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Singular pseudoparticles in Higgs theories

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We present a new class of pseudoparticles in four-dimensional Euclidean non-Abelian gauge theories with Higgs fields. These pseudoparticles, in the classical limit, have nonzero Lagrangian density at only one single point. We discuss the topological and analytic construction of these pseudoparticles and the related fractional-topological-charge configurations. The action of a unit-topological-charge pseudoparticle is the usual $8\pi^2/g^2$. These pseudoparticles are not field configurations, but are limiting cases of field configurations. Owing to quantum fluctuations, their size is proportional to $\sqrt{\hbar}$ if $\hbar$ is small.

I. INTRODUCTION

In this paper we present a new class of classical Euclidean pseudoparticles in some gauge theories in four space-time dimensions. The theories we consider include Higgs fields:

$$\mathcal{L} = \tfrac{1}{4}G^i_{\mu\nu}G^i_{\mu\nu} + \tfrac{1}{2}D_\mu\varphi^i D_\mu\varphi^i + V(\varphi)\,. \qquad (1)$$

Our objects are determined by nontrivial topological mappings of the direction in space-time into the direction of the scalar field in internal-symmetry space. (Given $\varphi \neq 0$, the gauge field is determined by $D_\mu\varphi \to 0$ as $R\to\infty$.) Such nontrivial mappings are readily apparent if the symmetry group is SO(4), as discussed in Ref. 1, and indeed the configurations described in Ref. 1 are examples of our objects although SO(4) is not our example in the present paper.

Since $SO(4) = SU(2)\times SU(2)$, we expressed the SO(4) solution of Ref. 1 in terms of the SU(2)'s, and found the second SU(2) to be irrelevant. Thus our example will be to take the internal-symmetry group to be SU(2).

Let us suppose for the moment that the scalar field with nonzero vacuum expectation value is an isovector. Its value at large distances is a point on a sphere in three-dimensional space, S^2. The directions in space-time are given by points on a sphere in four-dimensional space, S^3. Thus the topology is specified by mappings of S^3 into S^2 and $\Pi_3(S^2) = Z$. [It is difficult to picture these mappings because the one lower-dimensional analog, nontrivial mappings of S^2 into the circle, do not exist, $\Pi_2(S^1) = 0$.] The unit-topological-charge mapping and from it the solution can be obtained from projective geometry. In addition, the singular half-topological-charge configuration[2] can be similarly obtained. The idea is that four-dimensional space-time is considered as a two-complex-dimensional space, which is projected onto the complex projective line, which can then be considered as the real projective plane and projected back to the sphere.

For other topological charges, and for other representations for the scalar field, we describe the solutions instead by a matrix technique. With this technique we are also able to construct additional configurations with more than two centers, each of fractional topological charge, and with possibly unequal divisions of the charge between the centers.

A scaling argument[3] suggests that the objects we are describing do not exist, and in a strict sense they do not exist as solutions of the classical equations of motion. However, for practical purposes they exist. What we actually find is a set of configurations whose action is slightly higher than, but arbitrarily close to, the greatest lower bound on the action for the given topology. Our classical object is pointlike, with vacuum at all other points, and a particular "structure" at that point. The true size of these objects is given by quantum effects and is simple to estimate.

II. PROJECTIVE MAPPING

In this section we assume that the Higgs field φ is an isovector under the SU(2) gauge group, and that it has vacuum expectation value $\varphi_0 \neq 0$. The topological properties of the solution are specified by a mapping, $\hat{R} \to \varphi_i$, of directions in four-space to φ_i, where $\varphi_i\varphi_i = \varphi_0^2$. This mapping specifies the Higgs field at large distances from the pseudoparticle.

Topologically nontrivial mappings can be obtained from projective geometry. We consider four-space R^4 as a two-dimensional complex space C^2. We project this space into the complex projective line, which in turn can be regarded as a real projective plane. As is well known, the projective plane is topologically identical to a sphere, which can be obtained by a stereographic projection.

We start with a gnomonic projection, shown in Fig. 1. Topologically, we would obtain the same configuration projecting from any point inside the

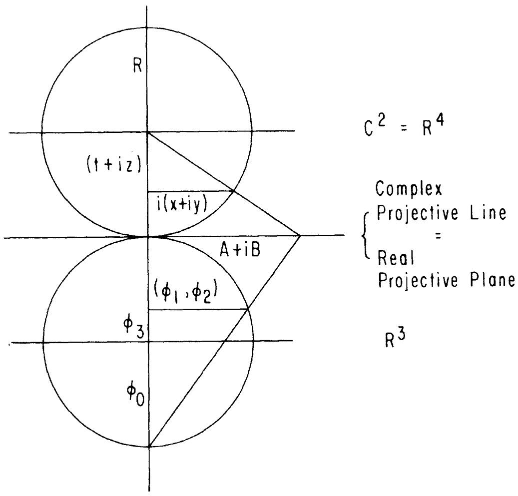

FIG. 1. Projective construction of the pseudoparticle mapping $S^3 \to S^2$. The top half of the figure is four-dimensional space, R^4, considered as two-dimensional complex space C^2. Below the projective plane is three-dimensional space R^3. $\vec{\varphi}$ is found by similar triangles; $A+iB$ is scaled down by R and up by $2\varphi_0$.

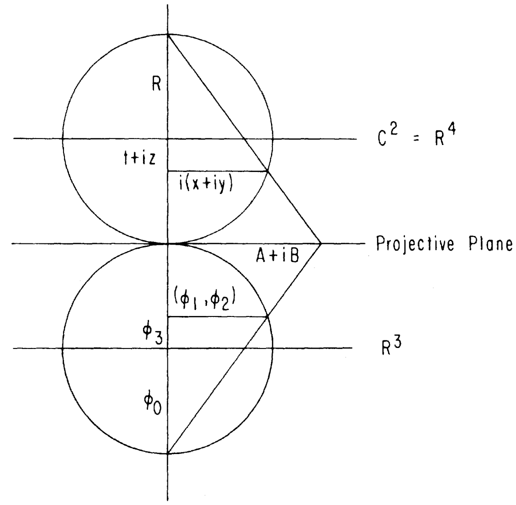

FIG. 2. Projective construction of the half-charge analog of the pseudoparticle, similar to Fig. 1, but using a complex stereographic projection. $A+iB$ is scaled down by $2R$ and up by $2\varphi_0$.

sphere $R=$const. Anticipating a simplicity that will result in the matrix approach, we choose our complex coordinates to be $z_1=t+iz$, $z_2=i(x+iy)$. From two pairs of similar triangles in Fig. 1, we see that the mapping is given by

$$A+iB=\frac{i(x+iy)}{t+iz}=\frac{\varphi_1+i\varphi_2}{\varphi_0+\varphi_3}\ , \tag{2}$$

where an appropriate scale change is made on the projective plane.

Solving for $\varphi_1, \varphi_2, \varphi_3$ we find

$$\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}=\begin{pmatrix}2(xz-yt)/R^2\\ 2(yz+xt)/R^2\\ 2(z^2+t^2)/R^2-1\end{pmatrix}\varphi_0, \tag{3}$$

which is the unit-topological-charge mapping of S^3 onto S^2.

We can obtain other topologically equivalent mappings by cyclic permutations of x, y, z and simultaneous permutations of $\varphi_1, \varphi_2, \varphi_3$. We arrange the coefficients of φ_0 into a matrix, where the columns are arranged by whether (x, φ_1), (y, φ_2), or (z, φ_3) occur in the denominator of Eq. (2). This matrix

$$M_1=\begin{pmatrix}2(x^2+t^2)/R^2-1 & 2(xy+zt)/R^2 & 2(xz-yt)/R^2\\ 2(xy-zt)/R^2 & 2(y^2+t^2)/R^2-1 & 2(yz+xt)/R^2\\ 2(xz+yt)/R^2 & 2(yz-xt)/R^2 & 2(z^2+t^2)/R^2-1\end{pmatrix} \tag{4}$$

is the starting point for the development in the next section. (The subscript 1 refers to the isospin of the Higgs field.) The infinity at $R=0$ represents the absence of the vacuum at the position of the pseudoparticle.

If we were to project C^2 through a point outside the hypersphere, we would obtain a topologically trivial mapping. Projection through a point on the sphere is intermediate between a pseudoparticle and nothing. In Fig. 2, we show a stereographic projection. Similar triangles give us

$$A+iB=\frac{i(x+iy)}{R+t+iz}$$
$$=\frac{\varphi_1+\varphi_2}{\varphi_0+\varphi_3}\ . \tag{5}$$

Solving for φ and making the cyclic permutations we find a matrix

$$M_1^{1/2} = \begin{pmatrix} \frac{x^2+Rt+t^2}{R(R+t)} & \frac{xy+z(R+t)}{R(R+t)} & \frac{xz-y(r+t)}{R(R+t)} \\ \frac{xy-z(R+t)}{R(R+t)} & \frac{y^2+Rt+t^2}{R(R+t)} & \frac{yz+x(R+t)}{R(R+t)} \\ \frac{xz+y(R+t)}{R(R+t)} & \frac{yz-x(R+t)}{R(R+t)} & \frac{z^2+Rt+t^2}{R(R+t)} \end{pmatrix}. \tag{6}$$

This matrix is infinite, and therefore cannot represent the vacuum either at $R=0$ or along the half-line $t=-R$. It represents a half-topological-charge object, thus the superscript $\frac{1}{2}$.

For isospin $\frac{1}{2}$, the projection is not needed, as the representation is just two complex numbers. Configurations for φ/φ_0 can be chosen as a column of the matrix

$$M_{1/2} = \frac{1}{R}\begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}. \tag{7}$$

In the next section we will relate the matrices M_1, $M_1^{1/2}$, $M_{1/2}$, and discuss the contruction of all M_T^q, where T is the Higgs isospin and q is the topological charge.

III. MATRIX APPROACH

Regions of space occupied by the vacuum in a gauge theory can be described as follows. At each point in such regions there is a unitary matrix M in the representation of the gauge group to which the Higgs field belongs. The Higgs field is given by

$$\vec{\varphi} = M\hat{e}\varphi_0 , \tag{8}$$

where e is any fixed unit vector. The field A_μ^i is given by

$$gA_\mu^i T^i = (\partial_\mu M)M^{-1} , \tag{9}$$

where the T^i's are the gauge group generators in the same representation as M. The matrices defined in the preceding section are examples of such matrices.

The isovector generators of SU(2) have the property

$$(T^i)^3 = T^i \tag{10}$$

since the eigenvalues of T^i are $0, \pm 1$. Therefore any unitary matrix in the isovector representation is

$$\exp(i\theta\hat{e}\cdot\vec{T}) = 1 + i\sin\theta\hat{e}\cdot\vec{T} + (1-\cos\theta)(\hat{e}\cdot\vec{T})^2. \tag{11}$$

Examination of Eq. (4) shows M_1 to be of this form, with $\hat{e}=\hat{r}$, $\vec{r}=(x, y, z)$, and $\cos\theta = (t^2-r^2)/R^2$, $\sin\theta = 2tr/R^2$. This result is more simply expressed in terms of the half angle:

$$\cos\tfrac{1}{2}\theta = t/R ,\qquad \sin\tfrac{1}{2}\theta = r/R . \tag{12}$$

Now that we have this result, we can find pseudoparticles for any representation for the Higgs field. We use

$$M = \exp(i\theta\hat{r}\cdot\vec{T}) , \tag{13}$$

where $\vec{T}$ is the vector of generators in the desired representation, and θ is given by Eq. (12). The right side of Eq. (13) can be simplified by using the equation similar to Eq. (10) for the given representation.

For isospin $\frac{1}{2}$ the condition is

$$T_i^2 = \tfrac{1}{4} . \tag{14}$$

So

$$M_{1/2} = \cos\tfrac{1}{2}\theta + i\hat{r}\cdot\vec{\tau}\sin\tfrac{1}{2}\theta = (t+i\vec{r}\cdot\vec{\tau})/R , \tag{15}$$

which is identical to Eq. (7).

A gauge transformation is performed by multiplying the matrix M by another unitary matrix, either on the right or on the left. Thus a configuration of multiple topological charge can be obtained by multiplying together the matrices for the individual single-charge pseudoparticles. The matrix

$$\frac{(t+i\vec{r}\cdot\vec{\tau})[t-t_0+i(\vec{r}-\vec{r}_0)\cdot\vec{\tau}]}{R|\vec{R}-\vec{R}_0|} \tag{16}$$

represents two pseudoparticles, one at $R=0$ and one at $\vec{R}=\vec{R}_0 \equiv (\vec{r}_0, t_0)$.

In particular, a topological-charge-q configuration concentrated at $R=0$ can be obtained by

$$M^q = \exp(iq\theta\hat{r}\cdot\vec{T}) , \tag{17}$$

and again $\vec{T}$ can be in any representation. If q is a nonzero integer, this expression is singular at $R=0$, since θ is. Negative q's give antipseudoparticles.

If q is not an integer, additional configurations are constructed. Since these do not have integer topological charge, they cannot be singularity-free on a sphere surrounding $R=0$. The unit vector $\hat{r}$ appearing in Eq. (17) is singular at $r=0$. This singularity must occur in M^q, at least along one of the two half lines $t=\pm R$. In fact, it occurs at $t=-R$.

As an example, we can choose $q=\frac{1}{2}$, $T=1$. Analogous to Eq. (11), we have

$$M_1^{1/2} = 1 + i\sin\tfrac{1}{2}\theta\hat{r}\cdot\vec{T} + (1-\cos\tfrac{1}{2}\theta)(\hat{r}\cdot\vec{T})^2$$
$$= 1 + i\frac{\vec{r}\cdot\vec{T}}{R} + \frac{1-t/R}{R^2-t^2}(\vec{r}\cdot\vec{T})^2$$
$$= 1 + \frac{i\vec{r}\cdot\vec{T}}{R} + \frac{(\vec{r}\cdot\vec{T})^2}{R(R+t)} . \tag{18}$$

This is the same as the matrix, Eq. (5), constructed by the projection method.

Consider a number of noninteger-charge configurations placed on the line $r=0$, such that the sum of their q's is an integer. The product of their matrices is constructed, from which $\vec{\varphi}$ and $\vec{A}_\mu$ are constructed. This matrix and these fields are singular on the line segment joining these, but are regular elsewhere. Such combinations generalize the objects described by Callan *et al.*[2] in that the restriction $q=\frac{1}{2}$ is unnecessary. We conjecture that the properties of such configurations are very similar to those of the pairs of objects described in Ref. 2.

The restriction to these combinations occurring with the singularity line pointing in the t direction is obviously artificial. Singularity lines made of straight-line segments in any direction and possibly joining together (with zero q at the vertex, if desired) are easily written down by multiplying the appropriate matrices. Curved connections are constructed in principle by limits of such straight-line connections.

The problem of smearing out the singularities of these configurations in order to obtain minimal action is formidable.

IV. MINIMIZING THE ACTION

As will become clear shortly, one does not minimize the action by attempting to solve the dynamical equations. We begin by considering a scaling

$$\vec{\varphi}(\vec{R}) \to \vec{\varphi}(\vec{R}/\lambda) ,$$
$$\vec{A}(\vec{R}) \to \frac{1}{\lambda}\vec{A}(\vec{R}/\lambda) . \tag{19}$$

Under this scaling, the three terms in the action behave as

$$\int \tfrac{1}{4}(\vec{G}_{\mu\nu})^2 d^4\vec{R} \sim \text{const} ,$$
$$\int \tfrac{1}{2}(D_\mu\vec{\varphi})^2 d^4\vec{R} \sim \lambda^2 , \tag{20}$$
$$\int V(\varphi) d^4\vec{R} \sim \lambda^4 .$$

Therefore, as the scale λ is reduced, the action is made smaller. The classical solution thus consists of the fields concentrated at a point, with vacuum everywhere else. The scale invariance of the pure gauge pseudoparticle is removed by the existence of the Higgs field with nonzero vacuum expectation value (so the coefficients of λ^2 and λ^4 are nonzero). Moreover, the minimum action is just the usual pure gauge action obtained from $G_{\mu\nu}=\tilde{G}_{\mu\nu}$. For single topological charge, our gauge is the usual one, $\partial_\mu A_\mu^i=0$, and the solution was shown by Belavin *et al.*[4] to be

$$g\vec{A}_\mu\cdot\vec{T} = \frac{R^2}{R^2+R_0^2}(\partial_\mu M)M^{-1} . \tag{21}$$

In our case, the limit $R_0 \to 0$ is to be understood. The first term in the action is then $8\pi^2/g^2$. (In order to make the λ^2 term finite, the factor in front must be modified to go to zero faster than $1/R^2$ for $R \gg R_0$.)

The interpretation of our object requires some discussion, as the limiting process is not normally a part of finding a classical solution. If the question asked is: What is the solution of the classical field equations with unit topological charge? or even the weaker question: What is the minimal action for unit topological charge? then the precise answer is that a solution or a minimum does not exist. Yet these questions, understood in the precise sense, are not the physically interesting question. This particular question would be: What is the nature of the paths which are important in the path integral. In this sense our object is relevant.

The weighting factor in the path integral is the exponential of the action divided by $\hbar$. The important configurations are those for which the action is within about $\hbar$ of the greatest lower bound. Our solution is the greatest lower bound, since the gauge action is minimal and the remainder tends to zero as R_0 approaches zero.

The actual expected size of our configuration is thus given by the quantum fluctuations. For example, consider the case with the Higgs field φ_0 being an isospinor. With $\vec{A}_\mu$ as given by Eq. (21), the term $\frac{1}{2}\int|D_\mu\varphi|^2 d^4\vec{R}$ in the action is minimized by

$$\vec{\varphi} = \frac{R}{(R^2+R_0^2)^{1/2}} M\hat{e}\varphi_0 \tag{22}$$

[compare this with Eq. (8)]. The action is then

$$S = \frac{8\pi^2}{g^2} + \pi^2 R_0^2\varphi_0^2 + O(R_0^4) . \tag{23}$$

The size $\langle R_0\rangle$ is determined by $\delta S \approx \hbar$, from which

$$\langle R_0\rangle \approx \frac{\sqrt{\hbar}}{\pi\varphi_0} . \tag{24}$$

The fact that our solution has this unusual nature has one important consequence: It is not possible to estimate the path integral by means of a Gaussian integral in the perturbed path. A possible approximation to attempt is a Gaussian integral expanding from the configuration whose size is given by Eq. (24).

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