm(r) \\ 0 & 0 & g_{12}^\pm(r) & \lvert g_{12}^\pm(r)\rvert \end{bmatrix},$$

which we drop because it only decreases the number of bound states. We then use the fact that $0<s(r)<1$, to find that the operator in (5.3) is greater than the following diagonal matrix operator (an operator A is greater than an operator B if A-B is positive):

$$\begin{bmatrix} -\frac{d^2}{dr^2}+\frac{j(j-2)+\sqrt{2}(2-\sqrt{2})}{r^2}+V_1^\pm & 0 & 0 & 0 \\ 0 & -\frac{d^2}{dr^2}+\frac{j(j+1)-\sqrt{2}(\sqrt{2}+1)}{r^2}+V_2^\pm & 0 & 0 \\ 0 & 0 & -\frac{d^2}{dr^2}+\frac{j(j-1)}{r^2}+W_1^\pm & 0 \\ 0 & 0 & 0 & -\frac{d^2}{dr^2}+\frac{j^2}{r^2}+W_2^\pm \end{bmatrix}.$$

The potentials $V_i^\pm(r)$, $W_i^\pm(r)$ tend to zero exponentially outside the radius of the monopole, while at the origin they behave at worst like $1/r$. We are thus comparing our problem to a system of decoupled Schrödinger-type equations, to which we can apply the bounds (5.3) (which hold regardless of whether l is an integer or not). The minimax principle then implies that the number of bound states of (5.3) has an upper bound of the form

$$\frac{1}{\alpha(j)}\sum_{\pm,i}\int_0^\infty r\,dr(|V_i^\pm(r)|+|W_i^\pm(r)|), \qquad (5.4)$$

where $\alpha(j)>0$ and $\alpha(j)\to j$ as $j\to\infty$.

From this bound we learn two things, which constitute a proof of the results anticipated in the beginning of this section.

(1) For large enough J, the bound (5.4) becomes less than 1, and therefore the corresponding partial waves possess no bound states. Thus the Dirac equation (2.15) for the true monopole potentials has only a finite number of bound states and, in particular, the zero-energy mode is discrete.

(2) An examination of the potentials $V_i^\pm$, $W_i^\pm$ has shown that the integral in (5.4) tends to zero as $F\to\infty$, i.e., as the radius of the monopole approaches zero. Thus, if the radius is sufficiently small, the only bound state is the one at zero energy. Just as in the examples of Secs. III and IV, we of course have a continuous spectrum for $|E|\geq|\mu|$.

In conclusion, in this paper we have done two things. First we have shown that fermions in the field of the magnetic monopole of 't Hooft and Polyakov can only be found in a finite number of bound states. Our experience with topological solitons allows us to distinguish two kinds of binding: Topological binding, which gives rise to the zero-energy modes, and nontopological binding, which has been shown to arise from the interaction of the fermion with the Higgs field. The effect of the gauge field is to eliminate this kind of binding.

The second contribution is the exactly soluble models that we discussed. Quite apart from their relevance to the physical problem, these models are sufficiently simple and already well understood to provide a laboratory for investigating the behavior of fermions in the presence of topological magnetic monopoles. It is clear that we cannot carry out calculations in a closed form for the exact monopole solutions, and, on the other hand, the models of Secs. III and IV are already expressed in terms of exhaustively known problems.

Lastly, we remark that the limiting cases we are able to deal with in this particular case of an isospin -$\frac{1}{2}$ fermion in the field of an SU(2) monopole can also be studied in all variations of the group structure of the background field or of the isospin of the fermions. The essence of the results will be similar. In particular, we would in this way encounter fermion-monopole systems with $eg=\frac{1}{2}n$, n integral.

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[5]The author has encountered an example of a Dirac equation which possesses a countable set of eigenvalues of infinite multiplicity which is dense in the interval $(-\mu,\mu)$, $\mu>0$. The example is

$$(i\gamma^\mu\partial_\mu+\mu\hat{r}\cdot\vec{\sigma})\psi=0\,.$$

This only calls for caution even with the most innocuous-looking Dirac equations.

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[7]R. Jackiw and C. Rebbi, Phys. Rev. Lett. 36, 1116 (1976).

[8]In the unitary gauge the fields corresponding to (4.1) $\Phi_a=\delta_{a3}\mu$, $A_{a\mu}=\delta_{a3}(1/g)A_\mu(\vec{r})$, where $A_\mu(\vec{r})$ is the vector potential of an Abelian magnetic monopole of unit charge (see e.g. Ref. 6), and the Dirac equation is

$$i\gamma^\mu(\partial_\mu-i\tau^3A_\mu(\vec{r}))\psi+\tau^3\mu\psi=0\,.$$

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[11]The charge symmetry is only one of the generators of an O(4) symmetry group, which the Dirac equation possesses besides the O(3) space symmetry. The remaining generators can be read off quite easily from the harmonic analysis (4.2). We hope to discuss the meaning of this group in a future work.

[12]It is pointed out in Ref. 10 that the self-adjoint extensions that have zero as a bound state are the ones that preserve the odd character of the Hamiltonian under CP; i.e., the domains $D(K_0)$ for $\mu>0$ and $D(K_\infty)$ for $\mu<0$ are invariant under CP.

[13]See, for example, the excellent books by M. Reed and B. Simon, *Methods of Mathematical Physics* (Academic, New York, 1976), Vol. II; or T. Kato, *Perturbation Theory for Linear Operators*, 2nd edition (Springer, New York, 1976).

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