Comments and Addenda

The section Comments and Addenda is for short communications which are not appropriate for regular articles. It includes only the following types of communications: (1) Comments on papers previously published in **The Physical Review** *or* **Physical Review Letters**. *(2) Addenda to papers previously published in* **The Physical Review** *or* **Physical Review Letters**, *in which the additional information can be presented without the need for writing a complete article. Manuscripts intended for this section must be accompanied by a brief abstract for information-retrieval purposes. Accepted manuscripts follow the same publication schedule as articles in this journal, and page proofs are sent to authors.*

Real parts of the pion-proton forward scattering amplitude*

T. R. Engelmann and R. E. Hendrick
Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213
(Received 4 April 1977; revised manuscript received 17 June 1977)

The most recent total-cross-section data are used to calculate real parts of the forward elastic $\pi^{\pm}p$ scattering amplitudes from threshold to 240 GeV/c. Using statistical and systematic uncertainties of the total cross sections and their momenta, along with uncertainties of the subtraction and coupling constants, unphysical cuts, and cross-section extrapolations, we calculate the uncertainties of the real amplitudes. Our results are compared to experimental and other theoretical determinations of the $\pi^{\pm}p$ forward real amplitudes.

I. INTRODUCTION

Over the last few years, the $\pi^{\pm}p$ total cross sections have been accurately measured from near threshold to 240 GeV/c. The low-energy data of Carter *et al.*[1] determining the total cross sections near threshold, along with their low-energy differential cross-section data,[2] have allowed more accurate determinations of the πp subtraction and coupling constants.[3] Recent high-energy data from Fermilab[4] have extended earlier measurements[5] to 240 GeV/c, showing a clear increase in $\pi^{\pm}p$ total cross sections at high energy.

Several recent dispersion-relation calculations of the $\pi^{\pm}p$ real parts have made use of some, or all, of these new data. Carter and Carter[6] used their low-energy total-cross-section data to calculate real amplitudes assuming constant $\pi^{\pm}p$ total cross sections at high energies. Their real-part calculations assumed the usual subtracted symmetric dispersion relation, but unlike most, also assumed a subtracted antisymmetric dispersion relation. No uncertainties on the real parts were estimated by Carter and Carter.

Höhler, Jacob, and Kaiser[7] have calculated the real $\pi^{\pm}p$ amplitudes using both the new low-energy and high-energy data, assuming an unsubtracted antisymmetric dispersion relation. Real parts have been evaluated numerically below 10 GeV/c and from asymptotic expansions above 10 GeV/c. Uncertainties on the real parts were estimated by choosing a new high-energy extrapolation or a new set of low-energy data and recalculating.

A recent work of Hendrick and Lautrup[8] included all the recent total-cross-section data except those of Carroll *et al.*[4] which extend the $\pi^{\pm}p$ data from 200 to 240 GeV/c and fill in some points below 200 GeV/c. Hendrick and Lautrup calculated the real parts and their uncertainties between 1 and 200 GeV/c, assuming an unsubtracted antisymmetric dispersion relation and using a straight-line interpolation between data points.

The present paper is an attempt to improve on the above-mentioned work[8] by extending the energy range of the calculation and by eliminating the assumption of a straight-line interpolation between data. Furthermore, careful attention has been given to sources contributing to uncertainties in the $\pi^{\pm}p$ real amplitudes. Errors to the real parts have been determined by a Monte Carlo variation of the data, subtraction constant, coupling constant, unphysical cuts, and high-energy extrapolations. Both the statistical and systematic uncertainties of the total-cross-section data have been taken into account, as well as the uncertainties in the momenta at which the measurements were made. Interpolation between data points and smoothness of the cross sections were achieved by making cubic spline fits to the $\pi^{\pm}p$ cross sections, which interpolate the cross section between each pair of data points by a cubic polynomial. Twice-subtracted symmetric and unsubtracted antisymmetric dispersion relations have been assumed.

The results of our real-parts calculations do not correspond precisely to the results of either Höhler *et al.*[7] or Carter and Carter,[6] though the results of Höhler *et al.* tend to agree more closely with our results. The results of Carter and Carter exhibit a crossover of $\pi^{\pm}p$ real parts at 22 GeV/c which seriously conflicts with our results, and conflicts with behavior generally expected from an unsubtracted antisymmetric dispersion relation.[9]

Comparison of our results with experimental determinations of the $\pi^{\pm}p$ real parts shows general agreement for the $\pi^{+}p$ amplitude, but a serious continuing disagreement for the $\pi^{-}p$ amplitude in the 20-GeV/c to 40-GeV/c region. This disagreement also exists between other recent calculations and the experimental data, strongly indicating the need for reanalysis of existing experiments, and possibly additional $\pi^{-}p$ real-part and total-cross-section measurements, in that energy range.

We include a comparison of the calculated antisymmetric πp real amplitude with recent charge-exchange results. A slight but persistent discrepancy exists at high energies, reflecting different Regge intercepts for the ρ trajectory found in total-cross-section and charge-exchange measurements.

The organization of this paper is as follows: Section II describes the calculation of the real amplitudes; Sec. III discusses the calculation of uncertainties; and Sec. IV contains our results and conclusions.

II. CALCULATIONS

The optical theorem we use is

$$\mathrm{Im}F_{\pi p}(E)=\frac{1}{4\pi}(E^2-m_\pi^{\,2})^{1/2}\sigma_{\pi p}(E)\,, \tag{1}$$

where E is the laboratory energy of the incident pion and $\hbar=c=1$. The symmetric and antisymmetric amplitudes are defined as

$$\begin{aligned}F_S(E)&=\tfrac{1}{2}[F_{\pi^-p}(E)+F_{\pi^+p}(E)]\,,\\ F_A(E)&=\tfrac{1}{2}[F_{\pi^-p}(E)-F_{\pi^+p}(E)]\,,\end{aligned} \tag{2}$$

with similar relations for the cross sections. The forward dispersion relations for the symmetric and antisymmetric amplitudes are

$$\begin{aligned}\mathrm{Re}F_S(E)=C-f^2\frac{m_\pi^{\,2}}{M[E^2-(m_\pi^{\,2}/2M)^2]}+I_s(E)\\ +\frac{E^2}{2\pi}\mathrm{P}\int_{m_\pi}^{\infty}\frac{dE'\sigma_S(E')(E'^2-m_\pi^{\,2})^{1/2}}{E'(E'^2-E^2)}\,,\end{aligned} \tag{3}$$

$$\begin{aligned}\mathrm{Re}F_A(E)=\frac{2Ef^2}{E^2-(m_\pi^{\,2}/2M)^2}+I_A(E)\\ +\frac{E}{2\pi}\mathrm{P}\int_{m_\pi}^{\infty}\frac{dE'\sigma_A(E')(E'^2-m_\pi^{\,2})^{1/2}}{E'^2-E^2}\,,\end{aligned} \tag{4}$$

where M is the proton mass, f^2 is the pion-nucleon coupling constant, $C=\mathrm{Re}F_S(0)$ is the symmetric subtraction constant, and $I_S(E)$ and $I_A(E)$ are the symmetric and antisymmetric integrals over the unphysical cuts.

The subtraction and coupling constants used in this calculation have been taken from the recent determinations made by Carter *et al.*, and by Samaranayake and Woolcock.

The contribution from unphysical processes is calculated following the parametrizations of Samaranayake and Woolcock.[3] The authors use the scattering-length approximation to compute the amplitude for the charge-exchange reaction, $\pi^-+p\to\pi^0+n$. For the radiative process, $\pi^-+p\to n+\gamma$, they use the multipole amplitudes of Berends, Donnachie, and Weaver.[10] The unphysical cut integrals in Eqs. (3) and (4) are

$$\begin{aligned}I_S(E)=\frac{E^2}{\pi M}\int_{\bar m_{ce}}^{m_\pi}\frac{dE'\frac{2}{9}\nu_0(E')(m_\pi^{\,2}+ME')(a_1-a_3)^2}{E'(E'^2-E^2)}\\ +\frac{E^2}{8\pi^2M}\int_{\bar m_{rc}}^{m_\pi}\frac{2k(E')|E_0(m_\pi)|^2W(E')dE'}{E'(E'^2-E^2)}\,,\end{aligned} \tag{5}$$

$$\begin{aligned}I_A(E)=\frac{E}{\pi M}\int_{\bar m_{ce}}^{m_\pi}\frac{dE'\nu_0(E')(m_\pi^{\,2}+ME')(a_1-a_3)^2}{E'^2-E^2}\\ +\frac{E}{8\pi^2M}\int_{\bar m_{rc}}^{m_\pi}\frac{2k(E')|E_0(m_\pi)|^2W(E')dE'}{E'^2-E^2}\,,\end{aligned} \tag{6}$$

where ν_0 is the speed of the π^0 in the charge-exchange reaction, k is the magnitude of the momentum in the center-of-momentum frame for the radiative reaction, and $\bar m_{ce}$ and $\bar m_{rc}$ are the unphysical thresholds for each of these reactions. $W(E')$ is the center-of-momentum energy, $(a_1-a_3)=0.27m_\pi^{\,-1}$, and $E_0(m_\pi)=0.0033m_\pi^{\,-1}$.

The primary contribution to the calculation of $\mathrm{Re}F(E)$ arises from the principal-value integrals. The $\pi^{\pm}p$ cross sections used in the calculation are fitted by a series of "flexible" spline curves, by which the data are smoothed and interpolated. The spline curve can be thought of as a flexible beam with a certain stiffness. The beam is connected to the data points by springs, each spring constant being inversely proportional to the uncertainty of the datum. The stiffness of the beam and the resulting fit to the data are determined in accordance with the maximum-likelihood principle. By this method, the spline curve interpolates the fits at each pair of data points by a cubic polynomial. The principal-value integral is then evaluated using Gaussian integration.

For laboratory momenta above 45 GeV/c the cross section is approximated by a simple high-energy parametrization of the $\pi^{\pm}p$ data[11]:

$$\sigma_S(p_{\rm lab}) = C_1 \ln\left(\frac{p_{\rm lab}+206}{0.33}\right) + \frac{16.8}{p_{\rm lab}{}^{0.57}}, \qquad (7)$$

$$\sigma_A(p_{\rm lab}) = \frac{C_2}{p_{\rm lab}{}^{0.45}}, \qquad (8)$$

where C_1 and C_2 are adjusted to ensure a smooth transition between the spline fit up to 45 GeV/c and the parametrization above 45 GeV/c.

III. ERROR ESTIMATES

Uncertainties in the real amplitudes are the result of uncertainties from each of several sources: (1) the value of the πN coupling constant, f, (2) the value of the symmetric subtraction constant, C, (3) parametrization of the unphysical cut integral, (4) the high-energy total-cross-section extrapolation, and (5) systematic and statistical errors in the measurements of the total cross sections and the momenta at which those cross sections are measured. The uncertainty in Re$F(E)$ is computed by varying each of these five sources in accordance with the individual errors of each source. The variation is performed using random points taken from a Gaussian distribution with a width corresponding to one standard deviation of the appropriate data point or parameter. In all, 14 separate evaluations of Re$F(E)$ are computed at each energy using different randomized values for the cross sections and parameters. These are then statistically combined to determine the mean and standard deviation of Re$F(E)$ at each energy.

The subtraction and coupling constants are varied according to their appropriate uncertainties.[3] To determine the error contribution due to the unphysical cut integral, the uncertainty is estimated to be 66% of the total contribution from the symmetric unphysical integral in accordance with the procedure of Samaranayake and Woolcock.[3] The error contribution from the high-energy extrapolation is determined by varying the estimate of the symmetric total cross section above 240 GeV/c within a region bounded above by a function increasing as $\ln^2 E$ and bounded below by a constant. The error due to the total cross section arises from two sources. The first is a systematic error in the measurements of both the laboratory momenta and total cross sections. These errors are uniform for cross-section data from the same experiment. The systematic error is taken into account by varying points from the same set of experimental data by the same random amount in accordance with the experimenter's estimated systematic uncertainty in momentum and cross-section measurements.

The second source of error is the statistical error in each point. The statistical errors are taken into account by randomly varying each individual data point with a Gaussian of width determined by the quoted statistical uncertainty. The terminology used here defines statistical error as that error which is completely uncorrelated from point to point. The systematic error is defined to be fully correlated among data points from the same experiment. Since "systematic" experimental uncertainties are not always completely correlated and correlations are not known precisely, our usage does not always correspond to the definition of systematic and statistical errors used by experimenters. We have attempted to take this discrepancy into account.

It is of interest to determine the uncertainty induced into the real amplitudes by each of these five sources individually. By varying each of these error sources separately, holding the others constant, we determine the uncertainty in the real amplitudes due to each source. The results of this calculation are strongly energy dependent. At low scattering momenta ($0 \le p_{\rm lab} \le 0.5$ GeV/c), uncertainties in the πNN coupling constant, the subtraction constant, and the total-cross-section data all contribute roughly equally to the real-part uncertainties. The unphysical-cut uncertainty is of the same order as these contributions near threshold, but decreases rapidly in effect, becoming negligible compared to data uncertainties at 0.5 GeV/c. Above 1 GeV/c the uncertainties in the total-cross-section data dominate, until 30 to 50 GeV/c, where uncertainties in the cross-section extrapolation (used for $p_{\rm lab} \ge 240$ GeV/c) begin to dominate uncertainties in the real amplitudes.

A Monte Carlo determination of uncertainties provides a direct method for studying correlations between real parts at different energies. Overall parameters which enter into the dispersion relations, such as the subtraction and coupling constants, unphysical cuts and high-energy extrapolations, of course serve to correlate the real amplitudes. We also find that total-cross-section data provide relatively strong correlations between real amplitudes over a significant energy range. This is particularly true between two energies which fall within a single experimental range due to the common systematic errors involved. A correlation coefficient of 0.9 exists for Ref_{π^+p} between 0.8 and 1.0 GeV, while the correlation between these energies for Re$f_{\pi^- p}$ is essentially zero. This reflects a dependence on the structure of the total-cross-section data as well. Correlations between $\pi^+ p$ and $\pi^- p$ real amplitudes at the same energies are rather weak.

IV. RESULTS AND CONCLUSIONS

Results of our calculations are shown in Figs. 1 and 2. The shaded band represents one standard deviation in the calculated ratio of real to imaginary amplitudes. (A complete list of $\pi^\pm p$ amplitudes may be obtained from the authors by direct request.)

Several features of a previous calculation[8] have been improved. Replacing a straight-line interpolation between data points by a "flexible" cubic spline interpolation has improved the continuity of the total-cross-section data in the dispersion integral. Careful attention to systematic and statistical errors has decreased the problems of overlapping data sets inducing spurious structure into the forward real parts.[12]

The results of Höhler *et al.*[7] tend to agree fairly well, within one standard deviation at most energies, with our results. One possible source of discrepancy with our results is that Höhler *et al.* perform fits to the symmetric and antisymmetric πp cross sections rather than the $\pi^\pm p$ cross sections. This can ignore some available data and introduce spurious correlations into the $\pi^\pm p$ real parts. The real amplitudes calculated by Carter and Carter[6] appear to have a more serious discrepancy resulting in a crossover of the $\pi^+ p$ and $\pi^- p$ real amplitudes at 22 GeV/c. This is due to a sizable negative subtraction constant whose effect increases linearly with energy in the antisymmetric dispersion relation. As pointed out in Ref. 7, such a crossover is not found using an unsubtracted antisymmetric dispersion relation. In fact, a crossover at high energies will result from an unsubtracted dispersion relation only when the imaginary antisymmetric amplitude decreases monotonically.[9]

Our results are compared with experimental determinations of the $\pi^\pm p$ real parts in Figs. 1 and 2. Two areas of disagreement are noteworthy. Our

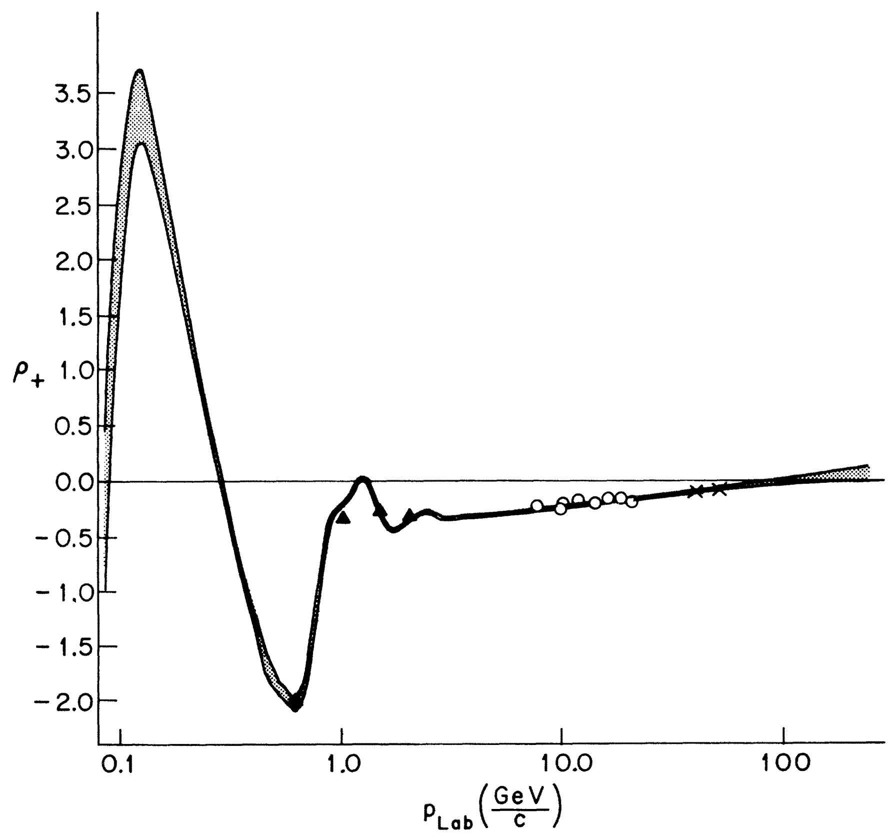

FIG. 1. The ratio of real to imaginary forward amplitudes for $\pi^+ p$ scattering. The shaded band represents one standard deviation in the calculated real amplitudes. Data points represent experimental determinations of the real amplitudes cited in Ref. 13.

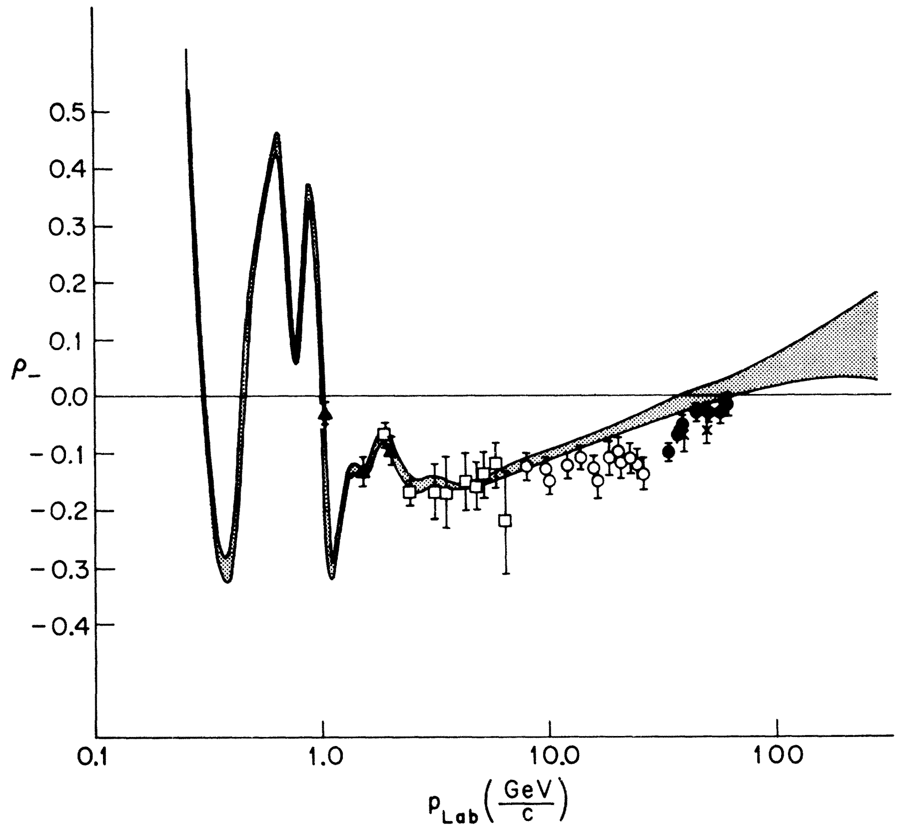

FIG. 2. The calculated ratio of real to imaginary forward amplitudes for $\pi^- p$ scattering, compared to experimental measurements of Ref. 13.

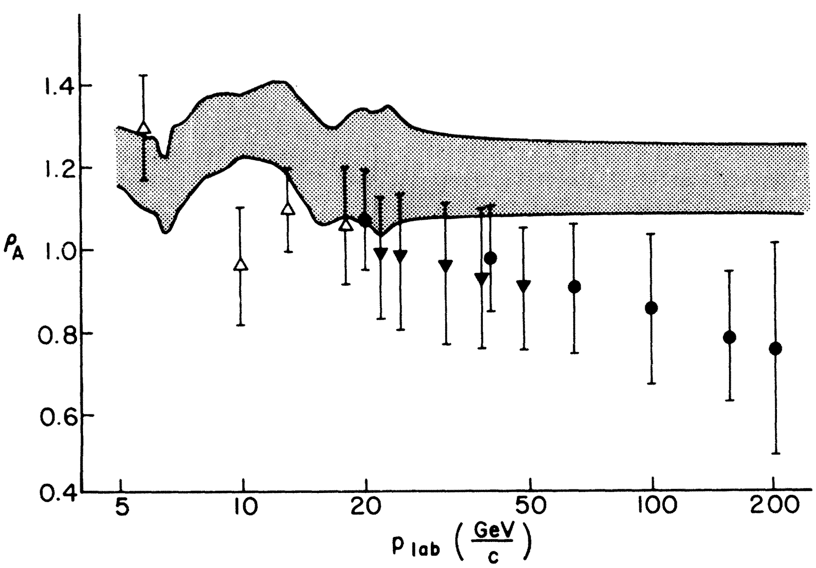

FIG. 3. The calculated ratio of real to imaginary antisymmetric amplitudes, with one standard deviation, is represented by the shaded band. Data points represent experimental determinations of ρ_A from the charge-exchange experiments of Ref. 14. The apparent structure in the shaded band at these relatively high energies should not be taken seriously. It results primarily from normalization discrepancies in different sets of total-cross-section data, to which the calculated ρ_A is particularly sensitive.

calculations, as well as most recent dispersion-relation calculations, seriously disagree with experimental determinations of the π^-p real amplitudes in the region between 20 and 40 GeV/c. This suggests the need for reanalysis of the π^-p real and imaginary amplitudes in this region. A more localized disagreement exists between recent low-energy calculations and the accurate new π^+p real-part measurement by Baillon *et al.*,[13] at 1.009 GeV/c. Our calculation falls several standard deviations from the quoted experimental value, and suggests possible difficulties in the simple parametrization used by Baillon *et al.*, in extrapolating their differential cross-section measurements to $t=0$.

Our antisymmetric real-part results are compared to results from the charge-exchange reaction $\pi^-p\to\pi^0n$ in Fig. 3. The quantity plotted is $\rho_A(E)=\mathrm{Re}F_A(E,0)/\mathrm{Im}F_A(E,0)$. A discrepancy persists at high energies, especially in the region of the Caltech-LBL data. This discrepancy reflects the different ρ-trajectory intercepts found from total-cross-section and charge-exchange measurements. The measured antisymmetric total cross sections are best parametrized with $\alpha_\rho(0)\approx 0.55$, while charge-exchange measurements prefer $\alpha_\rho(0)\approx 0.48$.

ACKNOWLEDGMENT

We are grateful to Professor R. E. Cutkosky for numerous helpful discussions.

---

*Work supported by the U. S. Energy Research and Development Administration.

[1]A. A. Carter *et al.*, Nucl. Phys. B26, 445 (1971).

[2]J. R. Carter, D. V. Bugg, and A. A. Carter, Nucl. Phys. B58, 378 (1973).

[3]V. K. Samaranayake and W. S. Woolcock, Nucl. Phys. B48, 205 (1972); D. V. Bugg, A. A. Carter, and J. R. Carter, Phys. Lett. 44B, 278 (1973).

[4]A. S. Carroll *et al.*, Phys. Rev. Lett. 33, 928 (1974); Phys. Lett. 61B, 303 (1976).

[5]Other πp total cross-section references: A. A. Carter *et al.*, Phys. Rev. 168, 1457 (1968); A. Citron *et al.*, *ibid.* 144, 1101 (1966); corrected by G. Giacomelli, Prog. Nucl. Phys. 12, Part 2 (1970); K. J. Foley *et al.*, Phys. Rev. Lett. 19, 330 (1967); S. P. Denisov *et al.*, Nucl. Phys. B65, 1 (1973).

[6]A. A. Carter and J. R. Carter, Rutherford Lab Report No. 73-024, 1973 (unpublished).

[7]G. Höhler, H. P. Jacob, and F. Kaiser, Karlsruhe Report No. KFK 2457, 1977 (unpublished); G. Höhler *et al.*, Phys. Lett. 58B, 348 (1975).

[8]R. E. Hendrick and B. E. Lautrup, Phys. Rev. D 11, 529 (1975). Other earlier πp real amplitude calculations include: J. S. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963); G. Höhler, G. Ebel, and J. Giesecke, Z. Phys. 180, 430 (1964); B. Lautrup and P. Olesen, Phys. Lett. 17, 62 (1965); B. Lautrup, P. M. Nielsen, and P. Olesen, Phys. Rev. 140, B984 (1965); K. J. Foley *et al.*, Phys. Rev. 181, 1775 (1969); G. Höhler *et al.*, Z. Phys. 233, 430 (1970); G. Höhler and H. P. Jacob, Lett. Nuovo Cimento 6, 389 (1973); Z. Phys. 268, 75 (1974).

[9]H. Cornille and R. E. Hendrick, Phys. Rev. D 10, 3805 (1974).

[10]F. A. Berends, A. Donnachie, and D. L. Weaver, Nucl. Phys. B4, 54 (1967).

[11]R. E. Hendrick *et al.*, Phys. Rev. D 11, 536 (1975).

[12]This problem was pointed out in Ref. 7.

[13]Real-parts measurements shown in Figs. 1 and 2: ▲ P. Baillon *et al.*, Phys. Lett. 50B, 387 (1974); CERN Report No. 75-10, 1975 (unpublished); ○ K. J. Foley *et al.*, Phys. Rev. 181, 1775 (1969); × V. D. Apokin *et al.*, Phys. Lett. 56B, 391 (1975); in *Proceedings of the XVIII International Conference on High Energy Physics, Tbilisi, 1976,* edited by N. N. Bogolubov *et al.* (JINR, Dubna, U.S.S.R., 1976), p. A1-29; □ G. G. Vorobev *et al.*, Yad. Fiz. 19, 849 (1974) [Sov. J. Nucl. Phys. 19, 433 (1974)]; ● S. Nurushev *et al.* Nucl. Phys. B106, 413 (1976).

[14]Charge-exchange experiments shown in Fig. 3: △ A. V. Stirling *et al.*, Phys. Rev. Lett. 14, 763 (1965); P. Sonderegger *et al.*, Phys. Lett. 20, 75 (1966); ▼ V. N. Bolotov *et al.*, Nucl. Phys. B73, 365 (1974); ● A. V. Barnes *et al.*, Phys. Rev. Lett. 37, 76 (1976); see also I. Mannelli *et al.*, *ibid.* 14, 408 (1965); M. A. Wahling and I. Mannelli, Phys. Rev. 168, 1515 (1968). The value ρ_A is determined from

$$\frac{1}{1+\rho_A{}^2}\,\frac{d\sigma_{\pi^-p\to\pi^0n}}{dt}\bigg|_{t=0}=\frac{\sigma_A{}^2}{8\pi}\,.$$

The ρ_A vlaues of Bolotov *et al.* come directly from their paper, which uses σ_A measured by S. P. Denisov *et al.*, Ref. 5. The ρ_A values of Barnes *et al.* have been extracted using σ_A determined by A. S.Carroll *et al.*, Ref. 4. This accounts for the apparent consistency of ρ_A values from the two experiments.