Supersymmetric form of the nonlinear σ model in two dimensions

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A supersymmetric form of the two-dimensional nonlinear σ model is described; the primary purpose is to explore further the analogy between the σ model and four-dimensional Yang-Mills theory. In the case of three components, the theory contains an enlarged supersymmetry algebra and is equivalent to the ordinary σ model with an additional logarithmic interaction between pseudoparticles.

INTRODUCTION

The purpose of this paper is to describe a natural extension of the two-dimensional nonlinear σ model to include a supersymmetric coupling to fermions. The primary reason for studying the two-dimensional σ model is that it has many similarities with four-dimensional gauge theories.[1] One motivation for incorporating supersymmetry is a suspicion that the supersymmetric form of the theory may be easier to understand. Another motivation is that the behavior of the σ model with fermions may shed some light on the interaction of fermions with four-dimensional gauge fields.

CONSTRUCTION OF THE MODEL

I will try to describe the construction of a supersymmetric σ model in a self-contained way.

In the superspace approach[2] to supersymmetry, we consider a spacetime described by the usual coordinates x^μ but also by additional anticommuting coordinates θ_α. Here we will consider a two-dimensional spacetime and take θ to be a single two-component Majorana spinor. [By a Majorana spinor I mean a two-component real spinor. The γ matrices I take to be $\gamma^0=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\gamma^1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$, $\gamma^5=\gamma^0\gamma^1$, and the adjoint of a spinor is, as usual, $\bar\psi=\psi^T\gamma^0$. Note that, with these choices, $\bar\psi$ is pure imaginary while ψ is real.]

For purposes of orientation, consider a scalar field ϕ defined on the space with coordinates x^μ, θ. If we expand this field in a power series in θ, we find that terms cubic or higher in θ vanish because θ is an anticommuting spinor with only two components. So the most general form is

$$\phi(x^\mu,\theta_\alpha)=A(x^\mu)+\bar\theta\psi(x^\mu)+\tfrac{1}{2}\bar\theta\theta F(x^\mu)\ . \qquad (1)$$

A field ϕ defined on the space (x^μ,θ_α) is thus equivalent to three ordinary fields A, ψ, and F that are functions of x^μ only.

Now we wish to construct theories that are invariant not only under the ordinary translation

$$x^\mu\to x^\mu+a^\mu\ ,$$

$$\theta\to\theta\ ,$$

but also under the additional transformations

$$x^\mu\to x^\mu+i\bar\epsilon\gamma^\mu\theta\ ,$$

$$\theta\to\theta+\epsilon\ ,$$

where ϵ is a constant Majorana spinor. These transformations are usually called supersymmetry transformations. Just as the ordinary translations are generated by the derivative operator, $\partial/\partial x^\mu$, the supersymmetry transformations are generated by the operator

$$Q_\alpha=\frac{\partial}{\partial\bar\theta_\alpha}+i(\gamma^\mu\theta)_\alpha\frac{\partial}{\partial x^\mu}\ .$$

To construct supersymmetric Lagrangians, we make use of the additional operator

$$P_\alpha=\frac{\partial}{\partial\bar\theta_\alpha}-i(\gamma^\mu\theta)_\alpha\frac{\partial}{\partial x^\mu}\ .$$

The basic property of P is that it anticommutes with Q:

$$\{P_\alpha,Q_\beta\}=0\ .$$

The second-order operator $\epsilon_{\alpha\beta}P_\alpha P_\beta$ (where $\epsilon_{\alpha\beta}$ is the antisymmetric tensor) therefore commutes with Q,

$$[Q_\lambda,\epsilon_{\alpha\beta}P_\alpha P_\beta]=0\ , \qquad (2)$$

so it is supersymmetric as well as Lorentz invariant.

We may now consider as a Lagrangian density the quantity $\frac{1}{2}\phi\epsilon_{\alpha\beta}P_\alpha P_\beta\phi$. Because of (2), under a supersymmetry transformation with $\delta\phi=Q_\lambda\phi$, the variation of this Lagrangian density is simply $Q_\lambda(\frac{1}{2}\phi\epsilon_{\alpha\beta}P_\alpha P_\beta\phi)$. This is a total derivative, and so vanishes when integrated over all the coordinates (θ_α as well as x^μ). Therefore, the action integral

$$\int d^2x\,d\theta_1 d\theta_2\,\tfrac{1}{2}\phi\epsilon_{\alpha\beta}P_\alpha P_\beta\phi \qquad (3)$$

is supersymmetric and Lorentz invariant.

The meaning of (3) is more transparent if one carries out the integration over θ_1 and θ_2. According to the standard rules for fermion integration,[3] the value of the integral is just the coefficient of the quadratic term in the expansion of the integrand in powers of θ. This is

$$\frac{1}{2}\int d^2x[(\partial_\mu A)^2+\bar{\psi}i\partial\!\!\!/\psi+F^2] \ . \tag{4}$$

Thus, we have constructed the supersymmetric form of a free field theory. We have a massless free scalar A, a massless free spinor ψ, and an auxiliary field F which in this case decouples—its equation of motion is simply $F=0$. The construction of the supersymmetric σ model is exactly analogous.

The usual nonlinear σ model involves a d-component real scalar field n^a constrained to satisfy $n^a n^a=1$. To construct a supersymmetric version of the σ model we simply consider a d-component real field ϕ^a defined on the superspace (x^μ, θ_α). This field will have an expansion

$$\phi^a=n^a+\bar{\theta}\psi^a+\tfrac{1}{2}\bar{\theta}\theta F^a \ , \tag{5}$$

where n^a and F^a are real, scalar, isovector fields, and ψ^a_α is a Majorana spinor and isovector.

We now impose the constraint $\phi^2=1$, which, when expanded in powers of θ, yields three constraints:

$$\begin{aligned} &n^2=1 \ , \\ &n^a\psi^a_\alpha=0 \ , \\ &n^aF^a=\tfrac{1}{2}\bar{\psi}^a\psi^a \ . \end{aligned} \tag{6}$$

As before, we consider the Lagrangian density $(1/2g^2)\phi^a\epsilon_{\alpha\beta}P_\alpha P_\beta\phi^a$. After integration over the anticommuting coordinates, we find that the corresponding action is

$$\frac{1}{g^2}\int d^2x[\tfrac{1}{2}(\partial_\mu n^a)(\partial_\mu n^a)+\tfrac{1}{2}\bar{\psi}^a i\partial\!\!\!/\psi^a+\tfrac{1}{2}(F^a)^2] \ . \tag{7}$$

Exactly as for the ordinary nonlinear σ model, the action is that of a free field theory—but subject to constraints (6) that induce interactions.

I will make some general comments about this theory before moving on in the next section to discuss some special properties of the three-component model.

The supersymmetry transformation laws are

$$\begin{aligned} &\delta n^a=\bar{\epsilon}\psi^a \ , \\ &\delta\psi^a=-i(\gamma^\mu\epsilon)\partial_\mu n^a+F^a\epsilon \ , \\ &\delta F^a=\bar{\epsilon}i\partial\!\!\!/\psi^a \ , \end{aligned} \tag{8}$$

where ϵ is a constant spinor.[4] These are found most easily by expanding the relation $\delta\phi=\bar{\epsilon}Q\phi$ in powers of θ. The conserved supersymmetry current is $S_\mu=(\partial_\lambda n^a)\gamma^\lambda\gamma_\mu\psi^a$.

Because derivatives of F appear neither in the Lagrangian nor in the constraints (6), it is possible to eliminate F and to consider the equivalent theory defined by the action

$$\frac{1}{g^2}\int d^2x[\tfrac{1}{2}(\partial_\mu n)^2+\tfrac{1}{2}\bar{\psi}i\partial\!\!\!/\psi+\tfrac{1}{8}(\bar{\psi}^a\psi^a)^2] \tag{9}$$

and the constraints

$$\begin{aligned} &n^2=1 \ , \\ &n\cdot\psi=0. \end{aligned}$$

This theory is renormalizable consistently with supersymmetry, because it is possible to regularize the theory by adding higher derivative terms to the Lagrangian in a way that preserves supersymmetry. For example, one may add to the Lagrangian a term $(\epsilon_{\alpha\beta}P_\alpha P_\beta\phi)^2$ as a regulator.

This theory is asymptotically free; in fact, one can show that its one-loop β function coincides with that which has been calculated for the nonlinear σ model without supersymmetry.[5]

In the case of the nonlinear σ model, it is known from the very interesting exact solution for the S matrix by Zamolodchikov that the physical particles have a nonzero mass.[6] (This had been suspected from the high-temperature expansion, the asymptotic freedom, and the $1/N$ expansion.) The nonlinear σ model possesses no symmetry that would forbid the appearance of physical masses (even though it is not possible to add an explicit mass term to the Lagrangian, and it may not seem intuitively obvious that physical masses should appear). The supersymmetric model, however, contains a discrete chiral symmetry $\psi\to\gamma_5\psi$ which will forbid the appearance of masses unless it is spontaneously broken. Alvarez[7] has studied the $1/N$ expansion of this theory, and finds chiral-symmetry breaking and the appearance of masses.

The most interesting fact is that it is probably possible, using the method of Zamolodchikov,[6] to solve for the exact S matrix of this theory. An attempt to do this is under way.

THE THREE-COMPONENT THEORY

The nonlinear σ model is particularly interesting in the case of three components, because it is this case that has the most striking similarities with four-dimensional gauge theories. Here I would like to point out some special properties that the supersymmetric model has when the number of components equals three.

The first observation is that for three components, the theory has an enlarged supersymmetry algebra. In addition to the supersymmetry current $S_\mu=\partial_\nu n^a\gamma^\nu\gamma_\mu\psi^a$, there is a second conserved supersymmetry current $\tilde{S}_\mu=\epsilon^{abc}n^a\partial_\nu n^b\gamma^\nu\gamma_\mu\psi^c$. In addition, there is a conserved vector current V_μ

$=\epsilon^{abc}n^a\bar{\psi}^b\gamma_\mu\psi_c$, and axial-vector current $A_\mu=\epsilon_{\mu\nu}V^\nu$, which do not have analogs for more than three components.

Designating as Q_α, $\tilde{Q}_\alpha$, and K the charges corresponding to S_μ, $\tilde{S}_\mu$, and V_μ, these generate the algebra

$$\begin{aligned}&\{Q_\alpha,\bar{Q}_\beta\}=\{\tilde{Q}_\alpha,\bar{\tilde{Q}}_\beta\}=P_\mu\gamma^\mu_{\alpha\beta}\ ,\\&\{Q_\alpha,\bar{\tilde{Q}}_\beta\}=0\ ,\\&[K,Q_\alpha]=\tilde{Q}_\alpha\ ,\\&[K,\tilde{Q}_\alpha]=-Q_\alpha\ .\end{aligned}\tag{10}$$

One can recognize this as the algebra of complex supersymmetry transformations or supersymmetry transformations with internal symmetry O(2).

One may wonder whether this enlarged symmetry algebra is present in the quantum theory. In fact, I do not know how to regularize the theory in a way that preserves this algebra. Nevertheless, the algebra is present in the renormalized quantum theory, as we can see as follows. First of all, the current V_μ is conserved in the renormalized theory because there is simply no operator with dimension two and with the correct quantum numbers to be a candidate as the anomalous divergence of V_μ. [The situation is analogous to that of the SU(4) nonsinglet chiral currents in four-dimensional gauge theories. Any regularization breaks the conservation of these currents; nevertheless, the conservation is restored in the continuum limit because there is no operator with the correct dimension and the correct quantum numbers to be the anomalous divergence.] Therefore the charge K is conserved, along with the charges Q_α whose conservation is preserved by regularization. It now follows from $\tilde{Q}_\alpha=[K,Q_\alpha]$ that $\tilde{Q}_\alpha$ is also conserved, so the entire algebra (10) is present.

The phenomenology of this theory will have to be somewhat unusual. In the ordinary σ model, the simplest possibility consistent with the symmetry is that there might be a multiplet of d bosons with the quantum numbers of n^a. In the supersymmetric model for $d>3$ there could be just d bosons and d fermions with the quantum numbers of n^a and ψ^a. But for $d=3$ there must be at least two d bosons and two d fermions, with the quantum numbers of the operators n^a, ψ^a, $(n\times\psi)^a$, and the isospin current $(n\times\partial_\mu n)^a+i(\bar{\psi}\times\gamma_\mu\psi)^a$, to form a multiplet under the symmetry. Incidentally, if a multiplet of such particle exists, it will probably be possible to interpret any one as a bound state of two others.

Although there is no anomaly in the algebra (10), there is one for the axial-vector current $A_\mu=\epsilon_{\mu\nu}V^\nu$. It is easy to see that the only operator with the correct dimension and quantum numbers to be the anomalous divergence of A_μ is the pseudoparticle density $\epsilon^{abc}n^a\epsilon_{\mu\nu}\partial_\mu n^b\partial_\nu n^c$. We will soon see that such an anomaly exists.

Actually, it turns out that the only role of the fermions for $d=3$ is to provide a logarithmic interaction between pseudoparticles. To see this, we can integrate explicitly over the fermions while keeping the field n^a fixed. We introduce a fixed isovector $n_0=(0,0,1)$ and write $n(x)$ as $U(x)n_0$, where U is a spacetime-dependent rotation matrix (which always exists and is not uniquely determined). We also write ψ as $U(x)\chi$, where $\chi=(\chi^1,\chi^2,0)$ and χ^1 and χ^2 are two Majorana spinors; note that $n_0\cdot\chi=0$ because $n\cdot\psi=0$. Now if we introduce the Dirac spinor $\lambda=(\chi^1+i\chi^2)/\sqrt{2}$ and let $A_\mu=\epsilon^{ijk}n_0^i(U^{-1}\partial_\mu U)^{jk}$ then the fermionic part of the action becomes just

$$\frac{1}{g^2}[\bar{\lambda}i\gamma^\mu(\partial_\mu+iA_\mu)\lambda+\tfrac{1}{2}(\bar{\lambda}\lambda)^2]\ .$$

This is the massless Thirring model coupled to an electromagnetic field. According to the standard boson representation of fermions,[8] it is equivalent to the Lagrangian

$$\tfrac{1}{2}(\nabla\phi)^2+\frac{1}{(\pi+g^2)^{1/2}}\phi\epsilon_{\mu\nu}\partial_\mu A_\nu\ ,\tag{11}$$

where ϕ is a real scalar field. Also, with the above definition of A_μ, one finds that $\epsilon_{\mu\nu}\partial_\mu A_\nu$ is equal to the pseudoparticle density $\epsilon^{abc}n^a\epsilon_{\mu\nu}\partial_\mu n^b\partial_\nu n^c$. Thus, the supersymmetric σ model for $d=2$ is equivalent to the theory

$$L=\frac{1}{2g^2}(\partial_\mu n^a)^2+\tfrac{1}{2}(\nabla\phi)^2+\frac{1}{(\pi+g^2)^{1/2}}\phi\epsilon^{abc}n^a\epsilon_{\mu\nu}\partial_\mu n^b\partial_\nu n^c\ .\tag{12}$$

If desired, one can integrate explicitly over ϕ, giving (in Euclidean space) a logarithmic interaction between pseudoparticles. In a fashion similar to the discussion by Polyakov,[9] one can see that mass generation for the field ϕ will depend on the behavior of the pseudoparticle system. Because of the supersymmetry, mass generation for ϕ is equivalent to mass generation for all the fields in the theory.

In terms of ϕ, the axial-vector current A_μ is simply $\partial_\mu\phi$, and one sees from (12) that the divergence of $\partial_\mu\phi$ is not zero but is proportional to the pseudoparticle density. This is the axial-vector anomaly.

In terms of the superspace formalism by which we constructed the supersymmetric σ model, the existence for $d=3$ of the enlarged algebra (10) is not manifest—the existence of the charges Q_α is manifest, but $\tilde{Q}_\alpha$ and K appear as if by accident. It is interesting to write the theory in such a way that the full symmetry algebra is manifest. From the point of view of superspace, this can be achieved by considering a superspace with bosonic coordinates x^μ and with *two* Majorana spinors θ^i_α, $i=1,2$ as anticommuting coordinates. In this space the supersymmetry generators corresponding to Q_α and $\bar{Q}_\alpha$ are

$$D^i_\alpha = \frac{\partial}{\partial\bar{\theta}^i_\alpha} + i(\gamma^\mu\theta^i)_\alpha \frac{\partial}{\partial x^\mu} ,$$

and the internal-symmetry generator, corresponding to K, is $R=\epsilon_{ij}\theta^{i\alpha}\partial/\partial\bar{\theta}^j_\alpha$. The differential operators

$$P^i_\alpha = \frac{\partial}{\partial\bar{\theta}^i_\alpha} - i(\gamma^\mu\theta^i)_\alpha \frac{\partial}{\partial x^\mu}$$

anticommute with D.

We now again consider a three-component field ϕ^a with $\phi^2=1$. However, because we have enlarged the superspace, the most general such field will contain many more degrees of freedom than we wish (and it will probably be hard to construct a physically sensible theory involving this field). Therefore, we wish to impose a supersymmetric constraint that will remove the extra degrees of freedom. The necessary constraint turns out to be

$$P^a_\alpha\phi^i + \epsilon^{ab}\epsilon^{ijk}\phi^j P^b_\alpha\phi^k = 0 \tag{13}$$

and the equation of motion turns out to be

$$\epsilon_{\alpha\beta}P^a_\alpha P^b_\beta\,\phi = \tfrac{1}{2}\delta^{ab}\epsilon_{\alpha\beta}P^k_\alpha P^k_\beta\phi \ . \tag{14}$$

Equations (13) and (14) together are equivalent to the supersymmetric σ model, and in this form the full supersymmetry algebra is manifest. To verify this it is necessary to check the following: If (13) and (14) are satisfied, then the expansion of ϕ in powers of θ^i_α involves only a single scalar n^a and spinor ψ^a_α, and (13) and (14) imply that these fields satisfy the equations of motion that one would derive from the Lagrangian (9).

Equations (13) and (14) are rather striking, even though it is not clear that they will be useful. Equation (13) is a fermionic form of the pseudoparticle equation that was introduced by Belavin and Polyakov,[1] while (14) is just a linear equation. Equation (13) can be solved the way Belavin and Polyakov solved the bosonic form of this equation (or it can be solved by expanding in powers of θ), and (14), since it is linear, can of course be solved, but it is difficult to solve the two equations simultaneously.

The above results are mostly formal and do not clarify very much the nature of the theory. However, they do suggest that the solution of the supersymmetric σ model may be in some way singular or qualitatively different at $d=3$. It will be interesting to see if this is so.

Note added. The supersymmetric model has also been constructed, in an interesting paper, by P. Di Vecchia and S. Ferrara. I would like to thank R. Jackiw for pointing this out to me.

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[1]A. M. Polyakov, Phys. Lett. 59B, 79 (1975); A. A. Belavin and A. M. Polyakov, Zh. Eksp. Teor. Fiz. Pis'ma Red. 22, 503 (1975) [JETP Lett. 22, 245 (1975)].

[2]A. Salam and S. Strathdee, Nucl. Phys. B76, 477 (1974); S. Ferrara, J. Wess, and B. Zumino, Phys. Lett. 51B, 239 (1974).

[3]F. A. Berezin, *The Method of Second Quantization* (Academic, New York, 1966).

[4]Actually, because of the conformal symmetry, this transformation is a symmetry of the classical theory if ϵ is any solution of the free, massless Dirac equation. However, the conservation laws for nonconstant ϵ will be ruined in the quantum theory by the conformal anomaly, and we will not discuss them further.

[5]A. M. Polyakov, Ref. 1.

[6]A. B. Zamolodchikov, ITEP report in preparation.

[7]O. Alvarez, Harvard report (unpublished).

[8]S. Coleman, Phys. Rev. D 11, 2088 (1975).

[9]A. M. Polyakov, Nucl. Phys. B120, 429 (1977).