Pseudoparticle parameters for arbitrary gauge groups*

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The number of parameters entering a Euclidean Yang-Mills solution with topological charge k is determined for a theory constructed from an arbitrary Lie group G. It is shown that this number is precisely that required to specify the position, scale, and relative group orientation of k independent solutions each with minimum topological charge 1. Such minimal single-pseudoparticle solutions can be obtained by embedding the familiar SU_2 pseudoparticle of Belavin *et al.* into the general Lie group.

I. INTRODUCTION

A considerable amount of information is now known about self-dual solutions[1-4] to the Euclidean, Yang-Mills field equations. Of particular interest is the recent application of the Atiyah-Singer index theorem which determines the number of parameters entering the general solution with given topological charge k.[5-7] For SU_2 the number of parameters, $8k-3$, can be readily interpreted as resulting from the combination of k "elementary" SU_2 pseudoparticle solutions with topological charge 1—each with a particular size, space-time location, and SU_2 orientation.

In this paper we ask whether a similar interpretation is possible for self-dual solutions in a Yang-Mills theory with arbitrary gauge group G. This question is answered in three steps: First, it is observed that a solution with minimum topological charge (normalized to $k=1$) can be obtained by a particular embedding of the SU_2 solution of Belavin, Polyakov, Schwartz, and Tyupkin[1] into the general gauge group G. Second, for a solution with arbitrary k, we apply the Atiyah-Singer[8] index theorem to determine the number of parameters on which such a solution depends. Finally, we show that this number can be interpreted as that required to describe the scale, position, and group orientation of k examples of the SU_2 embedding found in the first step. Thus it may well be possible to view an arbitrary self-dual Yang-Mills solution, even for a general Lie group, as an appropriate combination of familiar SU_2 pseudoparticles.

We begin by considering a Yang-Mills theory with simple gauge group G (Ref. 9) of dimension $d(G)$ and with action

$$S=\tfrac{1}{4}\int (F^i_{\mu\nu})^2 d^4x\,, \tag{1.1}$$

where

$$F^i_{\mu\nu}=\partial_\mu A^i_\nu-\partial_\nu A^i_\mu+f_{ijk}A^j_\mu A^k_\nu. \tag{1.2}$$

Repeated group indices are summed from 1 to $d(G)$, and the structure constants f_{ijk} are chosen completely antisymmetric—for SU_2, $f_{ijk}=\epsilon_{ijk}$. The topological charge k is defined as

$$k=\frac{1}{32\pi^2}\int F^i_{\mu\nu}\tilde{F}^i_{\mu\nu}d^4x\,, \tag{1.3}$$

where the dual of $F_{\mu\nu}$ is

$$\tilde{F}_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}\,. \tag{1.4}$$

For the case of SU_2, a self-dual solution with topological charge k depending on $5k$ parameters has been given by 't Hooft[10]:

$$A^i_\mu(x)=-\bar{\eta}^i_{\mu\nu}\partial_\nu \ln\left[1+\sum_{j=1}^{k}\frac{\lambda_j^{\,2}}{(x-x_j)^2}\right]. \tag{1.5}$$

The singularities of this solution at the points $x_1, \ldots, x_n$ are not physical and can be removed by a gauge transformation. Since all non-Abelian groups contain SU_2 as a subgroup, these SU_2 solutions can be used to generate self-dual solutions with various topological charges for a Yang-Mills theory with an arbitrary group. More can be learned about the space of self-dual solutions by considering small fluctuations $A^i_\mu+\delta A^i_\mu$ about a particular solution A^i_μ and asking that the resulting field strength continue to be self-dual. If expanded to first order in δA_μ, this requirement can be written

$$\delta F_{\mu\nu}=\delta\tilde{F}_{\mu\nu}\,, \tag{1.6}$$

where

$$\delta F_{\mu\nu}=D_\mu\delta A_\nu-D_\nu\delta A_\mu\,. \tag{1.7}$$

The gauge-covariant derivative D depends on the initial solution A^i_μ and is defined by

$$(D_\mu\delta A_\nu)^i=\partial_\mu\delta A^i_\nu+f_{ijk}A^j_\mu\delta A^k_\nu. \tag{1.8}$$

In addition, one must require that the modified solution $A^i_\mu+\delta A^i_\mu$ represents a new solution and not simply a gauge transformation

$$\delta A^i_\mu=\delta\Lambda^j f_{ijk}A^k_\mu-\partial_\mu\delta\Lambda^i=-(D_\mu\delta\Lambda)^i \tag{1.9}$$

of the original A^i_μ. This can be done by requiring that δA^i_μ be orthogonal to all functions of the form (1.9), i.e.,

$$\int d^4x (D_\mu \delta\Lambda)^i \delta A^i_\mu = 0 \qquad (1.10)$$

for all functions $\delta\Lambda^i(x)$. If integration by parts in Eq. (1.10) is allowed, this orthogonality requirement is equivalent to the usual background field gauge condition

$$D_\mu \delta A_\mu = 0 . \qquad (1.11)$$

For the case of SU_2, the analysis of small fluctuations about the solution (1.5) has been approached in three different ways. First, Jackiw and Rebbi[4] have found $8k-3$ solutions to Eq. (1.6) which are not gauge transformations of A^i_μ. Second, Schwartz[5] and Atiyah, Hitchin and Singer[7] transform the SU_2 Yang-Mills theory to the four-dimensional sphere S^4, apply the Atiyah-Singer index theory to the simultaneous linear differential equations (1.6) and (1.11), and show that there are precisely $8k-3$ parameters appearing in the general solution. Third, Brown, Carlitz, and Lee[6] combine equations (1.6) and (1.11), writing them as a single spinor equation. They then employ a variant of a method suggested by Coleman[11] to show directly that in Euclidean space, E^4, Eqs. (1.6) and (1.11) possess exactly $8k$ simultaneous solutions. This result does agree with the S^4 application of the Atiyah-Singer theorem. On S^4 the integration by parts relating Eq. (1.11) and the condition (1.10) is always permitted so that in the background gauge on S^4 all gauge freedom is eliminated. However, on E^4 there is a three-parameter family of gauge transformations which generate a δA^i_μ satisfying Eq. (1.11). Thus, as Brown *et al.* observe, there are only $8k-3$ physical modes for E^4.

Let us now consider self-dual solutions for an arbitrary simple compact Lie group G. Just as in the SU_2 case, each such solution (in a nonsingular gauge) approaches a direction-dependent gauge transformation at infinity

$$\lim_{|x|\to\infty} A_\mu(x) = g(\hat{x})\partial_\mu g^{-1}(\hat{x}) \qquad (1.12)$$

with $\hat{x}_\mu = x_\mu/(x^2)^{1/2}$. Hence to each solution there corresponds a mapping of directions in four dimensions (i.e., the three-dimensional sphere S^3) into the gauge group. These mappings fall naturally into equivalence classes, with elements in each class being continuously deformable into one another. The topological charge k is determined by the equivalence class to which the mapping belongs. The group of all such equivalence classes, $\Pi_3(G)$, has been thoroughly analyzed in the mathematical literature. In particular, the familiar result that $\Pi_3(SU_2)$ is isomorphic to the integers is valid for any simple Lie group. Furthermore, there is a particular minimal embedding of SU_2 into an arbitrary simple Lie group G such that each equivalence class of mappings in $\Pi_3(G)$ contains representatives obtained by mapping S^3 into that particular SU_2 subgroup.[12] Thus each topologically distinct set of boundary conditions (1.12) for an arbitrary simple group G can be obtained by embedding one of the SU_2 solutions (1.5), transformed to a regular gauge, into G. We have normalized the definition (1.3) of the topological charge so that for an arbitrary field configuration in G, k takes on the value of the corresponding topologically equivalent SU_2 embedding.

The remainder of this paper is arranged as follows. In Sec. II we analyze a general embedding of the SU_2 pseudoparticle of Belavin *et al.* in an arbitrary simple Lie group G. We show that the minimum topological charge k for such an embedding is obtained when one uses an SU_2 subgroup of G generated by E_α, $E_{-\alpha}$, and $[E_\alpha, E_{-\alpha}]$, where α is a root of maximum length. These $k=1$ solutions correspond to the minimal SU_2 embeddings referred to in the paragraph above.

In Sec. III the Atiyah-Singer index theorem is introduced. This theorem relates the index $\mathcal{I}$,

$$\mathcal{I} = h^0 - h^1 + h^2 , \qquad (1.13)$$

of the simultaneous equations (1.6) and (1.11) to the topological charge k. Here h^0 is the number of linearly independent solutions to the equation

$$D_\mu \delta\Lambda = 0 , \qquad (1.14)$$

h^1 is the number of linearly independent simultaneous solutions of Eqs. (1.6) and (1.11), and $h^2=0$ for S^4. Following Schwartz,[5] we do not evaluate $\mathcal{I}$ directly but instead use its linear dependence on k and explicit evaluation of h^0 and h^1 for $k=0$ and 1 to determine it in general. The result is a formula determining $h^1 - h^0$ as a function of k.

Throughout this section and the remainder of the paper we are specifically discussing the Yang-Mills equations on the four-dimensional sphere. This eliminates the problem of surface terms and is necessary for the validity of the index theorem. It is well known that the conformal invariance of the Yang-Mills equations ensures that when any solution on S^4 is stereographically projected onto Euclidean space it will also solve the Euclidean space equations. Conversely, all known Yang-Mills solutions in Euclidean space are sufficiently regular at infinity that, when appropriately gauge transformed, they can be mapped onto S^4.

Thus we have a formula for $h^1 - h^0$, where the value of h^0 depends on the particular configuration $A^i_\mu(x)$. Two configurations with the same value of k may have different values of h^0. In Sec. IV

TABLE I. The rank, quadratic Casimir operator $C(G)$, dimension $d(G)$, and the quantities $M(G)$ and $I(G,k)$ are listed for all simple compact Lie groups. The number of parameters necessary to describe a configuration of topological charge k is given by $N(G,k) = 4C(G)k - d(G)$ if $k \geq M(G)$, and $N(G,k) = I(G,k)$ if $k < M(G)$. For any group, $I(G,1) = 5$. (Note that $SO_3 \cong SU_2$, $SO_5 \cong Sp_4$, $SO_6 \cong SU_4$, and SO_4 is not simple).

Group	Rank	$C(G)$	$d(G)$	$M(G)$	$I(G,k)$
SU_n	$n-1$	n	n^2-1	$\frac{1}{2}n$	$4k^2+1$
SO_n, n odd, $n \geq 7$	$\frac{1}{2}(n-1)$	$n-2$	$\frac{1}{2}n(n-1)$	$\frac{1}{4}(n-1)$	$8k^2-6k$ $(k \neq 1)$
SO_n, n even, $n \geq 8$	$\frac{1}{2}n$	$n-2$	$\frac{1}{2}n(n-1)$	$\frac{1}{4}n$	$8k^2-6k$ $(k \neq 1)$
Sp_{2n}	n	$n+1$	$n(2n+1)$	n	$2k^2+3k$
G_2	2	4	14	2	...
F_4	4	9	52	2	...
E_6	6	12	78	3	$I(2)=20$
E_7	7	18	133	3	$I(2)=20$
E_8	8	30	248	3	$I(2)=20$

we show that if a configuration has a certain regularity property, then its value of h^0 leads to a value of h^1 which gives the true number of parameters $N(G,k)$ necessary to specify the general self-dual solution of topological charge k. Furthermore, we show how to calculate these values of h^0. In Table I we display our final results for $N(G,k)$ for all simple compact groups G. Finally in Sec. V it is shown that the number of parameters we have obtained is precisely equal to the number necessary to describe the positions, scales, and relative group orientations of k independent pseudoparticles.

II. THE MINIMAL SU_2 EMBEDDING

In this section we explain how to embed the SU_2 pseudoparticle into an arbitrary, compact, simple Lie group to give the minimum topological charge

$$k = \frac{1}{32\pi^2} \int_M F^i_{\mu\nu} \tilde{F}^{i,\mu\nu} dx. \tag{2.1}$$

Here the $F^i_{\mu\nu}$ are the components of the field strength $F_{\mu\nu}$:

$$F_{\mu\nu} = F^i_{\mu\nu} T_i. \tag{2.2}$$

The T_i form a basis for the Lie algebra $\mathfrak{G}$ of the compact simple group G, and will be chosen to belong to the adjoint representation. The basis vectors T_i are chosen orthonormal with respect to the Cartan invariant inner product

$$\langle T_i, T_j \rangle = \frac{1}{C(G)} \operatorname{tr}(T_i T_j) = \delta_{ij}, \tag{2.3}$$

where $C(G)$ is a normalization constant that will be specified later. (Recall that it is for such a choice of basis that the structure constants f_{ijk} become completely antisymmetric.)

Given three matrices $\{J^i\}, 1 \leq i \leq 3$ in the adjoint representation of G which obey the commutation relations of angular momenta

$$[J^i, J^j] = i\epsilon_{ijk} J^k \tag{2.4}$$

or

$$[J^3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = J^3$$

with (2.5)

$$J^\pm = \frac{1}{\sqrt{2}}(J^1 \pm iJ^2),$$

we can easily obtain a pseudoparticle solution in G,

$$F_{\mu\nu} = F^{I,i}_{\mu\nu} J^i, \tag{2.6}$$

where $F^{I,i}_{\mu\nu}$, $1 \leq i \leq 3$ are the SU_2 components of the single-pseudoparticle solution obtained by setting $k=1$ in Eq. (1.5). The topological charge k of the solution (2.6) is simply the length of any one of the matrices J^i

$$k = \langle J^i, J^i \rangle$$

$$= \frac{1}{C(G)} \operatorname{tr}(J^i J^i) \quad \text{(no sum on } i). \tag{2.7}$$

Our problem then is to find an SU_2 subalgebra $\{J^i\}, 1 \leq i \leq 3$ of $\mathfrak{G}$, whose generators J^i have minimum length.

Let us first recall some properties of the root diagram of a simple Lie algebra $\mathfrak{G}$.[13,14] We choose a regular Abelian subalgebra H of $\mathfrak{G}$ which contains the maximum number of commuting generators. The dimension of H is called the rank r of $\mathfrak{G}$. We choose our basis T_i so that the first r elements, written as h_i, $1 \le i \le r$ form a basis for H. Next, raising and lowering operators, E_α, are constructed out of the remaining elements of $\mathfrak{G}$. The action of the E_α on $\hat{h}_i$ can be represented by a root diagram which is a vector diagram in a space of dimension r. The vectors are labeled by α and their components α_i are the amounts by which E_α changes $\hat{h}_i$. Thus we have n

$$[\hat{h}_i, \hat{h}_j] = 0, \qquad [\hat{h}_i, E_\alpha] = \alpha_i E_\alpha. \tag{2.8}$$

The E_α, with suitable normalization, obey

$$[E_\alpha, E_{-\alpha}] = \sum_{i=1}^{r} \alpha_i \hat{h}_i, \qquad [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}. \tag{2.9}$$

The constant $N_{\alpha\beta} = 0$ whenever $\alpha + \beta$ is not another root vector. Root vectors for all the compact simple Lie groups are listed by Racah.[13]

We conclude these preliminaries by discussing the normalization of the inner product (2.3) and hence of the basis T_i. The normalization of the T_i is fixed by the requirement that the maximal length of any root vector is one. Thus for such a root α^0

$$\sum_i (\alpha_i^0)^2 = 1. \tag{2.10}$$

With this choice of normalization, the constant $C(G)$ of Eq. (2.3),

$$C(G) = \mathrm{tr}(T_j^2) = \frac{1}{d(G)} \sum_{i=1}^{d(G)} \mathrm{tr}(T_i)^2 = \sum_{i=1}^{d(G)} T_i^2, \tag{2.11}$$

becomes the usual quadratic Casimir operator for the adjoint representation of G.

We can now determine the embedding of SU_2 in G which has the minimum topological charge. Suppose we have any embedding $\{J^i\}$, $1 \le i \le 3$ obeying Eq. (2.4). It is always possible to pick a regular Abelian subalgebra H so that it contains any given element of $\mathfrak{G}$;[15] in particular we can choose H so that J^3 is an element of H. Thus we have

$$J^3 = \sum_{i=1}^{r} \beta_i \hat{h}_i \tag{2.12}$$

for some set of coefficients β_i and seek an embedding with a J^3 whose length $|\beta| \equiv (\sum_i \beta_i^2)^{1/2}$ is a minimum. It follows that J_+ and J_- have the form

$$J^\pm = \sum_\alpha f_\alpha^\pm E_\alpha, \tag{2.13}$$

where f_α^+ and f_α^- are two sets of coefficients. Thus one of the commutation relations (2.5) becomes

$$\left[\sum_i \beta_i \hat{h}_i, \sum_\alpha f_\alpha^+ E_\alpha\right] = \sum_\alpha f_\alpha^+ E_\alpha. \tag{2.14}$$

Equation (2.8) and the linear independence of the E_α now yield

$$1 = \sum_i \beta_i \alpha_i = |\beta|\,|\alpha| \cos\theta, \tag{2.15}$$

whenever $f_\alpha^+ \ne 0$, where θ is the angle between α and β. To minimize $|\beta|$ we should, according to Eq. (2.15), choose β parallel to a root vector, α^0, of maximum length ($|\alpha^0| = 1$) and then choose $f_\alpha^+ = 0$ for all other α. Thus if α^0 is a root vector of maximum length, we have a minimum SU_2 subgroup given by

$$J^+ = E_{\alpha^0}, \quad J^- = E_{-\alpha^0}, \qquad J^3 = \sum_i \alpha_i^0 \hat{h}_i. \tag{2.16}$$

Consequently the generators J_i of this subgroup also have length 1 and the minimal SU_2 pseudoparticle constructed from them, Eq. (2.6), will have topological charge $k = 1$.

A minimal SU_2 subgroup obtained in this manner is easily described for the series SU_n, Sp_{2n}, and SO_n. For SU_n and SP_{2n}, it is just the obvious "upper-left-hand-corner" embedding of the two-dimensional representation of $SU_2 = Sp_2$ into the n-dimensional representation of SU_n or the $2n$-dimensional representation of Sp_{2n}. For SO_n, $n \ge 5$, a minimal SU_2 subgroup is obtained by embedding the four-dimensional representation of SO_4 into the n-dimensional representation of SO_n and then using one of the factors $SO_4 = SU_2 \times SU_2$. (The more obvious subgroup obtained by embedding the three-dimensional representation of SO_3 gives a J^i whose length is the $\sqrt{2}$ times that obtained by this method.

The embedding (2.16) has a simple property that will be extremely useful to us later on. First note that if we let the generators J^i of any SU_2 subgroup act on all the generators of the group T_a by

$$[J^i, T_a] = L_{ab}^i T_b, \tag{2.17}$$

then the matrices L_{ab}^i form a representation of SU_2. Now by (2.8) for the particular subalgebra (2.16) we have

$$[J^3, \hat{h}_j] = 0, \qquad [J^3, E_\alpha] = m_\alpha E_\alpha, \tag{2.18}$$

where

$$m_\alpha = \sum_{i=1}^{r} \alpha_i^0 \alpha_i.$$

Thus

$$|m_\alpha| \le 1. \tag{2.19}$$

This inequality implies that when the generators of the group are arranged into standard angular momentum representations under the action of J^i, these representations can have only $j=0,\ \frac{1}{2}$, or 1 and that the only $j=1$ piece consists of the original generators J^i. Further, if $p(G)$ is defined as the number of generators which belong to doublets and $s(G)$ the number which are singlets, then Eq. (2.11) and Eq. (2.18) imply the relation

$$C(G)=\mathrm{tr}(J_3{}^2)=2+\tfrac{1}{4}p(G)$$
$$=\tfrac{1}{4}[5+d(G)-s(G)], \qquad (2.20)$$

which will be referred to in Sec. III. The value of $C(G)$ for each of the simple Lie groups is listed in Table I.

III. APPLICATION OF THE INDEX THEOREM

We now apply the Atiyah-Singer index theorem[8] to our problem. Since the index theorem is valid only on compact manifolds, it cannot be applied directly to the Yang-Mills theory in Euclidean space. However, as explained in Sec. I, the conformal invariance of the Yang-Mills action makes it possible to project the theory onto the four-dimensional hypersphere, where the index theorem is applicable. Throughout this section we will understand the theory to be in this projected form.

Given a self-dual field strength $F_{\mu\nu}$ arising from a vector potential A_μ, we wish to investigate infinitesimal variations δA_μ which preserve to first order the self-duality of $F_{\mu\nu}$. The condition that the first-order change in $F_{\mu\nu}$ be self-dual may be written as

$$(D_1\delta A)_{\alpha\beta}\equiv\Pi_{\alpha\beta}{}^{\mu\nu}(D_\mu\delta A_\nu-D_\nu\delta A_\mu)=0, \qquad (3.1)$$

where $D_\mu\delta A_\nu$, defined in Eq. (1.8), is the gauge-covariant derivative using the unperturbed A_μ, and $\Pi_{\alpha\beta}{}^{\mu\nu}$ is a projection matrix which picks out the anti-self-dual part of a tensor. Among the solutions to Eq. (3.1) are those arising from infinitesimal gauge transformations of the unperturbed A_μ; these are of the form

$$(D_0\delta\Lambda)_\mu\equiv D_\mu\delta\Lambda. \qquad (3.2)$$

Two solutions of Eq. (3.1) are gauge equivalent if they differ by a field of the form of Eq. (3.2). Our problem is to find the number of linearly independent gauge-inequivalent solutions of Eq. (3.1).

A convenient device for formulating our problem and for applying the index theorem is the sequence of mappings[7]

$$0\xrightarrow{D_{-1}}M^0\xrightarrow{D_0}M^1\xrightarrow{D_1}M^2_-\xrightarrow{D_2}0. \qquad (3.3)$$

M^0, M^1, and M^2_- are the spaces of scalar, vector, and anti-self-dual antisymmetric rank-two tensors, respectively, all transforming under the adjoint representation of the group G. D_{-1} takes 0 to the scalar field which is identically 0, D_0 and D_1 are the differential operators defined by Eqs. (3.1) and (3.2), and D_2 takes all of M^2_- to 0. At each step in the sequence, the image of D_{i-1} is contained in the kernel of D_i. (The only nontrivial case is the application of D_1D_0 to a scalar field. This vanishes because a gauge transformation cannot change the self-duality of $F_{\mu\nu}$.) We may define equivalence classes of elements in the kernel of D_i by defining two elements to be equivalent if they differ by an element in the image of D_{i-1}. These equivalence classes form a vector space

$$H^i=\frac{\text{kernel } D_i}{\text{image } D_{i-1}} \qquad (3.4)$$

whose dimension we shall denote by h^i. In particular, H^1 consists of the classes of gauge-equivalent solutions to Eq. (3.1); its dimension, h^1, is precisely the quantity in which we are interested.

Since the image of D_{-1} is the field which is identically zero, H^0 is just the space of scalar fields in the adjoint representation with vanishing covariant derivative. This space may be simply described in terms of the holonomy group of the vector potential A_μ. (The holonomy group at a point x_0 is defined as follows: Given a vector potential, the operation of parallel transport along a closed path beginning and ending at x_0 determines an element of the group G. The holonomy group at x_0 is the subgroup of G obtained by considering all possible paths. For a connected manifold, the holonomy groups at different points are easily seen to be isomorphic.) Since the fields in H^0 have vanishing covariant derivative, they are unchanged by parallel transport about a closed path and thus at every point are left unchanged by the holonomy group Conversely, any element left unchanged by the holonomy group at a point x_0 determines, by parallel transport, an element at every point, thus giving a well-defined field $\phi(x)$ with vanishing covariant derivative. Thus h^0 is equal to the dimension of the subspace of the adjoint representation which is left unchanged by the holonomy group; this is equal to the dimension of the largest subgroup of G commuting with the holonomy group.

Since the kernel of D_2 is all of M^2_-, h^2 is equal to the dimension of the subspace of M^2_- orthogonal to the image of D_1. But this subspace is just the kernel of D_1^*, where D_1^* is the adjoint of D_1, so h^2 is the number of linearly independent solutions to $D_1^*T=0$. Any tensor field satisfying this equation also satisfies $D_1D_1^*T=0$; in Appendix B, we show that on a sphere (with the usual metric) $D_1D_1^*$ is a positive-definite operator, so $h^2=0$.

We now define an elliptic differential operator $\mathfrak{D}$, which takes ordered pairs of scalar and anti-self-dual tensor fields to vector fields, by

$$(\mathfrak{D}(S,T))_\mu = (D_0 S)_\mu + (D_1^* T)_\mu. \quad (3.5)$$

Using some linear algebra, one can show that its index, defined by

$$\mathscr{I}(\mathfrak{D}) = \dim(\text{kernel } \mathfrak{D}) - \dim(\text{kernel } \mathfrak{D}^*), \quad (3.6)$$

is related to the h^i defined above by

$$\mathscr{I}(\mathfrak{D}) = h^0 - h^1 + h^2. \quad (3.7)$$

(Keep in mind that for a hypersphere, the case in which we are interested, $h^2 = 0$.) On the other hand, the Atiyah-Singer index theorem gives an expression for $\mathscr{I}(\mathfrak{D})$ in terms of the topological charge k, the topological invariants of the manifold on which the fields are defined, and constants which depend on the group G. For a four-dimensional manifold, this expression will have the form

$$\mathscr{I}(\mathfrak{D}) = ak + b. \quad (3.8)$$

The constants a and b can be calculated by purely topological methods; instead, we shall obtain them by analytically determining $\mathscr{I}(\mathfrak{D})$ for $k=0$ and $k=1$.[16]

For $k=0$, any self-dual configuration must have zero action, and therefore $F_{\mu\nu}=0$. The holonomy group then consists of only the unit element, so h^0 is equal to the dimension of the group, $d(G)$. Since there are no solutions to Eq. (3.1) other than infinitesimal gauge transformations, $h^1 = 0$, and so $\mathscr{I}(\mathfrak{D}) = d(G)$.

For an example with $k=1$ we embed the one-pseudoparticle solution of Belavin *et al.*[1] into the group G via the minimal SU_2 subgroup described in Sec. II. It was shown in Sec. II that with respect to this subgroup the generators of G belong to one triplet (the generators of the SU_2 itself), $\frac{1}{2}p(G)$ doublets, and $s(G)$ singlets; clearly h^0 is equal to $s(G)$. We can obtain h^1 by a simple extension of 't Hooft's analysis[17] of the fluctuations about an SU_2 pseudoparticle. For the SU_2 case 't Hooft found eight modes corresponding to solutions of Eq. (3.1). Three of these correspond to gauge transformations, so only the remaining five, corresponding to translations and dilatation, contribute to h^1. Extending the analysis to vector fields belonging to doublets yields two modes per doublet, but both correspond to gauge transformations (see Appendix C). For singlet vector fields there are no modes. Thus for any group we obtain $h^1 = 5$, and $\mathscr{I}(\mathfrak{D}) = s(G) - 5$. The constants in Eq. (3.8) are now determined; using Eq. (3.7), we obtain

$$h^1 = [5 + d(G) - s(G)]k - [d(G) - h^0]. \quad (3.9)$$

Using Eq. (2.20) we may rewrite this as

$$h^1 = 4C(G)k - d(G) + h^0. \quad (3.10)$$

It should be noted that this result can also be obtained by using the method of Brown *et al.*[6] on E^4.

IV. DETERMINATION OF h^0

Given a group G and a value of the topological charge k, our goal is to find the dimension $N(G,k)$ of the manifold of self-dual configurations (modulo the action of the gauge group).[18] In this section we will show how to use the results of Sec. III, along with some knowledge of the structure of the Lie groups, to determine $N(G,k)$.

Given any initial self-dual configuration, the index theorem allows us to calculate h^1, the number of linearly independent gauge-inequivalent solutions to the infinitesimal variation problem. On the right-hand side of Eq. (3.10), however, appears the quantity h^0, the dimension of the largest subgroup of G which commutes with the holonomy group of the initial configuration. Two different configurations with the same value of k may have different values of h^0. Thus, the central problem is to find the appropriate value of h^0 such that $h^1 = N(G,k)$.

We begin by establishing some facts about the holonomy group. Given a configuration of Yang-Mills fields, let $G'(x)$ denote the holonomy group associated with the point x. Given two points x and y, $G'(x)$ and $G'(y)$ are equivalent in the following sense: There is an element $g_{xy} \in G$ which generates an isomorphism between the two groups of the form $g_x = g_{xy} g_y g_{xy}^{-1}$, where $g_x \in G'(x)$ and $g_y \in G'(y)$. (g_{xy} can be taken as the element of G defined by parallel transport along some fixed path between the two points.) We now show that it is possible to choose a gauge in which $G'(x)$ is identical for all points x, and in which $A_\mu^i(x)$ has nonzero values only within the holonomy group. On R^4 one imposes the following gauge conditions:

$$\begin{aligned} &A_1^i = 0 \text{ everywhere}, \\ &A_2^i = 0 \text{ if } x_1 = 0, \\ &A_3^i = 0 \text{ if } x_1 = x_2 = 0, \\ &A_4^i = 0 \text{ if } x_1 = x_2 = x_3 = 0. \end{aligned} \quad (4.1)$$

These conditions can be achieved (one at a time) by performing gauge transformations which are determined by simple first-order differential equations. (S^4 can be covered by two overlapping coordinate systems, and within each system one can impose the above gauge conditions.) In this gauge an arbitrary point x can be connected to the origin by a path along which $A_\mu dx^\mu = 0$, and hence parallel transport is trivial. Thus, $G'(x) = G'(x=0) \equiv G'$. One then chooses a basis for the Lie algebra $\mathfrak{G}$ such that G' is generated by the first m generators, where $m = \dim(G')$. Then $F_{\mu\nu}^i(x) = 0$ unless $i \leq m$, since G' contains the transformations obtained by parallel transport around infinitesimal loops. Finally, one notes that these gauge conditions allow one to express $A_\mu^i(x)$ in terms of integrals which are linear in $F_{\mu\nu}^i$. Thus,

$$A_\mu^i(x)=0 \text{ unless } i\leq m. \tag{4.2}$$

Within the manifold of self-dual configurations, we will call a particular configuration "regular" if it is contained in an open region of configurations which all have equivalent holonomy groups. By a gauge transformation, it is possible to write this entire family of configurations in a form consistent with Eq. (4.2). We will see that such regular configurations are exactly the ones which give us the desired value for h^0, and hence h^1.

Suppose we consider a regular initial configuration with a given value of k. Using the index theorem, we found in Sec. III that $h^1=4C(G)k-d(G)+h^0$. These infinitesimal variations can then be iterated to obtain a family of self-dual solutions with h^1 parameters, with the initial configuration taken as the origin of parameter space. If $h^2=0$, such an iteration exists to all orders and the series has a nonzero radius of convergence.[7,19] Since the initial configuration is regular, there exists a region about the origin of parameter space in which all the configurations can be written to obey Eq. (4.2). Within this region, it is clear that the configurations are gauge inequivalent. Recall that the infinitesimal variation problem was formulated to remove at the outset the possibility of equivalence by infinitesimal gauge transformations. One must, however, also consider the possibility that there is a class of finite gauge transformations which leave the initial configuration invariant, but which lead to an equivalence among the infinitesimal variations. With our choice of gauge, the class of gauge transformations which leave the initial configuration invariant is simply the h^0-parameter class of global gauge transformations which commute with G'. These gauge transformations, however, also leave invariant any configuration obeying Eq. (4.2), and thus the gauge inequivalence of the family of configurations is established. Thus, we have constructed a local manifold of dimension h^1. (If the initial configuration were not regular, only the argument about gauge inequivalence would break down.) If we let $L(G,k)$ be the value of h^0 corresponding to a regular configuration of topological change k, then

$$N(G,k)=4C(G)k-d(G)+L(G,k). \tag{4.3}$$

[Since $N(G,k)$ has a well-defined value, the above relation implies that h^0 has the same value for all regular configurations.]

We are now prepared to derive $N(G,k)$ for all compact simple Lie groups G. We begin with the series $G=\mathrm{SU}_n$.

Our proof will be based on mathematical induction, so we begin by stating the answer:

$$N(\mathrm{SU}_n,k)=\begin{cases}4k^2+1 & \text{if } k\leq\frac{1}{2}n,\\ 4nk-(n^2-1) & \text{if } k\geq\frac{1}{2}n.\end{cases} \tag{4.4}$$

[The lower expression is simply $4C(G)k-d(G)$. The upper expression is the maximum value of the lower expression for fixed k. The crossover point is the value of n for which this maximum occurs.] One first verifies the solution for SU_2: Clearly $L(\mathrm{SU}_2,k)=0$, since there are no non-Abelian proper subgroups. We now assume that the formula holds for $\mathrm{SU}_2,\ldots,\mathrm{SU}_{n-1}$, and consider the case of SU_n.

By Eq. (4.3),

$$N(\mathrm{SU}_n,k)=4nk-(n^2-1)+L(\mathrm{SU}_n,k). \tag{4.5}$$

Since any SU_{n-1} configuration can be embedded into an SU_n theory, it follows that

$$N(\mathrm{SU}_n,k)\geq N(\mathrm{SU}_{n-1},k)\,. \tag{4.6}$$

The problem will then be solved by proving one more relation:

$$N(\mathrm{SU}_n,k)=N(\mathrm{SU}_{n-1},k) \text{ if } L(\mathrm{SU}_n,k)\neq 0. \tag{4.7}$$

To prove Eq. (4.7), imagine constructing a family of self-dual configurations with the same holonomy group G'. Then $L(\mathrm{SU}_n,k)\neq 0$ implies that there is at least one generator τ_0 which commutes with G'. One can diagonalize τ_0 (in the fundamental representation). Suppose that each repeated eigenvalue λ_i, $i=1,\ldots,r$, occurs with multiplicity p_i; let s be the number of unrepeated eigenvalues. Then

$$s+\sum_i p_i=n,\quad p_i<n. \tag{4.8}$$

If G'' is defined as the group generated by all elements of $\mathfrak{G}$ which commute with τ_0, then

$$G'\subset G''=\mathrm{SU}_{p_1}\times\cdots\times\mathrm{SU}_{p_r}\times(U_1)^{r+s-1}. \tag{4.9}$$

One can then write the entire family of configurations in a gauge satisfying Eq. (4.2), and one can further choose a basis for $\mathfrak{G}$ in which the generators of the subgroups $\mathrm{SU}_{p_1},\ldots,\mathrm{SU}_{p_r}$ occur as elements. Each configuration then decomposes into a superposition of r mutually independent Yang-Mills configurations. (Self-duality implies that the U_1 fields must vanish.) Within each subgroup SU_{p_i} the configuration will have a topological charge k_i, with $k=\sum k_i$. It follows that

$$N(\mathrm{SU}_n,k)\leq\sum_i N(\mathrm{SU}_{p_i},k_i). \tag{4.10}$$

Regardless of the values of the k_i's and p_i's, the above inequality will hold if the right-hand side is replaced by its maximum possible value. It is a straightforward exercise to verify from Eq. (4.4) that this maximum value is $N(\mathrm{SU}_{n-1}, k)$. Equation (4.7) is then obtained by recalling the inequality (4.6).

One can now carry out the induction in two steps First, suppose $k \geq \frac{1}{2}n$. Then

$$N(\mathrm{SU}_n, k) \geq 4nk - (n^2 - 1) > N(\mathrm{SU}_{n-1}, k). \tag{4.11}$$

This contradicts Eq. (4.7), so one must have $L(\mathrm{SU}_n, k) = 0$. Then suppose $k < \frac{1}{2}n$. It follows that

$$4nk - (n^2 - 1) < N(\mathrm{SU}_{n-1}, k). \tag{4.12}$$

Equations (4.5), (4.6), and (4.12) imply that $L(\mathrm{SU}_n, k) > 0$, and then Eq. (4.7) determines $N(\mathrm{SU}_n, k)$. Thus, the induction hypothesis has been shown for SU_n.

For the other groups, the answer can again be stated and then proved by induction:

$$N(G, k) = \begin{cases} I(G, k) & \text{if } k \leq M(G), \\ 4C(G)k - d(G) & \text{if } k \geq M(G), \end{cases} \tag{4.13}$$

where all of the quantities on the right are listed in Table I.

The proof of Eq. (4.13) for the symplectic groups follows the same pattern as the proof given for the unitary groups. The induction begins at $\mathrm{Sp}_2 = \mathrm{SU}_2$. Equation (4.9) must of course be revised. Here the analysis is facilitated by using the root diagrams rather than the fundamental representations and choosing a regular subalgebra containing τ_0.[15] For Sp_{2n} one can show that

$$G' \subset \mathrm{SU}_p \times \mathrm{Sp}_{2r} \times (U_1)^{n+1-p-r}, \tag{4.14}$$

where

$$p + r \leq n, \quad r < n. \tag{4.15}$$

There is one further complication: When the SU_p root diagram is looked at within the root diagram of Sp_{2n}, its maximum root length is only $1/\sqrt{2}$. This means that a given configuration, when viewed in the Sp_{2n} theory, will have some $k = k_{\mathrm{SU}} + k_{\mathrm{Sp}}$. However, when the SU_p fields are viewed as a configuration in an SU_p theory, they have topological charge $k' = \frac{1}{2}k_{\mathrm{SU}}$. This fact is necessary to show that

$$N(\mathrm{Sp}_{2n}, k) = N(\mathrm{Sp}_{2n-2}, k) \text{ if } L(\mathrm{Sp}_{2n}, k) \neq 0. \tag{4.16}$$

The proof for the orthogonal groups is then a straightforward exercise. The induction begins at $\mathrm{SO}_5 \cong \mathrm{Sp}_4$ for the odd orthogonal groups, and $\mathrm{SO}_6 \cong \mathrm{SU}_4$ for the even orthogonal groups. For SO_{2n+1} one can show that

$$G' \subset \mathrm{SU}_p \times \mathrm{SO}_{2r+1} \times (U_1)^{n+1-p-r} \tag{4.17}$$

where again (4.15) holds. For SO_{2n} one can show

$$G' \subset \mathrm{SU}_p \times \mathrm{SO}_{2r} \times (U_1)^{n+1-p-r}, \tag{4.18}$$

where (4.15) also holds. Using these relations, one can show that for $n \geq 8$

$$N(\mathrm{SO}_n, k) = N(\mathrm{SO}_{n-2}, k) \text{ if } L(\mathrm{SO}_n, k) \neq 0. \tag{4.19}$$

For $n = 7$ the right-hand side of the above equation is replaced by $N(\mathrm{SU}_3, k)$.

Only the exceptional groups remain, and fortunately it is necessary to know only a few facts to determine the answer. Using the fact that $\mathrm{D}_5 \subset \mathrm{E}_6 \subset \mathrm{E}_7 \subset \mathrm{E}_8$, Eq. (4.6) may be replaced by

$$N(\mathrm{E}_8, k) \geq N(\mathrm{E}_7, k) \geq N(\mathrm{E}_6, k) \geq N(\mathrm{D}_5, k). \tag{4.20}$$

Since SU_2 is always a subgroup,

$$N(\mathrm{G}_2, k) \geq N(\mathrm{SU}_2, k), \tag{4.21}$$

$$N(\mathrm{F}_4, k) \geq N(\mathrm{SU}_2, k). \tag{4.22}$$

(One must of course check that the above embeddings do not involve a rescaling of the topological charge.) Equation (4.9) may be replaced by the general statement that if $L(G, k) \neq 0$, then G' is contianed in a group G'' which, apart from U_1 factors, must have a rank which is less than the rank of G. The decomposition of G'' may involve a rescaling of topological charge, but general arguments guarantee that such a rescaling must always be in the same direction as it was for the symplectic groups. In these cases such a rescaling has no effect on the derivation.

V. CONCLUSION

A self-dual configuration of topological charge k is often thought of as a kind of nonlinear superposition of k single pseudoparticles. (In Appendix A we discuss two examples which support this hypothesis.) This superposition interpretation makes a definite prediction for the number of parameters in the general solution: For each pseudoparticle there should be one scale parameter, four position parameters, and some number of parameters to specify the relative orientation of the pseudoparticle in group space. We now show that the number of parameters which we have calculated is in agreement with this interpretation.

To clarify the basis for this agreement, we summarize some of the logic used in Sec. III. Using the Atiyah-Singer index theorem, one asserts that

$$h^1 = -ak - b + h^0. \tag{5.1}$$

For $k = 0$ one knows that $h^1 = 0$, $h^0 = d(G)$. For $k = 1$ one must show that there exists an embedding of the elementary SU_2 pseudoparticle into the group G for which $h^1 = 5$. Using the basis for the Lie algebra adopted in Sec. II, the value of h^0 for this embedding is simply equal to the number $s(G)$ of singlet generators (those which commute with the J_i used in the embedding). These two cases determine the constants a and b, leading to

$$h^1=[5+d(G)-s(G)]k-[d(G)-h^0]\ . \tag{5.2}$$

As discussed in Sec. IV, h^1, the number of solutions to the linearized equations, is equal to the number of parameters in the general solution, $N(G,k)$, provided we are expanding about a regular initial configuration, in which case $h^0=L(G,k)$.

The terms in Eq. (5.2) can be easily recognized if such a configuration is viewed as a superposition of k, SU_2 pseudoparticles. Since $s(G)$ is the number of independent generators which commute with an arbitrary minimal SU_2 embedding $\{J_i\}$, $1\le i\le 3$, the quantity $d(G)-s(G)$ in Eq. (5.2) is precisely the number of parameters necessary to specify the orientation of the J_i within the group G.

Thus, the first term on the right-hand side of Eq. (5.2) is just the number of parameters required to fix the scale, position, and group orientation (relative to a fixed basis) of each pseudoparticle. The second term simply subtracts out the number of nontrivial global gauge transformations of the entire configuration—thus, only the relative group orientations are counted.

In fact the value of the second term in Eq. (5.2), $d(G)-L(G,k)$, determined explicitly in Sec. IV, is also correctly given by the hypothesis that the solution is made up of k, SU_2 pseudoparticles. Consider a subgroup G_k of G generated by k minimal embeddings of SU_2. Again, such a subgroup G_k might be called regular if it commutes with the minimum number of independent generators of G. A simple rewording of the arguments in Sec. IV shows that the number of independent generators which commute with such a regular subgroup G_k is the same quantity $L(G,k)$ appearing above. Consequently $d(G)-L(G,k)$ is also the number of parameters entering a global gauge transformation which rotates a regular embedding of k minimal SU_2 subgroups.

Thus the superposition interpretation is in precise agreement with the number of parameters necessary to describe a self-dual configuration of any topological charge k, in any gauge group G. This agreement certainly suggests that it may be possible to parametrize the general configuration of topological charge k by the variables appropriate to this superposition interpretation.

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APPENDIX A

We will now describe two examples which illustrate the hypothesis that a self-dual solution with $k>1$ can be viewed as a combination of $k=1$, SU_2 pseudoparticles. We first consider the one family of SU_2 multipseudoparticle solutions which is completely known[3]—the $k=2$ solution obtained by conformal transformation of Eq. (1.5):

$$A^i_\mu(x)=-\bar\eta^i_{\mu\nu}\partial_\nu \ln\left[\sum_{i=0}^{2}\frac{\lambda_i{}^2}{(x-x_i)^2}\right]. \tag{A1}$$

Consider the configuration in which the separations $|x_i-x_j|$ are comparable and the ratios λ_1/λ_0 and λ_2/λ_0 are very small. With this choice of parameters the solution (A1) looks much like two isolated pseudoparticles located at points x_1 and x_2. In particular, for x very near x_1, i.e.,

$$|x-x_1|\sim\frac{\lambda_1}{\lambda_0}\,|x_1-x_0|\ ,$$

the argument of the logarithm becomes

$$\frac{\lambda_0{}^2}{(x_1-x_0)^2}+\frac{\lambda_1{}^2}{(x-x_1)^2}+O\left(\frac{\lambda_0\lambda_1}{(x_1-x_0)^2}\right)+O\left(\frac{\lambda_2{}^2}{(x_1-x_2)^2}\right), \tag{A2}$$

which approximates a $k=1$ solution at x_1 with scale $(\lambda_1/\lambda_0)|x_1-x_0|$. On the other hand, for x outside two small regions about x_1 and x_2, i.e.,

$$|x-x_i|\gg\frac{\lambda_i}{\lambda_0}|x-x_0|\ ,$$

the argument of the logarithm is approximately the single term $\lambda_0{}^2/(x-x_0)^2$ which corresponds to a pure gauge transformation

$$A^i_\mu\frac{\sigma^i}{2}=-g\partial_\mu g^{-1} \tag{A3}$$

with

$$g(x)=\frac{x^4-x_0{}^4-i(\vec{x}-\vec{x}_0)\cdot\vec{\sigma}}{[(x-x_0)^2]^{1/2}}\ . \tag{A4}$$

If we perform the inverse of that gauge transformation, the solution looks precisely like the superposition of two $k=1$ pseudoparticles of the type (1.5) at the positions x_i, with scales $\lambda_i'=(\lambda_i/\lambda_0)|x_i-x_0|$ but each with a different SU_2 orientation,

$$\sigma^jA^j_\mu\simeq\sum_{i=1}^{2}-g^{-1}(x_i)\sigma^jg(x_i)\bar\eta^j_{\mu\nu}\partial_\nu\ln\left(1+\frac{(\lambda_i')^2}{(x-x_i)^2}\right). \tag{A5}$$

The terms omitted from the right-hand side of Eq. (A5) are smaller than those retained by at least a factor of λ_i/λ_0 for $i=1$ or 2.

A complementary test of this interpretation of self-dual solutions with $k>1$ can be found by generating such solutions from nonminimal embeddings of SU_2 into larger groups. Consider for example SU_3, generated by the standard eight matrices λ_i,

$$[\lambda_i, \lambda_j] = 2if_{ijk}\lambda_k \tag{A6}$$

with f_{ijk} normalized according to our convention. Since f_{ijk} vanishes when only two of its indices lie between one and three and since $f_{123}=+1$ we can immediately construct an SU_3 solution using the $k=1$, SU_2 solution of Eq. (1.5),

$$A_\mu^i = \begin{cases} -\bar{\eta}^i_{\mu\nu}\partial_\nu \ln(1+\lambda^2/x^2), & 1 \le i \le 3 \\ 0, & 3<i \end{cases} \tag{A7}$$

which, from Eq. (1.3), must also have topological charge 1. The three matrices λ^2, λ^5, and λ^7 also form an SU_2 subalgebra but with $f_{257}=\frac{1}{2}$. Thus a second SU_3 solution[20,21] can be written

$$A'^i_\mu = -2\rho^i_{\mu\nu}\partial_\nu \ln(1+\lambda^2/x^2)\,, \tag{A8}$$

where

$$\rho^7_{\mu\nu} = \eta^1_{\mu\nu}, \quad \rho^5_{\rho\nu} = -\eta^2_{\mu\nu}, \quad \rho^2_{\mu\nu} = \eta^3_{\mu\nu}\,,$$

and

$$\rho^i_{\mu\nu} = 0 \text{ for } i \ne 2,\ 5,\ \text{or } 7. \tag{A9}$$

Because of the factor 2 in Eq. (A8), this solution has topological charge 4.

We now observe[20] that this $k=4$, SU_3 solution can be written as a simple superposition of four minimal solutions of the type (A7) with various gauge orientations:

$$\sum_{i=1}^{8} A'^i\lambda^i = \sum_{l=1}^{4}\sum_{i=1}^{3} \tau^i_l A^i_\mu\,. \tag{A10}$$

Here the four sets of three matrices τ^i_l obey the SU_2 commutation relations

$$[\tau^i_l, \tau^j_l] = 2i\epsilon_{ijk}\tau^k_l \tag{A11}$$

and are given by

$$\tau^i_1 = M^{-1}\begin{pmatrix} u\sigma^i u^{-1} & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0 & 0\end{matrix} & 0\end{pmatrix} M\,,$$

$$\tau^i_2 = M^{-1}\begin{pmatrix} u^{-1}\sigma^i u & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0 & 0\end{matrix} & 0\end{pmatrix} M\,,$$

$$\tau^i_3 = M^{-1}\begin{pmatrix} 0 & \begin{matrix}0 & 0\end{matrix} \\ \begin{matrix}0\\0\end{matrix} & u\sigma^i u^{-1}\end{pmatrix} M\,, \tag{A12}$$

$$\tau^i_4 = M^{-1}\begin{pmatrix} 0 & \begin{matrix}0 & 0\end{matrix} \\ \begin{matrix}0\\0\end{matrix} & u^{-1}\sigma^i u\end{pmatrix} M\,,$$

where $\sigma^i, 1 \le i \le 3$ are the Pauli matrices, $u = e^{i\pi\sigma^3/8}$ and

$$M = \begin{pmatrix} -\frac{1}{\sqrt{2}} & +\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \tag{A13}$$

is a matrix which diagonalizes λ^2.

APPENDIX B

In Sec. III it was stated that $D_1 D_1^*$ is a positive-definite operator on the four-dimensional hypersphere; in this appendix we prove this statement. In a curved space we may write ($\tilde{D}_\mu$ denotes the generally covariant and gauge covariant derivative)

$$\begin{aligned}(D_1 V)_{\alpha\beta} &= \tfrac{1}{2}\left(g^\mu_\alpha g^\nu_\beta - g^\nu_\alpha g^\mu_\beta - \frac{1}{\sqrt{g}}\epsilon_{\alpha\beta}{}^{\mu\nu}\right) D_\mu V_\nu \\ &= \tfrac{1}{2}\left(g^\mu_\alpha g^\nu_\beta - g^\nu_\alpha g^\mu_\beta - \frac{1}{\sqrt{g}}\epsilon_{\alpha\beta}{}^{\mu\nu}\right)\tilde{D}_\mu V_\nu\,. \end{aligned} \tag{B1}$$

Intergrating by parts to find the effect of D_1^* on an anti-self-dual tensor $T_{\mu\nu}$, we obtain

$$(D_1^* T)_\alpha = -2\tilde{D}_\beta T^\beta{}_\alpha\,. \tag{B2}$$

Combining Eqs. (B1) and (B2), and using the anti-self-duality of $T_{\mu\nu}$, we find after some manipulation

$$\begin{aligned}(D_1 D_1^* T)_{\mu\nu} = {}& -g^{\alpha\beta}\tilde{D}_\alpha\tilde{D}_\beta T_{\mu\nu} + R_{\nu\lambda}T^\lambda{}_\mu - R_{\mu\lambda}T^\lambda{}_\nu \\ & + 2R_{\mu\beta\nu\lambda}T^{\beta\lambda} + F_{\beta\mu}T^\beta{}_\nu - F_{\beta\nu}T^\beta{}_\mu\,. \end{aligned} \tag{B3}$$

Because $F_{\mu\nu}$ is assumed to be self-dual, the last two terms cancel. The Riemann tensor $R_{\mu\beta\nu\lambda}$ may be decomposed as[22]

$$\begin{aligned}R_{\mu\beta\nu\lambda} = {}& \tfrac{1}{2}(g_{\mu\nu}R_{\beta\lambda} - g_{\mu\lambda}R_{\beta\nu} - g_{\beta\nu}R_{\mu\lambda} + g_{\beta\lambda}R_{\mu\nu}) \\ & - \tfrac{1}{6}R(g_{\mu\nu}g_{\beta\lambda} - g_{\mu\lambda}g_{\beta\nu}) + C_{\mu\beta\nu\lambda}\,. \end{aligned} \tag{B4}$$

The tensor $C_{\mu\beta\nu\lambda}$ is called the Weyl, or conformal, tensor and vanishes whenever it is possible to choose a coordinate system in which $g_{\mu\nu}$ is proportional to a constant matrix throughout the manifold. Since the sphere has this property, we may set $C_{\mu\beta\nu\lambda}=0$ and substitute Eq. (B4) into Eq. (B3), obtaining

$$(D_1 D_1^* T)_{\mu\nu} = -g^{\alpha\beta}\tilde{D}_\alpha\tilde{D}_\beta T_{\mu\nu} - \tfrac{1}{3}RT_{\mu\nu}\,. \tag{B5}$$

Since $-g^{\alpha\beta}\tilde{D}_\alpha\tilde{D}_\beta$ is a positive operator, and R is everywhere negative on the hypersphere, $D_1 D_1^*$ is positive-definite.

APPENDIX C

The small oscillations about the SU_2 one-pseudoparticle solution have been investigated by 't

Hooft.[17] Working in a background gauge, he showed that for either scalar, spinor, or vector fields the normal modes are eigenmodes of the operator

$$\mathfrak{M} = -\left(\frac{\partial}{\partial r}\right)^2 - \frac{3}{r}\frac{\partial}{\partial r} + \frac{4}{r^2}L^2 + \frac{8}{(1+r^2)}\vec{T}\cdot\vec{L}_1 + \frac{4r^2}{(1+r^2)^2}\vec{T}^2 + \frac{16}{(1+r^2)^2}\vec{T}\cdot\vec{S}_1 , \qquad \text{(C1)}$$

where $\vec{L}_1$, $\vec{L}_2$, $\vec{S}_1$, and $\vec{S}_2$ are orbital and spin angular momentum operators for the two SU_2 components of SO_4 and $\vec{T}$ is the isospin operator. The orbital angular momenta satisfy $\vec{L}_1{}^2 = \vec{L}_2{}^2 \equiv L^2$. For scalars, $\vec{S}_1 = \vec{S}_2 = 0$, while (s_1, s_2) is $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ for right- and left-handed spinors and $(\frac{1}{2}, \frac{1}{2})$ for vectors. The multiplicity of a mode is $(2j_1+1)\times(2j_2+1)$, where $\vec{J}_1 = \vec{L}_1 + \vec{S}_1 + \vec{T}$ and $\vec{J}_2 = \vec{L}_2 + \vec{S}_2$.

This analysis may be used to study the small oscillations about self-dual solutions in a theory with a larger gauge group, G, which is obtained by embedding the SU_2 one-pseudoparticle solution. The generators of G can be classified into multiplets according to their transformation properties under the SU_2 of the embedding. In particular, if the minimal SU_2 is used, there will be one triplet, with the remaining generators belonging to doublets and singlets; thus we must consider vector field small oscillations with $t = 1$, $\frac{1}{2}$, or 0.

The modes which preserve the self-duality of $F_{\mu\nu}$ are precisely those with zero eigenvalue. Thus, to calculate h^1 for the $k=1$ configuration considered in Sec. III, we must determine the number of such modes, excluding those which correspond to gauge transformations. The $t=1$ modes are just those considered by 't Hooft; there are eight modes with zero eigenvalue, of which three correspond to gauge transformations. The $t=\frac{1}{2}$ case follows immediately from 't Hooft's result for right-handed spinors. [Since Eq. (C1) does not involve $\vec{S}_2$, the vector and right-handed spinor eigenvalues are the same, except for their multiplicity.] These modes have $j_1 = l_1 = l_2 = 0$, $j_2 = s_2 = \frac{1}{2}$, so there are two modes per doublet. However, these modes may be written in the form $D_\mu \delta\Lambda$, where $\delta\Lambda$ is an isospinor given by

$$\delta\Lambda = \left[\frac{x_4 - i\vec{x}\cdot\vec{\tau}}{(1+x^2)^{1/2}}\right] v \qquad \text{(C2)}$$

with v an arbitrary isospinor. Thus, when viewed in terms of the gauge group G, these modes correspond to gauge transformations and do not contribute to h^1. Finally, for $t=0$ there are no normalizable solutions of $\mathfrak{M}\psi = 0$.

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