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Lattice fermions

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The problem of formulating the field theory of Dirac particles on a spatial lattice is reviewed. In one dimension we construct free massless Dirac fields and Thirring fields and show their equivalence to the X-Y and asymmetric Heisenberg antiferromagnetic chains. In three dimensions we find that the simplest construction describes an isodoublet of massless Dirac fields. We discuss the incorporation of gauge degrees of freedom and illustrate how chiral symmetry is spontaneously broken by the interaction of gauge and fermion fields.

I. LATTICE FERMIONS IN 1 + 1 DIMENSIONS

In this paper we take the first step in formulating a realistic lattice gauge theory[1] of hadrons; we develop the lattice theory of fermion fields in three-dimensional space. As an introduction we will begin with the simpler one-dimensional case.[2]

In conventional one-dimensional continuum theory the Dirac field is a two-component spinor

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

and the Dirac matrices are

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.1}$$

The massless Dirac equation is

$$i\dot{\psi} = i\alpha\partial_z\psi . \tag{1.2}$$

Let us introduce a lattice into one-dimensional space as in Fig. 1.

The lattice spacing is a and the nth site has coordinate

$$z = na . \tag{1.3}$$

Now consider the discrete version of Eq. (1.2) obtained by replacing ∂_z by a discrete difference.

$$i\dot{\psi}(n) = \frac{i\alpha}{2a}\left[\psi(n+1) - \psi(n-1)\right] . \tag{1.4}$$

The dispersion laws for Eqs. (1.2) and (1.4) are respectively

$$\omega = \alpha k , \quad -\infty < k < \infty \tag{1.5}$$

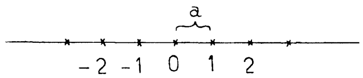

FIG. 1. A one-dimensional lattice with spacing a.

$$\omega = \alpha\frac{\sin l}{a} , \quad -\pi \le l \le \pi . \tag{1.6}$$

In Fig. 2 the dispersion laws corresponding to Eq. (1.2) for $\alpha = \pm 1$ are plotted, and it is seen that the solutions are right- and left-moving waves.

In Fig. 3 we see a similar description of Eq. (1.4). As $a \to 0$ the frequencies go as a^{-1} unless $l \to ka$, $l - \pi \to ka$, or $l + \pi \to ka$. Thus the region of finite frequency is concentrated near $l = 0$ and $l = \pm\pi$. Now the waves near $l = 0$ are the smooth, long-wavelength modes which we normally expect to survive in the continuum limit. But what of the low-energy modes near $l = \pm\pi$? Can these be excluded from consideration? After all, in the limit $a \to 0$ they carry infinite momentum.

In the free field theory we can always agree to populate only the region near $l = 0$ with fermions. However, in an interacting field theory the modes $l \approx \pm\pi$ may become excited without our permission. In fact, the excitation of these modes may not disappear with a going to zero. As an example, suppose the field couples to an external potential through the charge density $\psi^\dagger(n)\psi(n)$. In momentum space this is proportional to

$$\int_{-\pi}^{\pi} e^{in(l-l')}\psi^\dagger(l)\psi(l')dl\,dl' .$$

Thus it is as likely to excite a pair with momentum $\pm k$ as with $\pm(\pi/a - k)$. Any local coupling of ψ

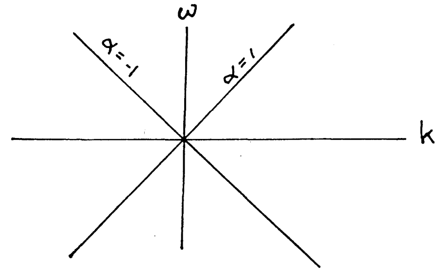

FIG. 2. Dispersion law for the continuum Dirac equation.

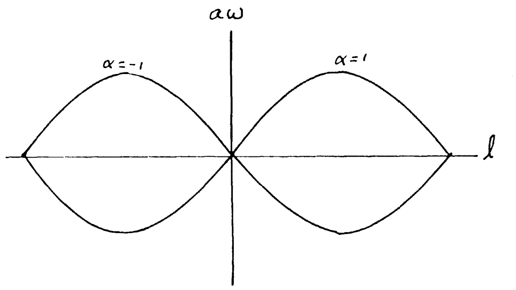

FIG. 3. Dispersion law for Eq. (1.4).

will in general excite the unwanted low-frequency modes near $\pm\pi$.

There are two simple ways to solve the problem. In the first method, due to Wilson, an additional term is added to the discrete Dirac Hamiltonian to raise the energy near $\pm\pi$. Equation (1.4) is modified to

$$i\dot{\psi}(n)=\frac{i\alpha}{2a}\left[\psi(n+1)-\psi(n-1)\right] + \frac{\gamma_0\kappa}{a}\left[2\psi(n)-\psi(n+1)-\psi(n-1)\right] , \qquad (1.7)$$

where κ is an arbitrary dimensionless parameter. The dispersion law for Eq. (1.7) is shown in Fig. 4. The gap $\mathcal{G}$ is proportional to the parameter κ.

The second method, due to Casher and this author, is to reduce the number of degrees of freedom by using a single component on each site of the lattice. Define a one-component field ϕ on each site. The Hamiltonian will be

$$H=\frac{i}{2a}\sum\left[\phi^\dagger(n)\phi(n+1)-\phi^\dagger(n+1)\phi(n)\right] . \qquad (1.8)$$

The field ϕ is a canonical fermion field satisfying

$$\{\phi(n),\phi(m)\}=0 , \quad \{\phi^\dagger(n)\phi(m)\}=\delta_{nm} . \qquad (1.9)$$

Using (1.8) and (1.9) the equation of motion for ϕ is found to be

$$\dot{\phi}(n)=\frac{1}{2a}\left[\phi(n+1)-\phi(n-1)\right] . \qquad (1.10)$$

For long wavelengths (1.10) may be replaced by the

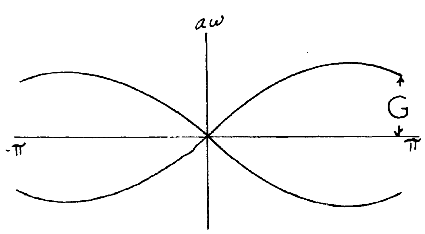

FIG. 4. Dispersion law for Eq. (1.7).

continuum equation

$$\dot{\phi}=\partial_z\phi , \qquad (1.11)$$

which describes left-moving waves. However, right-moving waves are also described in the spectrum of Eq. (1.10). To see this define

$$\phi'(n)=(-1)^n\phi(n) . \qquad (1.12)$$

The field $\phi'(n)$ satisfies

$$\partial_t\phi'(n)=-\frac{1}{2a}\left[\phi'(n+1)-\phi'(n-1)\right] , \qquad (1.13)$$

and hence for long wavelengths it is right-moving. The left- and right-moving waves are identified with the neighborhoods of $l=0$ and $l=\pm\pi$, respectively.

The two-component character of ϕ can be made manifest by defining odd and even sublattices and treating the fields on these two sublattices as independent fields, $\psi_2(n)$ for n odd and $\psi_1(n)$ for n even. The equation of motion for $\psi_{1;2}$ becomes

$$\dot{\psi}_1(n)=\frac{1}{2a}\left[\psi_2(n+1)-\psi_2(n-1)\right] , \quad \dot{\psi}_2(n)=\frac{1}{2a}\left[\psi_1(n+1)-\psi_1(n-1)\right] , \qquad (1.14)$$

which, in the continuum limit, becomes Eq. (1.2).

The usual fermion bilinears have lattice analogs which can be classified into two groups. Bilinears (bilinears such as $\psi^\dagger\psi$ will be assumed to be appropriately symmetrized) involving diagonal Dirac matrices ($\psi^\dagger\psi$ and $\psi^\dagger\gamma_0\psi=\bar{\psi}\psi$) do not mix ψ_1 and ψ_2,

$$\psi^\dagger\psi=\psi_1^\dagger\psi_1+\psi_2^\dagger\psi_2 , \quad \bar{\psi}\psi=\psi_1^\dagger\psi_1-\psi_2^\dagger\psi_2 . \qquad (1.15)$$

Thus they do not multiply an odd-site field with an even-site field. In particular, we may identify

$$\psi^\dagger\psi=\phi^\dagger(n)\phi(n) , \qquad (1.16)$$

$$\bar{\psi}\psi=\phi^\dagger(n)\phi(n) \quad (n=\text{even}) , \quad \bar{\psi}\psi=\phi^\dagger(n)\phi(n) \quad (n=\text{odd}) . \qquad (1.17)$$

Equation (1.17) may be summarized as

$$\bar{\psi}\psi(n)=\phi^\dagger(n)\phi(n)(-1)^n . \qquad (1.18)$$

The second class of bilinears involves off-diagonal matrices (in one dimension $\gamma_5=\alpha$) and therefore multiplies an odd and an even field. We may identify them with products of nearest-neighbor fields. The two bilinears of this type are $\psi^\dagger\gamma_5\psi$ and $\bar{\psi}\gamma_5\psi$:

$$\begin{aligned}\psi^\dagger\gamma_5\psi &= \psi_1^\dagger\psi_2 + \text{H.c.}\\ &= \phi^\dagger(n)\phi(n+1) + \text{H.c.}\ ,\\ \bar\psi\gamma_5\psi &= \psi_1^\dagger\psi_2 - \psi_2^\dagger\psi_1\\ &= [\phi^\dagger(n)\phi(n+1) - \text{H.c.}](-1)^n\ .\end{aligned} \tag{1.19}$$

The Hamiltonian (1.8) has lattice shift symmetry $\phi(n)\to\phi(n+m)$, where m is any integer: If m is an even integer we may interpret the shift as a translation

$$\psi_1(n)\to\psi_1(n+m)\ ,$$
$$\psi_2(n)\to\psi_2(n+m)\ .$$

But for m odd the fields ψ_1 and ψ_2 are interchanged. Since interchange of ψ_1 and ψ_2 is equivalent to multiplication by γ_5 we may interpret the odd-shift invariance as γ_5 symmetry. In this respect let us consider the effect of a mass term $m\bar\psi\psi$. According to Eq. (1.18) this is equivalent to adding

$$m\sum(-1)^n\phi^\dagger(n)\phi(n)\ , \tag{1.20}$$

which has *even*-but not *odd*-shift invariance. This of course reflects the breaking of γ_5 invariance by masses.

II. LATTICE FERMIONS AND HEISENBERG CHAINS

In this section we will describe some connections between one-dimensional fermion systems and one-dimensional spin lattices. The connections are only possible in one dimension, and therefore this section is not important to the reader who wishes to go on to three-dimensional fermion systems.

The trick of connecting one-dimensional chains of spins with one-dimensional chains of fermion fields is due to Jordan and Wigner. Consider a one-dimensional chain of spin-1/2 systems described by the operators $\sigma^\pm(n)$ and $\sigma_z(n)$. The σ's at different sites *commute*. Now define the canonically anticommuting fields $\phi(n)$ as follows:

$$\begin{aligned}\phi(n) &= \left\{\prod_{m<n}[i\sigma_z(m)]\right\}\sigma^+(n)\ ,\\ \phi^\dagger(n) &= \left\{\prod_{m<n}[-i\sigma_z(m)]\right\}\sigma^-(n)\ .\end{aligned} \tag{2.1}$$

Now consider the Hamiltonian in Eq. (1.8). Using (2.1) it can be reexpressed in terms of the σ variables

$$H=\frac{1}{2a}\sum[\sigma^+(n)\sigma^-(n+1)+\text{H.c.}]\ , \tag{2.2}$$

or defining $\sigma_x=\sigma^++\sigma^-$, $\sigma_y=-i(\sigma^+-\sigma^-)$ we may write (2.2) as

$$H=\frac{1}{4a}\,[\sigma_x(n)\sigma_x(n+1)+\sigma_y(n)\sigma_y(n+1)]\ . \tag{2.3}$$

This Hamiltonian is familiar to solid-state physicists who call it the X-Y model. The Dirac bilinears may be reexpressed in terms of spins:

$$\begin{aligned}\psi^\dagger\psi &= \tfrac{1}{2}\sigma_z\ ,\\ \bar\psi\psi &= \tfrac{1}{2}\sigma_z(-1)^n\ ,\\ \psi^\dagger\gamma_5\psi &= i[\sigma^+(n)\sigma^-(n+1)-\text{H.c.}]\ .\end{aligned} \tag{2.4}$$

Using these correspondences it is easy to construct spin lattice versions of familiar one-dimensional field theories. For example, the Thirring model is defined by the interaction

$$\begin{aligned}H_{\text{int}} &= g(\bar\psi\psi)^2\\ &= -g\psi_1^\dagger\psi_1\psi_2^\dagger\psi_2\\ &= -\tfrac{1}{4}g\sigma_z(n)\sigma_z(n+1)\ .\end{aligned} \tag{2.5}$$

This together with the free term (X-Y model) defines the general Heisenberg antiferromagnet. However, some caution must be exercised in comparing the long-wavelength behavior of the Heisenberg chain with that of the Thirring model. In particular, both the coupling constant and the speed of light must undergo finite renormalizations before comparison is made.

III. FERMIONS IN THREE DIMENSIONS

We introduce into three-dimensional space a simple cubic lattice with sites labeled by triplets of integers $\vec{r}=(x,y,z)$. At each site six unit vectors $\hat n_x$, $\hat n_y$, $\hat n_z$, $\hat n_{(-x)}$, $\hat n_{(-y)}$, $\hat n_{(-z)}$ are defined as in Fig. 5.

The naive four-component Dirac equation can be written as

$$\dot\psi(\vec r)=\frac{1}{2a}\sum_{\hat n}\vec\alpha\cdot\hat n\psi(\vec r+\hat n)\ , \tag{3.1}$$

where $\vec\alpha$ is the three-vector of Dirac matrices. We work in the representation

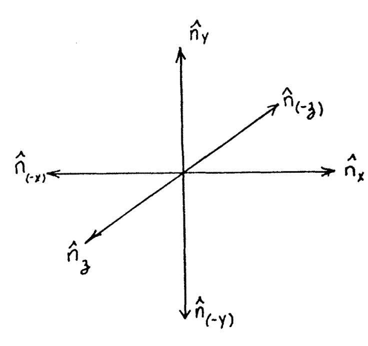

FIG. 5. The six unit lattice vectors.

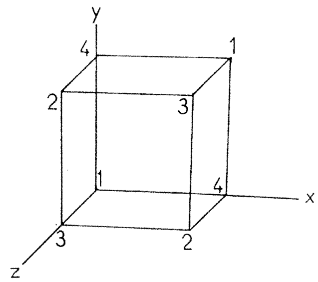

FIG. 6. Labeling of lattice sites.

$$\gamma_0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \vec{\alpha}=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}. \tag{3.2}$$

The dispersion law for (3.1) is

$$\omega=(\alpha_x \sin l_x+\alpha_y \sin l_y+\alpha_z \sin l_z)/a\ , \tag{3.3}$$

replacing the continuum dispersion law

$$\omega=\vec{\alpha}\cdot\vec{k}\ . \tag{3.4}$$

In this case the low-energy spectrum is found at eight points of momentum space corresponding to the corners of a cube in l space. The degeneracy must again be removed, either by the method of Wilson or by reducing the degrees of freedom. We shall choose the latter method.

Let us again reduce the number of degrees of freedom to one component, $\phi(r)$, per lattice site. In the one-dimensional case the lattice was subdivided into odd and even sublattices and the fields were renamed ψ_1 and ψ_2. In the present case we will subdivide the lattice into four sublattices to accommodate the four components of a conventional Dirac field. The subdivision is accomplished by first labeling the corners of the unit cube as shown in Fig. 6. The labeling is then continued periodically through the lattice. In Fig. 7 the planes $z=0$, $x=0$, and $y=0$ are illustrated. Now consider the Hamiltonian

$$H=\sum\left\{\frac{i}{2a}\left[\phi^\dagger(\vec{r})\phi(\vec{r}+\hat{n}_z)-\text{H.c.}\right](-1)^{x+y}+\frac{i}{2a}\left[\phi^\dagger(\vec{r})\phi(\vec{r}+\hat{n}_x)-\text{H.c.}\right]-\frac{1}{2a}\left[\phi^\dagger(\vec{r})\phi(\vec{r}+\hat{n}_y)+\text{H.c.}\right](-1)^{x+y}\right\}. \tag{3.5}$$

The equation of motion following from Eq. (3.5) is

$$\dot{\phi}(\vec{r})=\frac{1}{2a}\left[\phi(\vec{r}+\hat{n}_z)-\phi(\vec{r}-\hat{n}_z)\right](-1)^{x+y}+\frac{1}{2a}\left[\phi(\vec{r}+\hat{n}_x)-\phi(\vec{r}-\hat{n}_x)\right]+\frac{i}{2a}\left[\phi(\vec{r}+\hat{n}_y)-\phi(\vec{r}-\hat{n}_y)\right](-1)^{x+y}\ . \tag{3.6}$$

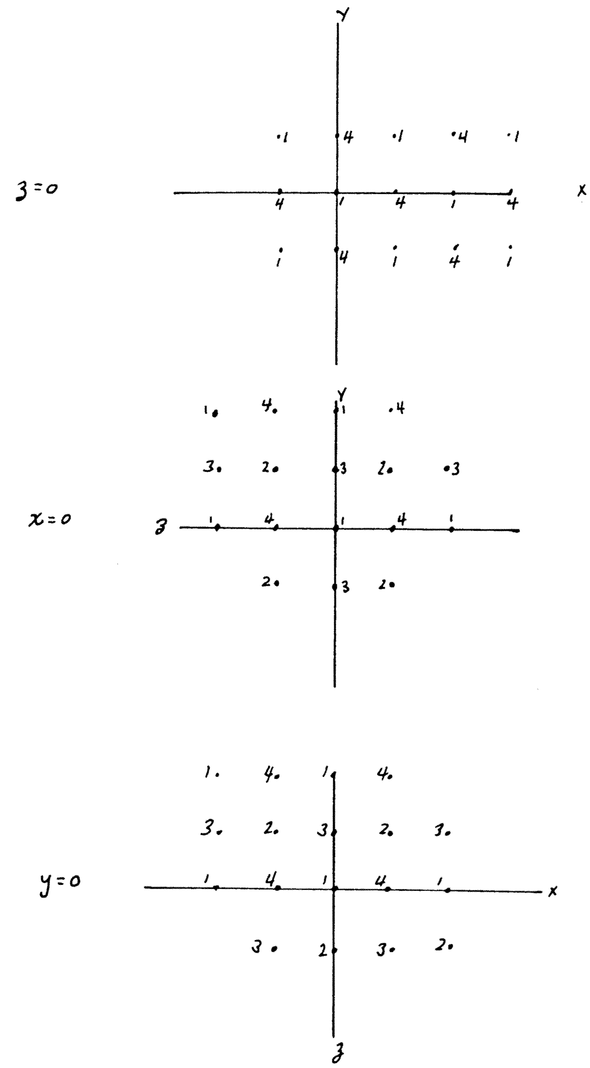

FIG. 7. The planes $z=0$, $x=0$, and $y=0$.

Now introducing the sublattice fields ψ_1, ψ_2, ψ_3, ψ_4 in analogy with the one-dimensional case, we can write Eq. (3.6) as

$$\begin{aligned}\dot{\psi}_1&=\Delta_z\psi_3+\Delta_x\psi_4+i\Delta_y\psi_4\ ,\\ \dot{\psi}_2&=-\Delta_z\psi_4+\Delta_x\psi_3-i\Delta_y\psi_3\ ,\\ \dot{\psi}_3&=\Delta_z\psi_1+\Delta_x\psi_2+i\Delta_y\psi_2\ ,\\ \dot{\psi}_4&=-\Delta_z\psi_2+\Delta_x\psi_1-i\Delta_y\psi_1\ ,\end{aligned} \tag{3.7}$$

where we have used the notation

$$\Delta_z\psi=\left[\psi(\vec{r}+\hat{n}_z)-\psi(\vec{r}-\hat{n}_z)\right]/2a\ .$$

Equation (3.7) may be written in the more compact form

$$\dot{\psi}=\vec{\alpha}\cdot\vec{\Delta}\psi\ . \tag{3.8}$$

For long wavelengths Eq. (3.8) is equivalent to the conventional Dirac equation. However, it does not follow that the long wavelengths exhaust the low-frequency spectrum of (3.8). In fact, we shall see that by further subdividing the lattice into eight sublattices, two complete and independent Dirac fields can be found. We call the two resulting species of fermions u and d and will later identify them with nonstrange quarks.

Let us further subdivide the lattice into those sites (f sites) for which y is even and g sites for which y is odd. The fields are relabeled

$$\psi_i = f_i \quad (y = \text{even}) ,$$

$$\psi_i = g_i \quad (y = \text{odd}) .$$

Next we write (3.7) in Fourier-transformed variables

$$\begin{aligned} a\dot{f} &= (\alpha_z \sin l_z + \alpha_x \sin l_x) f + (\alpha_y \sin l_y) g , \\ a\dot{g} &= (\alpha_z \sin l_z + \alpha_x \sin l_x) g + (\alpha_y \sin l_y) f . \end{aligned} \tag{3.9}$$

Now consider the combinations $f+g=u$ and $f-g=\tilde{d}$. Adding Eqs. (3.9) we find that u is a conventional Dirac field for long wavelengths:

$$\dot{u} = (\alpha_i k_i) u . \tag{3.10}$$

On the other hand, $\tilde{d}$ is not quite a conventional Dirac field since the y derivative term has the wrong sign.

$$\partial_t \tilde{d} = (\alpha_z k_z + \alpha_x k_x - \alpha_y k_y) \tilde{d} . \tag{3.11}$$

This can be straightened out by reidentification of the components of $\tilde{d}$. Define

$$\begin{aligned} d_1 &= \tilde{d}_2 = f_2 - g_2 , \\ d_2 &= -\tilde{d}_1 = -(f_1 - g_1) , \\ d_3 &= -\tilde{d}_4 = -(f_4 - g_4) , \\ d_4 &= \tilde{d}_3 = (f_3 - g_3) . \end{aligned} \tag{3.12}$$

Now d can be seen to satisfy a conventional Dirac equation.

The long-wavelength fermions described by u and d exhaust the finite-energy spectrum of Eq. (3.6) when $a \to 0$. These fermions are massless and free and therefore, in the continuum limit, the spectrum has the isospin symmetry expected of the nonstrange quarks. However, the shorter wavelengths most certainly do not exhibit continuous isospin symmetry. We shall discuss this further in the next section.

Equation (3.6) can be written in a form in which the symmetries are more readily seen. Define the following functions on the lattice sites:

$$\begin{aligned} D(x,y) &= \tfrac{1}{2}[(-1)^x + (-1)^y + (-1)^{x+y+1} + 1] , \\ A(n) &= [i^{n-1/2} + (-i)^{n-1/2}]/\sqrt{2} . \end{aligned} \tag{3.13}$$

On the sites D and A are either 1 or -1 and therefore

$$D^2 = A^2 = 1 . \tag{3.14}$$

They also satisfy the identities

$$\begin{aligned} D(y,x)D(y,x+1) &= (-1)^y , \\ D(y+1,x+1)D(y,x) &= (-1)^{x+y+1} , \\ A(y)A(y+1) &= (-1)^y . \end{aligned} \tag{3.15}$$

Now define

$$\chi(\vec{r}) = (i)^{z+x} A(y) D(x,z) \phi(r) . \tag{3.16}$$

Using (3.15) we find that (3.6) takes on the especially symmetric form

$$\begin{aligned} -i\dot{\chi}(\vec{r}) = {} & [\chi(\vec{r}+\hat{n}_z) + \chi(\vec{r}-\hat{n}_z)](-1)^y \\ & + [\chi(\vec{r}+\hat{n}_x) + \chi(\vec{r}-\hat{n}_x)](-1)^z \\ & + [\chi(\vec{r}+\hat{n}_y) + \chi(\vec{r}-\hat{n}_y)](-1)^x . \end{aligned} \tag{3.17}$$

IV. SYMMETRIES OF THE LATTICE DIRAC EQUATION

Generally the symmetries of Eq. (3.17) are discrete versions of various continuous symmetries in the conventional (continuum) theory. The phenomenon of discrete lattice symmetries being "promoted" to continuous symmetry in the continuum limit has already been met in one dimension. In this case, the unit lattice shift was identified as a γ_5 transformation. In the continuum limit which describes free massless fermions this is promoted to chiral symmetry. In the three-dimensional case the symmetries of rotation, isospin, and chirality will be of this type.

The manifest symmetries of Eq. (3.17) include the following:

(1) Lattice translations by even integers. These are interpreted as ordinary translations as they do not mix the internal indices.

(2) Cyclic interchanges of x, y, z.

(3) Rotations about any axis by angle π. The remaining symmetries are more subtle.

Notice that Eq. (3.5) has the symmetry $\phi(\vec{r}) \to \phi(\vec{r}+\hat{n}_z)$. In Eq. (3.17) this symmetry is somewhat hidden but it can be exposed using Eq. (3.16):

$$\chi(\vec{r}) \to (-1)^x \chi(\vec{r}+\hat{n}_z) . \tag{4.1}$$

Furthermore, applying cyclic permutations to (4.1) gives two new symmetries:

$$\begin{aligned} \chi(\vec{r}) &\to (-1)^z \chi(\vec{r}+\hat{n}_y) , \\ \chi(\vec{r}) &\to (-1)^y \chi(\vec{r}+\hat{n}_x) . \end{aligned} \tag{4.2}$$

In order to identify these symmetries in the conventional theory we reexpress them in terms of u and d fields. Since Eq. (4.1) is equivalent to

$$\phi(\vec{r}) \to \phi(\vec{r}+\hat{n}_z) , \tag{4.3}$$

we find (with some help from Figs. 6 and 7

$$f_1 \to f_3 ,\quad f_4 \to f_2 ,\quad g_1 \to g_3 ,\quad g_4 \to g_2 , \tag{4.4}$$

or

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \to \begin{pmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{pmatrix},$$

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} \to \begin{pmatrix} d_3 \\ d_4 \\ d_1 \\ d_2 \end{pmatrix},$$

or more compactly

$$q \to \gamma_5 \tau_3 q , \tag{4.5}$$

where q represents the isospinor $\binom{u}{d}$ and τ_3 is the usual diagonal isospin matrix. Similarly Eqs. (4.2) can be translated to

$$q \to \gamma_5 \tau_1 q ,\quad q \to \gamma_5 \tau_2 q . \tag{4.6}$$

These transformations are discrete versions of the axial isospin transformations, which together with isospin constitute the group $SU_2 \times SU_2$ (chiral). Invariance under the discrete transformations is sufficient to ensure that *no mass counterterms develop when the fermions are allowed to interact with a gauge field.*

An explicit mass term can be incorporated in the form

$$m \sum_r (\bar{u}u + \bar{d}d) ,$$

which is equivalent to

$$m \sum \chi^\dagger(\vec{r})\chi(\vec{r})(-1)^{x+y+z} . \tag{4.7}$$

This term destroys all the invariances in Eqs. (4.5) and (4.6).

Lattice shifts along the directions $\pm\hat{n}_x \pm \hat{n}_y \pm \hat{n}_z$ can be compounded from (4.1) and (4.2). An example is shown in Fig. 8, and the exact form of the symmetry is given in Eq. (4.8):

$$\chi(\vec{r}) \to (-1)^{x+y+z}\chi(\vec{r} + \hat{n}_x + \hat{n}_y + \hat{n}_z) . \tag{4.8}$$

The directions $\pm\hat{n}_x \pm \hat{n}_y \pm \hat{n}_z$ will be called the *large diagonals*. The conventional form of (4.8) is

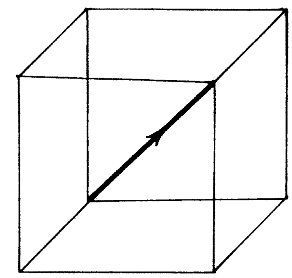

FIG. 8. A shift along a large diagonal.

$$q \to \gamma_5 q , \tag{4.9}$$

or the discrete singlet axial transformation. Finally the "small diagonal" shifts

$$\chi(\vec{r}) \to (-1)^{x+y}\chi(\vec{r} + \hat{n}_z + \hat{n}_x) . \tag{4.10}$$

and cyclic permutations are discrete isospin rotations

$$q \to \tau_{1,2,3}\, q . \tag{4.11}$$

It should be noted that the different isospin directions in Eqs. (4.5), (4.6), and (4.11) are related to different lattice directions. For example, the three symmetries in (4.5) and (4.6) are shift symmetries in three different lattice directions [Eqs. (4.1) and (4.2)]. In order to understand this we must consider the symmetries of lattice rotation by $\pi/2$.

Let us consider a rotation of $\pi/2$ about any axis. Every lattice site r is taken to an image r'. Inspection of Eq. (3.17) shows that it is not invariant under

$$r \to r' ,\quad \chi'(r') = \chi(r) . \tag{4.12}$$

In fact the transformed equation becomes

$$-i\dot{\chi}' = [\chi'(\vec{r} + \hat{n}_z) + \chi'(\vec{r} - \hat{n}_z)](-1)^x + \text{cyclic perm.} \tag{4.13}$$

However, by modifying the transformation law (4.12) to

$$\chi'(\vec{r}\,') = D(x, y)D(y, z)D(x, z)\chi(\vec{r}) , \tag{4.14}$$

we find that (3.17) transforms into itself. Thus the equations do have *cubic* symmetry. The only question is whether this cubic symmetry is associated with spatial rotations, isorotations, or something else. To this end we must again determine how the transformations work on u and d. Direct computation gives rotation about

$$\begin{aligned} z \text{ axis:}\quad & q'(r') = \{\exp[i\tfrac{1}{4}\pi(\tau_3 + \sigma_z)]\}q(r) ,\\ x \text{ axis:}\quad & q'(r') = \{\exp[i\tfrac{1}{4}\pi(\tau_1 + \sigma_x)]\}q(r) ,\\ y \text{ axis:}\quad & q'(r') = \{\exp[i\tfrac{1}{4}\pi(\tau_2 + \sigma_y)]\}q(r) . \end{aligned} \tag{4.15}$$

These transformations are discrete versions of the continuous transformations generated by $I+J$, where I = isospin and J = angular momentum. Thus we have cubic symmetry under simultaneous rotations of isospin and ordinary space. However, separate isospin and spatial rotations are only symmetries for $\theta=\pi$. Now we can understand the strange relation between spatial and isospin directions in Eqs. (4.5), (4.6), and (4.11). In the free-particle theory, it is obvious that the discrete symmetries are promoted to the full continuous symmetry of chiral $U_2 \times U_2$. We shall have to provide arguments that this is the case for an interacting theory. For now we shall just mention that the key to such arguments is asymptotic freedom.

We shall conclude this section by listing the correspondence between various lattice bilinears and the conventional Dirac bilinears:

$$q^\dagger q = u^\dagger u + d^\dagger d = \sum_i (f_i^\dagger f_i + g_i^\dagger g_i) = \phi^\dagger(\vec{r})\phi(\vec{r}) = \chi^\dagger(\vec{r})\chi(\vec{r}), \quad (4.16)$$

$$\bar{q}q = \phi^\dagger(\vec{r})\phi(\vec{r})(-1)^{x+y+z} = \chi^\dagger(\vec{r})\chi(\vec{r})(-1)^{x+y+z}, \quad (4.17)$$

$$q^\dagger\alpha_z q = i[(-1)^y \chi^\dagger(\vec{r})\chi(\vec{r}+\hat{n}_z) - \text{H.c.}]. \quad (4.18)$$

The other components of $q^\dagger\alpha q$ are obtained by cyclic permutation:

$$q^\dagger\gamma_5 q = \sum_\pm \chi^\dagger(\vec{r})\chi(\vec{r}\pm\hat{n}_x\pm\hat{n}_y\pm\hat{n}_z), \quad (4.19)$$

where the sum is over all eight large diagonals:

$$q^\dagger\tau_3 q = i[\chi^\dagger(\vec{r})\chi(\vec{r}+\hat{n}_x+\hat{n}_y) + \chi^\dagger(\vec{r})\chi(\vec{r}+\hat{n}_x-\hat{n}_y) - \text{H.c.}](-1)^{y+z}. \quad (4.20)$$

The other isospin components of $\psi^\dagger\tau\psi$ are obtained by cyclic permutation $x\to y\to z$, $1\to 2\to 3$:

$$q^\dagger\gamma_5\tau_3 q = i[(-1)^x\chi^\dagger(\vec{r})\chi(\vec{r}+\hat{n}_z) - \text{H.c.}], \quad (4.21)$$

$$q^\dagger\alpha_z\tau_3 q = \sum_\pm \chi^\dagger(\vec{r})\chi(\vec{r}\pm\hat{n}_x\pm\hat{n}_y+\hat{n}_z)(-1)^{x+y+z} - \sum_\pm \chi^\dagger(\vec{r})\chi(\vec{r}\pm\hat{n}_x\pm\hat{n}_y-\hat{n}_z)(-1)^{x+y+z}. \quad (4.22)$$

$q^\dagger\alpha_y\tau_2 q$ and $q^\dagger\alpha_x\tau_1 q$ are obtained from (4.22) by cyclic permutation. The important thing to note is that these "diagonal" components of $q^\dagger\alpha\tau q$ involve products of operators at opposite ends of large diagonals.

The off-diagonal isospin currents are exemplified by $q^\dagger\tau_3\alpha_x q$:

$$q^\dagger\tau_3\alpha_x q = (-1)^y[\chi^\dagger(\vec{r})\chi(\vec{r}+\hat{n}_y) + \text{H.c.}]. \quad (4.23)$$

These are operators which involve products at opposite ends of a single link. This structural difference between diagonal and off-diagonal components of $q^\dagger\alpha\tau q$ has its origin in the fact that spatial and isospin rotations are not separate symmetries on the lattice. However, the consequences of this asymmetry are restricted to length scales comparable to the lattice spacing and are irrelevant to the long-distance behavior.

V. LATTICE GAUGE THEORY AND THE SPONTANEOUS BREAKING OF CHIRAL SYMMETRY

In this section we will couple the fermion fields u and d to a gauge field and derive the structure of the ground state in the strong-coupling limit. For simplicity we will use an Abelian gauge field and ignore color. No differences are encountered in the more realistic non-Abelian SU_3 case.

We follow the methods outlined by Kogut and Susskind (Ref. 1). The gauge field on each link is represented by a quantity $U(\vec{r},\hat{n})$, which in the Abelian theory is just a phase factor $\exp(i\theta)$. Conjugate variables $\vec{E}(\vec{r})\cdot\hat{n}$ represent the electric flux. $\vec{E}$ is conjugate to the periodic variable θ and has integer spectrum.

The Hamiltonian for the coupled system of fermions and gauge fields is

$$H = \frac{g^2}{2a}\sum_{r,\hat{n}}(\vec{E}\cdot\hat{n})^2 - \sum_{\text{boxes}}\frac{1}{2ag^2}UUUU + \frac{1}{2a}\left\{\sum_{z\text{ links}}[\chi^\dagger(\vec{r})U(\vec{r},\hat{n}_z)\chi(\vec{r}+\hat{n}_z) + \text{H.c.}](-1)^y + \text{cyclic permutation}\right\}, \quad (5.1)$$

where the notations are those of Kogut and Susskind (Ref. 1).

We wish to determine the ground state of (5.1) for large g. In that limit the only important term in H is the electrostatic term $H_{es} = (g^2/2a)\sum(\vec{E}\cdot\hat{n})^2$. Thus for $g \gg 1$ the ground state $|0\rangle$ satisfies

$$\vec{E}(\vec{r},\hat{n})|0\rangle = 0. \quad (5.2)$$

Since the fermion field χ does not enter H_{es} the ground state is very degenerate. Most of this degeneracy is not associated with any symmetry of the full Hamiltonian and is lifted in higher orders. To see this we shall treat the kinetic term

$$H_K = \frac{1}{2a}\sum(\chi^\dagger U\chi + \text{H.c.})(-1)^y \quad (5.3)$$

as a perturbation.

Let us call the subspace satisfying (5.2) $|0,\overline{\Psi}\rangle$ to indicate that the electrostatic flux is zero but that the fermion content is arbitrary and described by $\overline{\Psi}$. Since the perturbation always excites a unit of flux it is evident that

$$\langle 0\overline{\Psi}|H_K|0\overline{\Psi}'\rangle = 0 \tag{5.4}$$

for all $\overline{\Psi}$ and $\overline{\Psi}'$. Therefore, we must go to second order in H_K to remove the degeneracy. The second-order energy shift is

$$-\left\langle 0\overline{\Psi}\,\middle|\, H_K \frac{1}{H_{\text{es}}} H_K \,\middle|\, 0\psi'\right\rangle \equiv h(\overline{\Psi}), \tag{5.5}$$

where $h(\overline{\Psi})$ is a matrix connecting states of the fermion system. We may regard $h(\overline{\Psi})$ as an effective Hamiltonian for determining the fermion content of $|0\rangle$.

By explicit computation the form of $h(\overline{\Psi})$ is found to be

$$h(\overline{\Psi}) \approx \sum_{\text{links}} \frac{[\psi^\dagger(\vec{r}), \psi(\vec{r})]}{2} \frac{[\psi^\dagger(\vec{r}+\hat{n}), \psi(\vec{r}+\hat{n})]}{2}. \tag{5.6}$$

That is, $h(\overline{\Psi})$ is a nearest-neighbor coupling of the fermionic charges on adjacent sites.

Let us now consider the space of states $\overline{\Psi}$ in detail. At each site we may define a state $|\text{down}\rangle$ satisfying

$$\chi|\text{down}\rangle = 0. \tag{5.7}$$

A second state $|\text{up}\rangle$ is defined by (the use of up and down is not related to the use of u and d for the two fermion species)

$$\chi^\dagger|\text{down}\rangle = |\text{up}\rangle. \tag{5.8}$$

Evidently

$$\begin{aligned} &\chi^\dagger(\text{up}) = 0, \\ &\chi(\text{up}) = |\text{down}\rangle. \end{aligned} \tag{5.9}$$

Now the states $|\text{up}\rangle$ and $|\text{down}\rangle$ are eigenvectors of $\rho = [\chi^\dagger, \chi]/2$ with eigenvalues $\pm\frac{1}{2}$. Obviously the way to minimize $h(\overline{\Psi})$ is to make every link terminate on a configuration $|\text{up}\rangle$ and $|\text{down}\rangle$ giving the minimum value to $\rho(\vec{r})\rho(\vec{r}+\hat{n})$. This is accomplished by dividing the lattice into two sublattices for which $x+y+z$ is odd or even. On the even sublattice we choose $|\text{down}\rangle$ and on the odd sublattice $|\text{up}\rangle$ (or vice versa). The ground state is still degenerate since the odd and even sublattices can be interchanged. However, this degeneracy is not lifted in higher order since it is associated with symmetries of the Hamiltonian (unit shift or chiral invariance). Thus we have exhibited spontaneous breakdown of the discrete unit shift invariances (4.1) and (4.2) or, better still, the chiral transformations (4.5), (4.6), and (4.8). In fact, our ground state obviously has a nonvanishing value of

$$\overline{q}q = \chi^\dagger(\vec{r})\chi(\vec{r})(-1)^r$$

even when averaged over $\vec{r}$.

Since the chiral transformations are only discrete in the lattice Hamiltonian no Goldstone bosons (pions) accompany the breakdown. Presumably as $a\to 0$ and the symmetries are promoted to continuous symmetries Goldstone particles will emerge.

VI. INTERACTIONS AND THE CONTINUUM LIMIT

Once interactions are included it is no longer trivial to prove that the continuum limit of the fermion field is correctly behaved. The purpose of this section is to illustrate one pitfall and to speculate on how it may be resolved. The chosen example is more or less characteristic of the difficulties.

Consider the quark field to be interacting with an external potential $V(r)$. The Hamiltonian is

$$H_0 + g\sum \phi^\dagger(r)\phi(r)V(r), \tag{6.1}$$

where H_0 is the free lattice Hamiltonian. We shall take $V(r)$ to be a plane wave e^{ilr} with $|l|\le\pi$. If a low-momentum quark with initial wave function $\phi(r)$ scatters off $V(r)$ it undergoes the transition

$$\phi(r)\to\phi(r)V(r) = \phi(r)e^{ilr}, \tag{6.2}$$

or in k space

$$\phi(k)\to\phi(k+l). \tag{6.3}$$

If l is small then this is merely a shift of the momentum of the quark. Suppose instead that l is large. For example, suppose that $l_x = l_z = 0$, $l_y=\pi$ ($l\equiv\pi_y$). Then the transition may be described in one of two ways. We may simply say the quark's final momentum is large ($\sim\pi$). However, we may also say that the final momentum of the quark is small but that the internal degrees of freedom have flipped.

To see how this second description works let us consider the relationship between ϕ and (u,d) in k space. Evidently

$$\begin{aligned} f_i(r) &= \psi_i(r)\left[\frac{1+(-1)^y}{2}\right], \\ g_i(r) &= \psi_i(r)\left[\frac{1+(-1)^{y+1}}{2}\right], \end{aligned} \tag{6.4}$$

or

$$\begin{aligned} f_i(k) &= \tfrac{1}{2}[\psi_i(k)+\psi_i(k+\pi_y)], \\ g_i(k) &= \tfrac{1}{2}[\psi_i(k)-\psi_i(k+\pi_y)], \end{aligned} \tag{6.5}$$

where $\pi_y = (0, \pi, 0)$.

Equations (6.5) may be rewritten in terms of u and and $\tilde{d}$:

$$u_i(k) = f_i(k) + g_i(k) = \psi_i(k),$$
$$\tilde{d}_i = f_i(k) - g_i(k) = \psi_i(k + \pi_y),$$

or

$$u_i(k + \pi_y) = \tilde{d}(k),$$
$$\tilde{d}_i(k + \pi_y) = u_i(k). \tag{6.6}$$

Thus when a low-energy u quark is scattered by a field with wave number $\sim\pi_y$ it becomes a low-momentum d quark. The precise transition for absorption of momentum π_y is proportional to

$$g(\bar{q}\sigma_y\tau_2 q). \tag{6.7}$$

For problems in an external field we need not worry about this effect. For a smooth external potential all the wave numbers k tend to zero with a. Thus in the conventional continuum limit of an external field the processes described in (6.7) do not occur.

In an interacting quantum field theory the situation is less clear. Quanta may be exchanged between quarks. These quanta may have $k \sim \pi_y$. In this case we describe the event as a transition between low-momentum quarks with effective vertex proportional to

$$\frac{g^2}{a}(\bar{q}\sigma_y\tau_2 q)(\bar{q}\sigma_y\tau_2 q). \tag{6.8}$$

Quanta with $k \sim \pi_y$ and $k \sim \pi_y$ may also be exchanged. The total effect is of the form

$$\frac{g^2}{a}[(\bar{q}\sigma_x\tau_1 q)^2 + (\bar{q}\sigma_y\tau_2 q)^2 + (\bar{q}\sigma_z\tau_3 q)^2]. \tag{6.9}$$

In other words, the effects of high-momentum exchanges induce an effective quartic nonrenormalizable coupling with coupling constant given by g^2a in conventional units.

There are two ways to deal with these unwanted effects. The first is simply to counter them by compensating terms in the original Hamiltonian. Thus we would add the negative of (6.9) into H. This is simple and direct, but it means that the counterterms must be carefully evaluated. The second way is more subtle and consists of ignoring the problem altogether. However, this only works in a asymptotically free theories. The point here is that in these theories the continuum limit is achieved by allowing the bare coupling g to tend to zero. Therefore, the strength of the induced nonrenormalizable vertex also goes to zero. This by itself does not mean that the nonrenormalization term becomes irrelevant. After all, the Yang-Mills coupling g tends to zero but is certainly not irrelevant to the long wavelengths. This is because the Yang-Mills coupling excites an "infrared instability" of the free field. On the other hand, nonrenormalizable couplings are very infrared stable and if they are made infinitesimally small their effect at long wavelength will typically be even smaller.

Obviously we can not pretend that this discussion is complete. It is clear that a careful study of the renormalization of lattice field theories is required. This is presently under investigation by S. Elitzur.

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