Allowed supersymmetric vacuums

M. H. Friedman and Y. Srivastava*
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 14 February 1977)

We construct the vacuum connection associated with a Yang-Mills theory of the graded Lie group of real, general linear transformations on superspace that is induced by supersymmetry transformations. This gives rise to a class of supersymmetric "vacuum metrics," most of which are noninvertible and/or involve theories with torsion. All of these connections are nonsingular and no limiting procedures are involved, even in the case of the original Wess-Zumino-Salam-Strathdee supersymmetric line element. Many of them are *a priori* compatible with the internal-symmetry group $SU(2)\times U(1)$ and extensions to include color.

In a previous paper[1] we developed a complete formalism for imposing Yang-Mills gauge invariance induced by general coordinate transformations on superspace (i.e., a space containing both commuting and anticommuting coordinates). The appropriate group was found to be the graded pseudo-Lie group of real, general linear transformations which was denoted by $GGL(N_B, N_F, R)$. Here N_B and N_F refer to the number of Bose and Fermi coordinates, respectively, with $N_B=4$ and N_F yet to be determined. In this earlier work we had exhibited an example of a vacuum symmetry breakdown that yielded the Lorentz metric. Here we present symmetry breakdowns that yield a general class of supersymmetric line elements. This investigation is important because some members of the above class have been shown[2] to produce inconsistencies in the equations of motion when dynamical fields are introduced.

As in all Yang-Mills theories we were led to introduce affine connections $\Gamma_{\hat{B}}{}^{A}{}_{C}(z)$ (Ref. 3) for the purpose of constructing covariant derivatives. It was shown in Ref. 1 that under infinitesimal transformation $\omega_B{}^A(z)$, the $\Gamma_{\hat{B}}{}^{A}{}_{C}$ were required to satisfy

$$\begin{aligned}\Gamma'_{\hat{B}}{}^{A}{}_{C}(z') \simeq \Gamma_{\hat{B}}{}^{A}{}_{C}(z) - (-1)^{c(1+d)}\Gamma_{\hat{B}}{}^{A}{}_{D}\omega_C{}^D \\ -(-1)^{(b+d)(a+c+d)}\Gamma_{\hat{D}}{}^{A}{}_{C}\omega_B{}^D \\ +(-1)^{c(a+d)}\Gamma_{\hat{B}}{}^{D}{}_{C}\omega_D{}^A - \partial_C\omega_B{}^A \ . \quad (1)\end{aligned}$$

Here z^A denotes collectively the Bose coordinates x^μ and the Majorana coordinates $\theta^{\alpha m}$ (α is a Dirac index, and m is an internal-symmetry index which will be suppressed whenever no confusion arises). All derivatives are right derivatives, and the Grassmann parity $a=0$ or 1 according to whether A is a Bose or Fermi index, respectively. The infinitesimal group parameters $\omega_B{}^A(z)$ are related to the elements U of the group by

$$U \simeq 1 + G_A{}^B\omega_B{}^A(z) , \quad (2)$$

where $G_A{}^B$ are the generators of the group satisfying the algebra

$$\begin{aligned}G_A{}^BG_C{}^D-(-1)^{(a+b)(c+d)}G_C{}^DG_A{}^B \\ =(\delta^B{}_C\,\delta^E{}_A\delta^D{}_F-(-1)^{(a+b)(c+d)}\delta^D{}_A\delta^B{}_F\delta^E{}_C)G_E{}^F \ . \quad (3)\end{aligned}$$

Specializing to the case of (global) supersymmetry we consider the transformations

$$\begin{aligned}\delta x^\mu &= -i\bar{\epsilon}\,\Gamma^\mu\theta \ , \\ \delta\theta^\alpha &= \epsilon^\alpha \ ,\end{aligned} \quad (4)$$

where ϵ^α is an infinitesimal constant Majorana spinor. The corresponding $\omega_B{}^A$ are as follows:

$$\omega_\mu{}^\nu = \omega_\alpha{}^\beta = \omega_\mu{}^\alpha = 0 \ ,$$

and (5)

$$\omega_\alpha{}^\mu = i(\bar{\epsilon}\Gamma^\mu)_\alpha \ ,$$

where

$$\Gamma^\mu = \gamma^\mu M_V + \gamma_5\gamma^\mu M_A \ . \quad (6)$$

M_V and M_A are real internal-symmetry matrices (with dimensions of mass) satisfying the conditions $M_V=M_V{}^T$ and $M_A=-M_A{}^T$ and hence $(\eta\Gamma_\mu)^T=(\eta\Gamma_\mu)$.[4]

The most general vacuum $\Gamma_{\hat{B}}{}^{A}{}_{C}$ that are form-invariant under Eqs. (1), (4), and (5) are found to be

$$\begin{aligned}&\text{(i) } \Gamma_\mu{}^\alpha{}_\nu=0, \quad \text{(ii) } \Gamma_\nu{}^\mu{}_\lambda=0 \ , \\ &\text{(iii) } \Gamma_\beta{}^\alpha{}_\mu=-(I_\mu)^\alpha{}_\beta \ , \quad \text{(iv) } \Gamma_\mu{}^\alpha{}_\beta=-(K_\mu)^\alpha{}_\beta \ , \\ &\text{(v) } \Gamma_\nu{}^\mu{}_\alpha=i(\bar{\theta}\Gamma^\mu K_\nu)_\alpha \ , \quad \text{(vi) } \Gamma_\alpha{}^\mu{}_\nu=-i(\bar{\theta}\Gamma^\mu I_\nu)_\alpha \ , \\ &\text{(vii) } \Gamma_\alpha{}^\delta{}_\beta=i(\bar{\theta}\Gamma^\lambda)_\alpha(K_\lambda)^\delta{}_\beta-i(\bar{\theta}\Gamma^\lambda)_\beta(I_\lambda)^\delta{}_\alpha \ , \\ &\text{(viii) } \Gamma_\alpha{}^\mu{}_\beta=(\bar{\theta}\Gamma^\mu I_\nu)_\alpha(\bar{\theta}\Gamma^\nu)_\beta \\ &\qquad -(\bar{\theta}\Gamma^\mu K_\nu)_\beta(\bar{\theta}\Gamma^\nu)_\alpha-i(\eta L^\mu)_{\alpha\beta} \ .\end{aligned} \quad (7)$$

In Eq. (7), I_μ, K_μ, and L_μ are constant matrices of the same form as Γ_μ in Eq. (6), e.g.,

$$I_\mu = \gamma_\mu I_V + \gamma_5\gamma_\mu I_A \ , \ \text{etc.}$$

However, the internal matrices I_V, I_A, etc., have no *a priori* symmetry constraints.

A general "vacuum metric" $g_{\hat{A}B}$ that is left form-invariant under the transformation given by Eq. (5) is

$$g_{\mu\nu}=\eta_{\mu\nu}\ ,\quad g_{\mu\alpha}=-g_{\alpha\mu}=i(\bar{\theta}\Gamma_\mu)_\alpha\ ,\qquad g_{\alpha\beta}=-g_{\beta\alpha}=i(\eta S)_{\alpha\beta}+(\bar{\theta}\Gamma^\lambda)_\alpha(\bar{\theta}\Gamma_\lambda)_\beta\ , \tag{8}$$

where S is an unspecified real internal-symmetry matrix with $S=S^T$.

We now require that the covariant derivative of $g_{\hat{A}B}$ formed with respect to the connection of Eq. (7) vanish, i.e.,

$$g_{\hat{A}B;\,C}=g_{\hat{A}B,\,C}-(-1)^{ce+cb}\Gamma_{\hat{A}}{}^{E}{}_{C}\,g_{\hat{E}B}-(-1)^{b+be}g_{\hat{A}E}\Gamma_{\hat{B}}{}^{E}{}_{C}=0\,. \tag{9}$$

This results in the following conditions:

$$\Gamma_\mu=SK_\mu+L_\mu \tag{10a}$$

and

$$(\eta S I_\mu)^T=(\eta S I_\mu)\,. \tag{10b}$$

We now turn our attention to the equations of motion proposed in Ref. 1 in terms of the contracted curvature tensor $R_{\hat{A}B}$, i.e.,

$$R_{\hat{A}B;\,C}=0\,. \tag{11}$$

There we had exhibited $R_{\hat{A}B}$ as

$$R_{\hat{A}B}=-(-1)^c\Gamma_{\hat{A}}{}^{C}{}_{C,\,B}+(-1)^{c(b+1)}\Gamma_{\hat{A}}{}^{C}{}_{B,\,C}+(-1)^{c+be}\Gamma_{\hat{A}}{}^{E}{}_{B}\Gamma_{\hat{E}}{}^{C}{}_{C}-(-1)^{ce}\Gamma_{\hat{A}}{}^{E}{}_{C}\Gamma_{\hat{E}}{}^{C}{}_{B}\,. \tag{12}$$

We note that the $\Gamma_{\hat{B}}{}^{A}{}_{C}$ of Eq. (7) satisfy $(-1)^c\Gamma_{\hat{B}}{}^{C}{}_{C}\equiv 0$. Thus, the first and third terms of Eq. (12) are identically zero.

Since we have already required that $g_{\hat{A}B;C}=0$, the equation of motion Eq. (11) is automatically satisfied if

$$R_{\hat{A}B}=\lambda g_{\hat{A}B}\,. \tag{13}$$

Equations (10), (12), and (13) yield

$$\lambda=\tfrac{1}{4}\operatorname{Tr}(I^\mu K_\mu)\ , \tag{14}$$

$$\lambda(\eta S)_{\alpha\beta}=(\eta SK^\mu K_\mu)_{\alpha\beta}-(\eta SK^\mu I_\mu)_{\beta\alpha}-(\eta L^\mu I_\mu)_{\beta\alpha}-(\eta L^\mu)_{\gamma\beta}(I_\mu)^\gamma{}_\alpha\,. \tag{15}$$

We reserve a complete analysis to a forthcoming paper and for the present restrict ourselves to the following interesting cases. In particular, we require that $[S,\Gamma_\mu]=0$, whence the invariance of the theory under coordinate transformations permits us to normalize S such that $S^2=S$.

Case I. $S=0$. This leads to a generalized non-invertible metric of the Wess-Zumino-Salam-Strathdee type.[5,6] As a result of Eq. (10a), $(\eta L_\mu)^T=(\eta L_\mu)$, since as stated earlier $(\eta\Gamma_\mu)$ had this symmetry. Equation (15) then gives

$$\Gamma^\mu I_\mu=0\,. \tag{16}$$

This is an example of a theory with "torsion" in superspace, because the $\Gamma_{\hat{B}}{}^{A}{}_{C}$ are "nonminimal" in that they do not satisfy the corresponding symmetry constraint.[1] However, we emphasize that this connection is *nonsingular* and *no* limiting procedures are required.

We briefly discuss some of the possible ways in which Eq. (16) may be satisfied. The simplest one is to have $I_\mu=0$. Then, through Eq. (14), $\lambda=0$ and our equation of motion reduces to $R_{\hat{A}B}=0$. It is interesting that in the absence of internal-symmetry indices (as in the original examples of Wess and Zumino,[5] and Salam and Strathdee[6]), this is the *only* possibility.

The case $I_\mu\neq 0$ is more complicated. Consider the internal-symmetry group U(1). Equation (16) now requires a breakdown of parity and charge conjugation, e.g.,

$$\Gamma_\mu=m_1\,\frac{(1+i\hat{C}\gamma_5)}{2}\,\gamma_\mu$$

and

$$I_\mu=m_2\,\frac{(1+i\hat{C}\gamma_5)}{2}\,\gamma_\mu\ ,$$

where the U(1) charge matrix $\hat{C}$ is the Pauli matrix ρ_2.

If Eq. (16) is to be satisfied (for $I_\mu\neq 0$) *without* a breakdown of parity and charge conjugation (in the vacuum sector) then the internal-symmetry group must be at least SU(2) × U(1). For example,

$$\Gamma_\mu=m_1\,\frac{(1+\hat{C}\Lambda_3)}{2}\,\gamma_\mu$$

and

$$I_\mu=m_2\,\frac{(1-\hat{C}\Lambda_3)}{2}\,\gamma_\mu\ ,$$

where $\hat{C}=\rho_2\otimes I$ and $\Lambda_3=\rho_2\otimes\tau_3$. This will induce a mass growth for the associated vector mesons as in the Weinberg-Salam[7] theory.

Case II. $S=1$. This case leads to an invertible metric. For the minimal theory, i.e., $I_\mu=K_\mu$ and $L_\mu=0$, the solution has been worked out in detail by Arnowitt and Nath.[8] Equations (10a), (14), and (15) confirm their conclusion that if $\Gamma^\mu\Gamma_\mu\neq 0$ (i.e., $\lambda\neq 0$), the Fermi dimensionality $N=2$, i.e., the group is U(1). However relaxing the minimality constraints leads to a whole class of new solutions. Amusingly enough, if $I_\mu=K_\mu=L_\mu$, Eqs. (14) and (15) give $N=4$, which is again the right dimensionality for an SU(2) × U(1) theory. On the other hand, if we wish to include color, then the appropriate choice is $I_\mu=K_\mu=(1/r)L_\mu$ with the result $N=2(1+r)$. Clearly for $SU(2)\times U(1)\times SU(3)_c$ $r=5$.

Case III. $S\neq 0,\ 1$ (Det$S=0$). This is the case of

S being a projection operator in the internal-symmetry space, which must be larger than U(1). In fact, U(2) [or SU(2)⊗U(1)] is the *smallest* group to accommodate this possibility. However, unlike case II, *any* non-Abelian group can be sustained (*without* $\lambda = 0$) *even* for a minimal theory, i.e., $I_\mu = K_\mu$, $L_\mu = 0$. The secret of such behavior lies in the fact that, again, as for case I, the metric is *not* invertible.

If parity or charge conjugation is not to be broken in the vacuum sector, then Eqs. (10), (14), and (15) force $K^\mu = mS\gamma^\mu$ and $\mathrm{tr}S = 2$. These conditions can be satisfied for SU(n) × U(1). Thus there exists the possibility of building an SU(2) × U(1) model of the *minimal* type with no P or C breakdown at this level.

In conclusion, using the formalism developed in Ref. 1, we have been able to construct whole new classes of supersymmetric vacuums. Most of these yield noninvertible (vacuum) metrics and/or give rise to theories with torsion. It is important to note that the connections are nonsingular for all of these examples and hence no limiting procedures need be invoked. Furthermore, we have shown that the Weinberg-Salam SU(2) × U(1) theory can be incorporated in many of the above schemes.

We now have to decide which of these is tenable when dynamical fields are introduced. Since it is a self-sourced theory, there are other severe constraints. For example, we have already found[2] that some of the possible vacuum solutions do *not* lead to consistent field equations for the gauge fields. We are currently investigating this problem and seeking to narrow the choice.

---

*Work supported in part by a grant from the National Science Foundation.

[1]M. H. Friedman and Y. Srivastava, Phys. Rev. D 15, 1026 (1977).

[2]M. H. Friedman and Y. Srivastava (unpublished).

[3]$\hat{B}$ denotes an index that transforms as the left derivative of a scalar (for details see Ref. 1).

[4]We work in the Majorana representation with $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$, $\gamma_5{}^2 = -1$, and $\eta_{\alpha\beta}$ is related to the charge-conjugation matrix (in Dirac space), $\eta_{\alpha\beta} = -(C^{-1})_{\alpha\beta}$. The Lorentz matric $\eta_{\mu\nu} = (-,+,+,+)$. The antisymmetric piece of $(\eta\Gamma_\mu)$ can be transformed away from the supersymmetric line element.

[5]J. Wess and B. Zumino, Nucl. Phys. B70, 39 (1974). For a recent, comprehensive review of the subject, see P. Fayet and S. Ferrara, Phys. Rep. (to be published).

[6]A. Salam and J. Strathdee, Nucl. Phys. B76, 477 (1974).

[7]S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium No. 8)*, edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 367.

[8]R. Arnowitt and P. Nath, Phys. Rev. Lett. 36, 1526 (1976).