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Disentangling the scattering matrix when there is a dominant reaction mechanism*

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Our previous strategy for obtaining the S matrix from polarization measurements is applied to a general spin-1/2, two-body reaction at fixed s and t. One may test the hypothesis of a dominant reaction mechanism by carrying out a *short list* of M measurements: $M \leq 4$. Subdominant mechanisms may be extracted by a *long list* of $2N$-2 measurements: N is the number of independent amplitudes and $N \leq 16$. The long list brings in all independent interferences of the dominant amplitude with the remaining amplitudes. The scale of the amplitudes is set by the differential cross section; the amplitudes can be determined only to within an overall phase. The cross section, short list, and long list are the elements of an *optimal strategy* for amplitude determination; its extension to arbitrary spins is discussed. These formal arguments do pay respect to practical aspects of amplitude determination.

I. INTRODUCTION

The dynamics of hadron-hadron scattering is fundamental and of some intrinsic interest. It was already recognized in early studies of NN scattering that hadronic interactions are complicated. The forces were seen to be strongly spin-dependent, and measurements were crucial keys to unraveling this spin structure.[1] A basic formalism for studying the spin dependence of polarized NN scattering was developed by Wolfenstein.[2] There emerged a systematic and widely accepted procedure for comparing theoretical models of the nuclear forces with the polarization data that gradually became available. The scattering matrix became the meeting ground for theorists and experimentalists.[3]

It is a problem of long-standing difficulty to determine the scattering matrix from polarization data in a way that is reliable, efficient, and expedient. At sufficiently low energies, where the scattering is mainly elastic, phase-shift models have been used to advantage.[3] (Phase-shift analyses do *indeed* impose certain model-dependent requirements, e.g., threshold behavior of the phase shifts, which is related to the long-range character of the nuclear force.) However, because of experimental uncertainties, gaps in data, or incomplete information, the scattering matrix is usually determined subject to both discrete and continuous ambiguities. Such ambiguities may prevent a meaningful comparison of theory with experiment. One suggestion to remove these ambiguities of phase-shift analysis is to do a sufficient number of measurements at each s and t for a complete reconstruction of the scattering matrix,[4] up to an overall phase. This suggestion becomes particularly important in the inelastic region, where a large number of phase shifts can contribute.

In more recent times, our conception of the fundamental forces has changed. They are not thought of as elementary-particle exchanges unitarized by a Schrödinger equation. One might view them as Regge-pole exchanges that carry distinct quantum numbers, subject to appropriate modifications for unitarity. There is no theoretically consistent formalism for the latter, although effective schemes that fit data do exist.[5] The currently popular view is that hadrons are made of quarks, held together by color-vector-gluon exchanges. Regge dynamics, which has been successful in describing two-body and quasi-two-body scattering data, is presumably a consequence of these fundamental forces.[6] Such theories, whatever their precise nature, should be subjected to experimental test to determine their limitations, if any. The central question we are addressing is how such tests may most efficiently be made.

In looking for reasonable constraints on theories of hadron-hadron scattering, we presuppose one dominant feature learned from Regge phenomenology[5]: Spin is an *essential* complication. Just as in low-energy NN scattering, at least some of the forces of interest are strongly spin-dependent. As a consequence, we feel that a necessary part of any strategy to test a theory must be to determine the scattering matrix for the process under consideration to *some* degree of accuracy. One can then ask what considerations are the most important for an amplitude reconstruction scheme. We feel that the following points are the most relevant:

(1) What basic hypotheses are to be tested?

(2) What is the best strategy for obtaining an answer to (1)?

(3) What experiments are possible, and to what accuracy can they be done?

(4) When one has determined the amplitudes from experiment, how can the information best be presented so as to display any discrete and continuous ambiguities which may be present?

We now elaborate upon these questions, and discuss various (incomplete) answers that have been given.

What basic hypotheses of the theory are to be tested? We feel that this question is often ignored by those who propose schemes for amplitude reconstruction. Suppose one believes there is only a *single* dominant mechanism, as one might expect for $pp \rightarrow pp$ at intermediate energies. It might be that the only question of interest is to verify or disprove that dominance. In a previous paper[7] we showed that from *two* special depolarization measurements, one may test the hypothesis that s-channel helicity is conserved at small momentum transfer. This statement is true even though there are *five* independent complex amplitudes for pp elastic scattering. On the other hand, the crucial dynamical or physical question may require isolating and understanding one of several competing mechanisms. In such a case, a general amplitude reconstruction would be called for. As an example, in pp elastic scattering, one would need at least nine measurements to obtain the scattering matrix up to an overall phase. Depending on the accuracy of the measurements and their choice, in general more than nine measurements would be needed. The latter point is discussed in more detail below. We emphasize that no matter what reconstruction scheme one follows, one must still be able to separate the competing physical mechanisms which are of interest.

What is the best strategy for answering question (1)? If there is a specific dynamical mechanism to be studied and compared with experiment, the procedure is usually obvious. For example, to test Regge dynamics in pp elastic scattering, one isolates and examines the dominant reaction mechanism, as well as the several subdominant ones.[7] If one has in mind no specific fundamental physical hypothesis, one can only rely on a formal mathematical construction of the scattering amplitude. Typically one chooses all those experiments that are feasible (an important consideration in any case) and verifies that, for data of infinite precision, the scattering amplitude can be determined unambiguously.[4,8] Of course such formal exercises are useful, but they do have obvious limitations. Since the data are all subject to intrinsic experimental uncertainties, not all "complete sets of observables" are equally suitable for the practical problem of amplitude determination. Moreover, not all bases are equally appropriate for expressing the amplitudes one determines. Rotational invariance ensures that the scattering matrix may be expressed in various equivalent bases. However, it may well happen that the errors on the amplitudes may be quite small in one basis, whereas they may be large in another basis. One cannot bypass question (1) merely by doing a "model-independent" determination of amplitudes. The information one needs to test a fundamental hypothesis may be expressed more conveniently in some natural basis, and one should perform a set of experiments to determine those *natural* amplitudes with small errors.

What experiments are feasible and to what accuracy can they be done? For the answer to this question, one must usually consult with the experimentalists. After obtaining information from them, one decides whether one can determine the amplitudes, in principle, from the feasible experiments. Probably the most important factor, and one often overlooked, is the precise effect of experimental error on the determination of the amplitudes. An otherwise reasonable set of experiments may give no useful results if the level of experimental error is too large. Unfortunately, it may well happen that *no* set of feasible experiments will give useful information at the level of precision that can be achieved. One practical suggestion for investigating the effect of experimental errors is to employ Monte Carlo techniques.[7] Analytic techniques that provide insights into the problem are certainly needed.[9]

Finally, when one has determined the amplitudes from measurements, how can the information best be presented so as to display the discrete and continuous ambiguities which may be present? If the data are incomplete, it is again quite practical to employ Monte Carlo techniques.[7] Thereby, one can study the effect of errors, as well as of insufficient information. One would expect to gain more insight on this matter from an analytic approach. Some analytic work has been done,[10] but many thorny questions remain. Properties of simultaneous quadratic equations either have not been exhaustively investigated, or else the results have not been applied effectively to this problem.

We will not attempt to provide general answers to questions (1)–(4) above, since the answers are apt to vary from context to context, as the character of the underlying physical situation changes. We have addressed ourselves to one small aspect of the problem of determining amplitudes from data, which does have a satisfactory solution.

When certain reasonable dynamical hypotheses suggest that there is one dominant mechanism, we believe all the above questions can be answered.

In a previous paper[7] we considered determination of the five independent amplitudes for proton-proton elastic scattering from polarization measurements at a particular energy and scattering angle. We described a set of eleven measurements from which the amplitudes can be deduced, to within an overall (energy and angle-dependent) phase. We developed a strategy for determining those amplitudes from an optimal set of measurements when one expects a particular amplitude (call it D) to dominate the scattering process. This strategy consists of the following steps:

(S) One should measure (relatively crudely) a sufficient number of observables that are linear in $|D|^2$, in order to verify that D is dominant.

(L) One should then measure (rather precisely) the observables which are linear in D and which involve all mutually independent interferences of D with the nondominant amplitudes.

Since the nondominant amplitudes are obtained from interferences with D, they are determined without substantial ambiguity. By contrast, if one depended upon "second-order small" observables, a considerably greater precision would be necessary to obtain comparable results. It is also possible that discrete ambiguities would occur.

Here we shall describe certain formalizations and generalizations of the above approach. In Sec. II, the strategy is posed in the general context of two-body scattering of spin-$\frac{1}{2}$ particles. We find it convenient to express the elements of the strategy in terms of the *observables* whenever possible, rather than directly in terms of a specific basis set of amplitudes. Thereby, the strategy can be elegantly expressed in a form directly applicable to experiment. In the Appendixes we describe some elements of the strategy for general two-body scattering of particles of arbitrary spins.

II. SPIN-$\frac{1}{2}$-SPIN-$\frac{1}{2}$ SCATTERING

The scattering of hadrons above the phase-shift region is more difficult to understand as one considers interactions of particles with higher spins. The spin-averaged cross section is an *incoherent* sum of terms involving the various spin amplitudes, so that one cannot unambiguously pick out trends in a given amplitude by looking at a single measurement. In fact, one would not expect to be able to draw any simple, firm conclusions from an incomplete set of measurements when there are several competing mechanisms of interest. Of course, one can gain certain valuable insights from incomplete spin measurements through the use of S-matrix models.[5] The focus of our work is to use models as guides in choosing suitable measurements, in order to disentangle the amplitudes in a model-independent and unambiguous way from a complete set of measurements.

There is a wide latitude of choice in constructing (algebraically) complete sets of measurements, and not all such complete sets are equivalent in a practical situation. In fact, we showed in Ref. 7 that there is considerable advantage to basing the choice of observables partly upon model expectations. Suppose one has a rough idea that a certain amplitude (D) dominates the scattering process. Then one can construct an optimal set of measurements consisting of a *short list* of measurements and a *long list*. The short list represents an implementation of item (S) of the strategy: If the measured values of these observables are in rough agreement with the anticipated results, it will have been established that D *is* the large amplitude. On the other hand, the long list corresponds to item (L) of the strategy: These latter measurements bring in each independent interference of a small amplitude with the dominant amplitude D.

The measurements on the short list serve primarily to establish that the amplitude D is dominant; they are not very effective at reducing ambiguities once that dominance has been established, since the nondominant amplitudes make only second-order contributions. By contrast, the measurements on the long list serve to reduce ambiguities by pinning down the various interferences with D. As the accuracy of the measurements on the long list is improved, the amplitudes can be determined with correspondingly greater precision. Furthermore, it is clear that *each* measurement on the long list plays an independent role in determining the amplitudes. Finally, we point out that the spin-averaged differential cross section establishes the absolute scale for the scattering amplitudes; it obviously must be well measured to set that scale. Without having measurements of the differential cross section, one can determine only the *relative* sizes of the various amplitudes.

For the general case of two-body interactions involving spin-$\frac{1}{2}$ particles, there are sixteen independent (complex) amplitudes. When the reaction conserves parity, there remain eight independent amplitudes. In proton-proton (pp) elastic scattering, for which one has parity conservation, time-reversal invariance, and identical particle symmetry, that number is reduced to five. Furthermore, since np and $\bar{p}p$ elastic scattering are related to pp by isospin and charge-conjugation invariance, respectively, they also have five independent amplitudes. In the subsequent discussion, many of the conclusions apply for general two-body reactions involving spin-$\frac{1}{2}$ particles with unequal masses. We shall mention specifically when use of a special symmetry is made. As in Ref. 7, we will consider the problem of amplitude reconstruction at a particular energy and scattering angle.

The usual observables of spin-correlated quantities, such as polarizations P_0, spin rotations R and A, and depolarizations D, may be expressed as traces of the scattering density matrix $\mathfrak{M}\mathfrak{M}^\dagger$ multiplied by polarization tensors for the various particles. For spin-$\frac{1}{2}$ particles, the polarization tensors are two-dimensional; the only independent ones are $\sigma_0 = I$, along with the Pauli matrices σ_1, σ_2, and σ_3. (Note that σ_3 lies along the axis of quantization, which may be chosen independently for each particle in the reaction.) A general observable for the process $m_a + m_b \to m_c + m_d$ may be written as

$$\begin{aligned} I(abcd) &= \mathrm{Tr}(\sigma_c \sigma_d \mathfrak{M} \sigma_a \sigma_b \mathfrak{M}^\dagger) \\ &= \sum_{i,i'} (\sigma_c)_{i_c i'_c} (\sigma_d)_{i_d i'_d} (\mathfrak{M})_{i'_c i'_d, i'_a i'_b} \\ &\qquad \times (\sigma_a)_{i'_a i_a} (\sigma_b)_{i'_b i_b} (\mathfrak{M})^*_{i_c i_d, i_a i_b} \end{aligned} \tag{2.1}$$

The quantity $I(0000)$ is the unpolarized differential cross section at the energy and angle in question. The scattering amplitude $\mathfrak{M}$ is expressed in a particular basis of spin states. For our purposes it is convenient to relabel the indices on $\mathfrak{M}$, and to form a (complex) column vector ψ from its 16 independent components. Similarly, we may define a 16-dimensional matrix V_{abcd} as the suitably indexed direct product $\sigma_a^t \otimes \sigma_b^t \otimes \sigma_c \otimes \sigma_d$. In that notation, we may write any observable as

$$I(abcd) = \psi^\dagger V_{abcd} \psi . \tag{2.2}$$

There are 256 linearly independent observables of the form (2.2), which correspond to the 256 linearly independent Hermitian matrices V_{abcd}. The two-dimensional matrices $\{\sigma_i\}$ satisfy the multiplication relation $\sigma_i \sigma_j = \eta_{ijk} \sigma_k$, with $\eta_{ijk}{}^2 = \pm 1$. As a consequence, the 16-dimensional matrices, as direct products of Pauli matrices, satisfy the matrix product identity

$$V_{abcd} V_{a'b'c'd'} = \eta V_{a''b''c''d''} , \tag{2.3}$$

where $\eta^2 = \pm 1$ here also. In particular, it follows from (2.3) that these V matrices either commute or anticommute with one another since the matrices $\{\sigma_j\}$ have such a property. The identity

$$V_{abcd} V_{abcd} = V_{0000} = I \tag{2.4}$$

is a special case of the product rule (2.3). Finally,

$$\mathrm{Tr} V_{abcd} = \mathrm{Tr}\sigma_c \, \mathrm{Tr}\sigma_d \, \mathrm{Tr}\sigma_a \, \mathrm{Tr}\sigma_b \tag{2.5}$$

so that all V's are traceless, except $V_{0000} = I$.

We establish a convenient scale for the amplitudes by setting

$$I(0000) = \mathrm{Tr}(\mathfrak{M}\mathfrak{M}^\dagger) = \psi^\dagger \psi = 1 . \tag{2.6}$$

In such a scale, one evidently has $I(abcd)^2 \le 1$ for any observable and any state ψ. In this scale, ψ becomes a (complex) vector of unit norm. Obviously, the physical scale is obtained by multiplying each component of ψ by the square root of the unpolarized cross section. We may define a density matrix ρ corresponding to this "wave function" ψ by the relation

$$\rho = \psi \psi^\dagger . \tag{2.7}$$

This density matrix satisfies the identity $\rho^2 = \rho$ and corresponds to the pure state with wave function ψ. Thereby, one may write the expression (2.2) for the observables as

$$I(abcd) = \mathrm{Tr}(\rho V_{abcd}) \tag{2.8}$$

and express the matrix ρ as a linear combination of the matrices V_{abcd}:

$$\rho = \sum K_{abcd} V_{abcd} , \tag{2.9}$$

the sum extending over all 256 V's. By using (2.3)–(2.5) and (2.8), we obtain

$$K_{abcd} = \tfrac{1}{16} I(abcd) . \tag{2.10}$$

The problem of determining amplitudes from observables amounts to determining ψ (or ρ) from the matrix elements $I(abcd)$ given in (2.2). There are 30 independent (real) numbers to be determined, because of (2.6), along with the fact that an overall phase in ψ cancels out of ρ, as well as the observables $I(abcd)$. With the goal of implementing item (S) of the strategy, we consider the case in which a set of observables $\{\mathcal{O}_k\}$ [corresponding to matrices D_k in (2.8)] take on extremal values, i.e.,

$$\mathcal{O}_k = \psi^\dagger D_k \psi = \pm 1 . \tag{2.11}$$

Because of the identities $D_k^2 = I$ and $\mathrm{Tr} D_k = 0$, each matrix D_k (except $V_{0000} = I$) has eight pairs of eigenvalues $+1$ and -1. From the extremal conditions (2.11), one may conclude that ψ is an eigenvector of each of the matrices D_k with eigenvalue $\mathcal{O}_k = \pm 1$. When a suitable set of observables D_k take on their extreme values, one can determine ψ, the common eigenvector, to within a phase. Notice that the matrices $\{D_k\}$ form a mutually commuting set, since the relations

$$D_i D_j \psi = \mathcal{O}_i \mathcal{O}_j \psi = D_j D_i \psi \tag{2.12}$$

exclude the possibility that D_i and D_j could anticommute.

Suppose that, upon the basis of model-dependent considerations, a specific amplitude is expected to dominate the scattering process under consideration. For convenience, we shall first discuss the case in which, with appropriately chosen axes of quantization for the spins of the various particles, the amplitude $\mathfrak{M}_{++,++}$ (which corresponds to the first component of ψ) is expected to be dominant. The set of mutually commuting matrices $\{D_k\}$ may be taken as $\sigma_a^t \otimes \sigma_b^t \otimes \sigma_c \otimes \sigma_d$, where each of the four indices a, b, c, d must be either 0 or 3. These 16 matrices are diagonal in the standard basis, and (2.11) is satisfied for the eigenvector ψ_0 that corresponds to dominance. To establish that $\mathfrak{M}_{++,++}$ is in fact dominant, it is sufficient to measure four suitably chosen observables from our set. The latter subset of four observables $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$ must be chosen so that the four-tuple assembled from the kth diagonal element of the corresponding D matrices, $w_k = (d_1^k, d_2^k, d_3^k, d_4^k)$, has the property $w_k \neq w_1$ for $k = 2, 3, \ldots, 16$. If one measures values near d_1^1, d_2^1, d_3^1, and d_4^1, respectively, for these observables, dominance is established. [This set of four observables must be chosen *in addition to* the differential cross section, $I(0000) = 1$.] As an example, one may choose the four observables $\mathcal{O}_1 = I(3000)$, $\mathcal{O}_2 = I(0300)$, $\mathcal{O}_3 = I(0030)$, and $\mathcal{O}_4 = I(0003)$.

The choice of this set may be dictated to an extent by convenience and practicality. Once four such observables have been measured, one knows that the values for all 16 observables $\mathcal{O}_k$ are equal to the first diagonal elements d_k^1 of the corresponding matrices D_k. Furthermore, because any matrix outside this set (call it U) anticommutes with *at least one* of these 16 matrices (call that one D_m), we have

$$d_m^1 \psi_0^\dagger U \psi_0 = \psi_0^\dagger U D_m \psi_0 = -\psi_0^\dagger D_m U \psi_0 = -d_m^1 \psi_0^\dagger U \psi_0. \qquad (2.13)$$

Because $d_m^1 = \pm 1$, we may conclude that for the state ψ_0 the observable corresponding to U is zero. The expression (2.6) for the density matrix consequently involves only a sum over these 16 diagonal matrices D_k:

$$\rho = \tfrac{1}{16} \sum_{k=1}^{16} d_1^k D_k. \qquad (2.14)$$

We have discussed the case in which a particular model predicts that a certain spin amplitude should dominate the scattering process. Of course, the model expectations may be of a different character. For example, they may suggest that a given exchange mechanism dominates the process, or perhaps that all "non-spin-flip" amplitudes are roughly equal, as well as much larger than any of the "spin-flip" amplitudes. In such cases, one might expect that the dominant amplitude would have several significant spin components; that is, the column vector ψ corresponding to the dominant amplitude has several nonvanishing components in *any* s-channel spin basis. One may still pick out a "short list" of observables, which take on appropriate extreme values if, and only if, that ψ is actually dominant. However, the observables on that list may come out as unwieldy and unfamiliar linear combinations of those observables usually encountered, whenever the model is relatively complex.

To illustrate and clarify the procedure, consider the simple case in which the scattering process is "spin-independent." We choose the *same* spin axes for all the particles, with the 3 direction along the momentum of the incident projectile, with the 2 direction normal to the scattering plane, and with $\hat{2} \times \hat{3} = \hat{1}$. Then one expects the 16 independent observables $\mathcal{O}_{ab} = I(abab)$, $a, b = 0, 1, 2, 3$, to be equal to $+1$, if the process is to be spin-independent. These observables correspond to mutually commuting matrices, D_{ab}. These latter matrices may be simultaneously diagonalized, and the vector ψ is one of their 16 simultaneous eigenstates. One must choose the short list of measurements in such a way that the set of their eigenvalues gives a unique signature for the eigenvector ψ. A suitably chosen set is $\mathcal{O}_{20} = I(2020)$, $\mathcal{O}_{30} = I(3030)$, $\mathcal{O}_{02} = I(0202)$, and $\mathcal{O}_{03} = I(0303)$.

We implement item (L) of the strategy by measuring observables that bring in all independent interferences of the dominant amplitude ψ_0 with each of the small amplitudes. [We adopt the notation of Eqs. (2.12)–(2.14) for this part of the argument.] In the general case there are 30 independent interference terms. The 16 members of the mutually commuting set $\{D_k\}$ do not contain any of these interferences, since they bring in the nondominant amplitudes $\{\psi_i,\ i = 1, 2, \ldots, 15\}$ only in second order. The interference terms, $\mathrm{Re}\,\psi_0\psi_i^*$ and $\mathrm{Im}\,\psi_0\psi_i^*$, must be determined by measuring a selected set of the 240 remaining observables. We shall establish that these 240 observables divide naturally into 30 equivalence classes of eight each, and that one should measure one observable from each class to determine ψ in our strategy.

To establish the eightfold equivalences, we consider any one of the 240 nondiagonal matrices, U. Of the sixteen matrices $\{D_k\}$, eight anticommute with U and eight (including I) commute with it. For each of the latter eight matrices, D_k, one has

$$D_k U = U D_k = \eta_k U'. \qquad (2.15)$$

It follows from (2.3) and commutativity that $\eta_k = \pm 1$. The matrix U' is also one of the 240 matrices. We express the amplitude to first order as $\psi = \psi_0 + \epsilon\phi$, where ψ_0 is the dominant part, and the parameter ϵ is small. Thereby, we obtain

$$\eta_k \psi^\dagger U' \psi = d_1^k \epsilon(\phi^\dagger U \psi_0 + \psi_0^\dagger U \phi) + O(\epsilon^2)$$

and (2.16)

$$\psi^\dagger U \psi = \epsilon(\phi^\dagger U \psi_0 + \psi_0^\dagger U \phi) + O(\epsilon^2).$$

We have used the identities $D_k \psi_0 = d_1^k \psi_0$ and $\psi_0^\dagger U \psi_0 = 0$ in establishing this result. Consequently, one has

$$\psi^\dagger U' \psi = \eta_k d_1^k \psi^\dagger U \psi + O(\epsilon^2). \tag{2.17}$$

As a result, we have established that, to within a determinable sign factor $\eta_k d_1^k$, the observables corresponding to U and U' give identical values, up to second order in ϵ. These observables are equivalent, in the sense that they involve the same interference term with ψ_0 (viz., $\mathrm{Re}\phi^\dagger U \psi_0$). The eightfold equivalence is thus established.

Having established that there are 30 equivalence classes for the 240 observables, we pick one observable from each class:

$$\mathcal{O}_i = \psi^\dagger U_i \psi, \quad i = 1, 2, \ldots, 30. \tag{2.18}$$

There are 30 interference terms to be determined, and each of the above 30 observables *must* determine an independent linear combination of them. (The set of 256 matrices is complete, and brings in all the interferences, as a consequence.) An appropriate set of observables is listed here:

$$\begin{aligned}
&I(1111), \quad I(1100), \quad I(1001), \quad I(0110), \quad I(0011),\\
&I(1112), \quad I(1200), \quad I(1002), \quad I(0120), \quad I(0012),\\
&I(1110), \quad I(1011), \quad I(1000), \quad I(0101), \quad I(0010),\\
&I(1120), \quad I(1012), \quad I(2000), \quad I(0102), \quad I(0020),\\
&I(1101), \quad I(1010), \quad I(0111), \quad I(0100), \quad I(0001),\\
&I(1102), \quad I(1020), \quad I(0112), \quad I(0200), \quad I(0002).
\end{aligned} \tag{2.19}$$

We obtain the eight observables equivalent to any one of them by one or more of the following operations:

(a) Freely replace 0's with 3's, or 3's with 0's.
(b) Exchange 1's and 2's, while keeping the total number of 2's relatively even.

Here is an example of eightfold equivalent sets of observables:

$$\begin{aligned}
&I(1120), \quad I(2110), \quad I(1123), \quad I(2113),\\
&I(1210), \quad I(2220), \quad I(1213), \quad I(2223).
\end{aligned} \tag{2.20}$$

In the remaining part of this section we specialize to pp elastic scattering data, where, because of invariance requirements, there are only five independent complex amplitudes. We make a specific application of the above arguments to reproduce the major conclusions of Ref. 7 concerning the choice of measurements on the short list.

Let us choose the overall scale of the observables so that the cross section $I(0000) = 1$. A direct consequence of parity conservation is that $I(nnnn) = 1$. One can easily verify this result in the transversity basis, for which the spins are quantized along $\hat{n}$, which is normal to the production plane. In that basis, parity requires all amplitudes involving an odd number of spin flips to vanish; in other words, eight of the sixteen amplitudes vanish. The parameter $I(nnnn)$ is the sum of squares of the even flip amplitudes minus the squares of the odd flip amplitudes, and hence is equal to $I(0000) = 1$. Conversely, if $I(nnnn) = 1$, the odd flip amplitudes vanish, and parity is, in effect, a good symmetry for this process.

The matrix V_{nnnn}, which corresponds to $I(nnnn)$ in Eq. (2.2), is $\sigma_n^t \otimes \sigma_n^t \otimes \sigma_n \otimes \sigma_n$; the scattering amplitude ψ is an eigenvector of V_{nnnn}, with maximal eigenvalue +1. All observables that anticommute with V_{nnnn} are zero. The remaining observables are equal in pairs, up to a determinable sign.

The basic physical hypothesis made in Ref. 7 is that, for small-t scattering, the dominant amplitude conserves s-channel helicity. The hypothesis is correct if, and only if, the depolarization parameters $D_{nn} = I(onon)$ and $D_{ls} = I(olos)$ are both +1. These give the requirements that when the target is polarized either along the normal, $\hat{n}$ or along the beam direction $\hat{l}$, the recoil proton will remain polarized in those same directions. For the recoil proton to have its spin parallel to the beam direction $\hat{l}$, one tests by double-scattering experiments that the spin lies in the production plane and transverse ($\hat{s}$) to its direction of motion. (Recall that in the laboratory frame, the recoil proton comes out at an angle of about 90° with respect to the beam direction.)

The operators $V_{onon} = \sigma_o^t \otimes \sigma_n^t \otimes \sigma_o \otimes \sigma_n$ and $V_{olos} = \sigma_o^t \otimes \sigma_l^t \otimes \sigma_o \otimes \sigma_s$ commute with V_{nnnn}. When the corresponding observables D_{nn} and D_{ls} take on their maximum eigenvalues, the scattering amplitude ψ is completely determined, as we shall show presently. The short list consists of D_{nn}, D_{ls} and $I(nnnn)$. The long list will be those inequivalent measurements which are exactly zero for $D_{nn} = D_{ls} = I(nnnn) = 1$ (they will anticommute with at least one of the members of the short list); it brings in linear interferences of the dominant amplitude with each of the remaining components of the scattering amplitude.

Through general arguments, one establishes that measurements of $D_{nn} = D_{ls} = 1$, along with the requirement $I(nnnn) = 1$, determine ψ to within two independent components. Parity requires 128 observables to be zero. The remaining 128 are equal in pairs, so that there are 64 independent observables and ψ has eight independent complex components. The constraint $D_{nn}=1$ requires 32 of the 64 observables to vanish; the remaining 32 are again equal in pairs, so that ψ now has four independent components. The requirement $D_{ls}=1$ leads to ψ having two independent components. Finally, we establish that identical-particle symmetry requires these components to be dependent.

We list below the observables which commute with V_{olos}, V_{onon}, and V_{nnnn}. Each row lists quantities which are equivalent when only $D_{nn}=1$ and $I(nnnn)=1$. Within each column, successive pairs are the equivalence classes when $D_{ls}=1$ as well:

$$\begin{array}{llll} I(oooo), & I(nnnn), & I(onon), & I(nono), \\ I(olos), & I(nsnl), & I(osol), & I(nlns), \\ I(nooo), & I(onnn), & I(nnon), & I(oono), \\ I(nlos), & I(osnl), & I(nsol), & I(olns), \\ I(soso), & I(lnln), & I(snsn), & I(lolo), \\ I(slss), & I(lsll), & I(sssl), & I(llls), \\ I(solo), & I(lnsn), & I(snln), & I(loso), \\ I(slls), & I(lssl), & I(ssll), & I(llss). \end{array} \tag{2.21}$$

There are only four independent measurements; representatives of these are $I(oooo)$, $I(nooo)$, $I(soso)$, and $I(solo)$. Although $I(nooo)$ can take on any value between -1 and 1, the observable $I(onoo)$, which is equal to $I(nooo)$ through identical-particle symmetry, must vanish. The latter result follows since V_{onoo} anticommutes with V_{olos} ($\{\sigma_n, \sigma_l\}=0$). The symmetry requires $I(nooo)$, as well as the seven equivalent observables, to vanish.

Let us express the density matrix $\rho=\psi\psi^\dagger$ in a 2×2 representation:

$$\rho=\tfrac{1}{2}I+\tfrac{1}{2}\alpha V_{soso}+\tfrac{1}{2}\beta V_{solo}. \tag{2.22}$$

We must have $\mathrm{Tr}\rho=1$ and $\rho^2=\rho$. The square of ρ is computed by using the direct product form $V_{abcd}=\sigma_a^t\otimes\sigma_b^t\otimes\sigma_c\otimes\sigma_d$:

$$\rho^2=\tfrac{1}{4}(1+\alpha^2+\beta^2)I+\tfrac{1}{2}\alpha V_{soso}+\tfrac{1}{2}\beta V_{solo}. \tag{2.23}$$

The requirement $\rho^2=\rho$ is reduced to $\alpha^2+\beta^2=1$, or equivalently

$$I(soso)^2+I(solo)^2=1. \tag{2.24}$$

We now investigate the further consequences of identical-particle symmetry, as it pertains to pp elastic scattering. The analysis can be done most easily using the c.m.-frame observables.[11] In particular, the combinations

$$\begin{aligned} I(xoxo)&=-I(solo)\sin\theta_L+I(soso)\cos\theta_L,\\ I(xozo)&=I(solo)\cos\theta_L+I(soso)\sin\theta_L \end{aligned} \tag{2.25}$$

are such observables. The quantity θ_L is the lab scattering angle of the fast final particle: $\tan\theta_L=(m/E)\tan\theta/2$ with c.m. angle θ and c.m. energy E for each proton. The "x" direction of the beam is identical to "s"; the "l" and "s" directions of the recoil proton are the rotations of "x" and "z" by angle θ_L. Analogously, we can also construct the c.m. quantities

$$\begin{aligned} I(oxox)&=-I(osos)\cos\theta_R-I(osol)\sin\theta_R,\\ I(oxoz)&=-I(osos)\sin\theta_R+I(osol)\cos\theta_R. \end{aligned} \tag{2.26}$$

Here θ_R is the lab angle of the recoil proton: $\tan(\pi/2-\theta_R)=(E/m)\tan\theta/2$. Identical-particle symmetry requires $I(oxox)=I(xoxo)$ and $I(oxoz)=-I(xozo)$.[11] Using these relations and the constraint $I(osos)=0$, we find

$$\begin{aligned} I(solo)&=-I(osol)\cos(\theta_R+\theta_L),\\ I(soso)&=-I(osol)\sin(\theta_R+\theta_L). \end{aligned} \tag{2.27}$$

We may determine the value of $I(osol)$ from the relation $V_{osol}=-V_{onon}V_{olos}$; it requires $I(osol)=-I(olos)=-1$. Thus the density matrix (3.2) is uniquely determined:

$$\begin{aligned} &\rho=\tfrac{1}{2}I+\tfrac{1}{2}\sin(\theta_R+\theta_L)V_{soso}+\tfrac{1}{2}\cos(\theta_R+\theta_L)V_{solo},\\ &I(soso)=\sin(\theta_R+\theta_L) \text{ and } I(solo)=\cos(\theta_R+\theta_L). \end{aligned} \tag{2.28}$$

To summarize, we have shown that D_{nn}, D_{ls} comprise the short list when parity and identical-particle symmetry are imposed. We can evidently make similar arguments to show that the long list consists of the eight measurements[7] P_0, C_{nn}, C_{ss}, C_{ll}, R, $I(nsos)$, $I(lson)$, and $I(slon)$. These anticommute with D_{nn}, D_{ls}, or both, and are inequivalent. The detailed proof that they are linear in the small components of the scattering matrix is left as an exercise.

Note added in proof. These results can also be proved for reactions such as $\bar{p}p\to\bar{\Lambda}\Lambda$, where C-conjugation invariance replaces identical-particle symmetry. The relations (2.22) and (2.24) hold for such diffractive processes as $pp\to N^*p$ with spin-$\frac{1}{2}^+$ N^*'s.

III. PERSPECTIVE

We have outlined a strategy for an optimal choice of experiments in the special case in which there is a dominant reaction mechanism. For the general two-body interaction of spin-$\frac{1}{2}$ particles, there are 16 independent (complex) scattering amplitudes and 256 linearly independent observables. In the special case of proton-proton scattering, for which one has the symmetries of parity, time-reversal invariance, and the Pauli principle, there are five amplitudes and 25 observables. Even in the latter case, considerable algebraic complication could conceivably arise in a direct implementation of our strategy. However, we have shown by means of a quite general, formal analysis that one can pick out the observables in the optimal set without too much effort. One need only be familiar with polarization operators (tensor products of Pauli matrices), and be able to construct a set of basis amplitudes, of which the dominant amplitude is a member. Simple and general arguments are then used to determine the observables that should take on values near their extremes (i.e., near the maximum or minimum permitted), and which observables should take on values near zero. The short list is a conveniently and somewhat arbitrarily chosen subset of the former class, whereas the long list is a subset of the latter class. As a consequence one can easily obtain an optimal set of observables, and to an extent its choice can even reflect the practical limitations of the experimental facilities.

We realize that the general problem of amplitude reconstruction still has no satisfactory solution. One of the main outstanding problems is to analyze the situation in which there are insufficient data. Are the measurements compatible with each other, and what constraints do they impose on the amplitudes? Will such constraints give answers to physically interesting questions? There is a need for systematic answers to these open questions.

Also, for realistic applications, symmetries such as parity, time reversal, isospin, etc. are expected to reduce the number of independent amplitudes. In our formal analysis, we found a natural and convenient method to impose parity conservation, but not the other symmetries. It would be interesting and useful to find a general, elegant treatment of symmetries.

Finally, for systems such as $NN\to NN$, there are distinct isospin amplitudes; the processes $pp\to pp$ and $np\to np$ are independent. A complete amplitude-reconstruction scheme will determine both isospin amplitudes and their relative phases. There arises the deeper question of how to fix the absolute phase of the scattering matrix in the inelastic region. Although the phase is not directly measurable (except in the forward direction by the optical theorems, and near-forward direction by Coulomb interference), it is crucial in comparing theories. Much more attention could be paid to this problem.

APPENDIX A: EXTENSIONS TO PARTICLES OF GENERAL SPIN: SPHERICAL TENSORS

It is convenient to express the density matrix for the tensor states of polarization of a particle of spin greater than $\frac{1}{2}$ in terms of a set of irreducible tensor basis states.[12] For particles of spin j, one defines $(2j+1)$-dimensional matrices as follows[10,13]:

$$(T_L^M)_{mm'}=\langle jm'|T_L^M|jm\rangle=\langle jm'LM|jm\rangle. \tag{A1}$$

We have applied the Wigner-Eckart theorem in relating the matrix elements of an irreducible tensor operator to appropriate Clebsch-Gordan coefficients (we follow the conventions of Condon and Shortley), and have chosen a convenient normalization. (With this convention, the elements of T_L^M are real.) The matrices T_L^M vanish identically for $L>2j$; so for spin j there are $(2j+1)^2$ independent matrices, which form a basis set of $(2j+1)$-dimensional matrices. The T_L^0, which are all diagonal, form a complete commuting set. The matrices T_L^M are superdiagonal for $M>0$, with nonvanishing elements *only* on the Mth row above the diagonal. One also has the reflection relation $T_L^{-M}=(-1)^M(T_L^M)^\dagger$.

As is required by their tensor character, the matrices T_L^M for spin j have the following simple commutation relations with the angular momentum matrices for spin j:

$$[J_z,T_L^M]=MT_L^M, \tag{A2}$$

$$[J_\pm,T_L^M]=[(L\mp M)(L\pm M+1)]^{1/2}T_L^{M\pm1}. \tag{A3}$$

One has $T_0^0=I$, the unit matrix. For $L=1$, we obtain the expressions

$$T_1^0=[j(j+1)]^{-1/2}J_z \tag{A4}$$

and

$$T_1^{\pm1}=\mp[2j(j+1)]^{-1/2}J_\pm. \tag{A5}$$

One may establish the following multiplication formula, which reflects the group-theoretical character of these spin-j matrices:

$$T_{L_1}^{M_1}T_{L_2}^{M_2}=\sum_L K(L_1,L_2,L;j)\langle L_1M_1L_2M_2|LM\rangle T_L^M, \tag{A6}$$

where K is defined in terms of the Racah $6j$-symbol[14] by the formula

$$\begin{aligned}K(L_1,L_2,L;j)&=K(L_2,L_1,L;j)\\&=(-1)^{L_1+L_2-L}[(2L+1)(2j+1)]^{1/2}\\&\quad\times W(jL_1,jL_2;jL).\end{aligned} \tag{A7}$$

It is useful to have a simple procedure for generating the matrices T_L^M. Since for a given value of L, the matrices T_L^M may be obtained from T_L^0 by repeated use of Eq. (A3), we need only consider determination of T_L^0. If we set $(L_1,M_1)=(L,0)$ and $(L_2,M_2)=(1,0)$ and evaluate the vector coupling coefficient in (A6), we obtain the recursion relation

$$T_L^0 T_1^0 = \frac{L+1}{2L+1}\left[\frac{(2j+L+2)(2j-L)}{4j(j+1)}\right]^{1/2} T_{L+1}^0 + \frac{L}{2L+1}\left[\frac{(2j+L+1)(2j-L+1)}{4j(j+1)}\right]^{1/2} T_{L-1}^0 . \tag{A8}$$

The matrices T_L^0 for $L \geq 2$ may be generated from the matrices $T_0^0 = I$ and T_1^0 [given in (A4)] by repeated use of the three-term recursion relation (A8). We thereby generate T_L^0 as a matrix polynomial of order L in the matrix J_z, for example,

$$T_2^0 = [(2j-1)j(j+1)(2j+3)]^{-1/2}[3J_z^2 - j(j+1)I], \tag{A9}$$

$$T_3^0 = [(j-1)(2j-1)j(j+1)(2j+3)(j+2)]^{-1/2} \times [5J_z^3 - 3j(j+1)J_z + J_z]. \tag{A10}$$

One may easily establish these trace relations for the spherical tensors from (A1) and (A6);

$$\begin{aligned} \mathrm{Tr}\, T_L^M &= (2j+1)\delta_{L0}\delta_{M0}, \\ \mathrm{Tr}(T_{L'}^{M'}(T_L^M)^\dagger) &= \frac{2j+1}{2L+1}\delta_{LL'}\delta_{MM'} . \end{aligned} \tag{A11}$$

The density matrix for a particle of spin j may be expanded in terms of the spin-j spherical tensors:

$$\rho = \sum_{L=0}^{2j}\sum_{M=-L}^{L} t_L^M \frac{2L+1}{2j+1} T_L^M, \tag{A12}$$

$$t_L^M = \mathrm{Tr}\rho(T_L^M)^\dagger . \tag{A13}$$

The normalization condition $\mathrm{Tr}\rho = 1$ is met if $t_0^0 = 1$. The density matrix must also be positive-definite.

We are especially interested in the case for which, for a particular choice of the axis of quantization, the component t_1^0 takes on its maximum value, $[j/(j+1)]^{1/2}$. Since the eigenvalue is unique [cf. Eq. (A4)], the density matrix must correspond to the pure state for which the z component of spin of the particle is $+j$. In the standard basis the matrix ρ is diagonal, with the only nonvanishing element being $(\rho)_{jj} = 1$. The expansion in (A12) involves only $M=0$ terms, and one may show from (A1), (A4), and (A8) that

$$t_L^0 = (T_L^0)_{jj} = \langle jjL0|jj\rangle = \prod_{k=1}^{L}\left(\frac{2j+1-k}{2j+1+k}\right)^{1/2} . \tag{A14}$$

A similar situation occurs if t_1^0 takes on its minimal value, $-[j/(j+1)]^{1/2}$. The only nonvanishing element of the density matrix is $(\rho)_{-j,-j} = 1$, and the value of t_L^0 is $(-1)^L$ multiplied by its value in (A14). For these special cases in which t_1^0 has its maximum or minimum value, the density matrix may be determined without additional information.

APPENDIX B: EXTENSIONS TO PARTICLES OF GENERAL SPIN: TWO-BODY REACTIONS

We describe the general two-body reaction $s_a + s_b \to s_c + s_d$, where the particles are labeled by their spins. The scattering amplitudes may be expressed in a suitable spin basis as $\mathfrak{M}_{cd,ab}$, with the subscripts representing the various spin components. The scattering amplitudes consist of $S = (2s_a+1)(2s_b+1)(2s_c+1)(2s_d+1)$ complex functions of the energy and scattering angle. The observables are bilinear functionals of these amplitudes, e.g.,

$$\begin{aligned} I(ABCD) &= \mathrm{Tr}(\mathfrak{M} T_A T_B \mathfrak{M}^\dagger T_C T_D) \\ &= \sum_{\text{all indices}} \mathfrak{M}_{cd,ab}(T_A)_{aa'}(T_B)_{bb'}\mathfrak{M}^*_{c'd',a'b'} \times (T_C)_{c'c}(T_D)_{d'd} . \end{aligned} \tag{B1}$$

The matrices T_A, T_B, T_C, and T_D, which describe the tensor state of polarization isolated for the corresponding particles, may be chosen *independently* from the set $\{T_L^M\}$. There are S^2 such observables. We choose the scale by setting the differential cross section at the energy and angle in question equal to $+1$; in our notation, $I(0000) = \mathrm{Tr}\mathfrak{M}\mathfrak{M}^\dagger = 1$, since T_A, T_B, T_C, and T_D are the identity matrices of appropriate dimension.

It is convenient in the general case, as for the scattering of spin-$\frac{1}{2}$ particles, to form the direct product of polarization tensors:

$$T(ABCD) = T_A^t \otimes T_B^t \otimes T_C \otimes T_D . \tag{B2}$$

We also define a density matrix in this direct-product space

$$\rho = \mathfrak{M}\mathfrak{M}^\dagger . \tag{B3}$$

The relation (B1) defining an observable may be written in terms of the S-dimensional matrices T and ρ as

$$I(ABCD) = \mathrm{Tr}(T(ABCD)\rho), \tag{B4}$$

since $\mathrm{Tr}\rho = 1$ with our choice of scale. We may use the identities (A11) to express the density matrix ρ as a linear combination of the S^2 independent matrices $T(ABCD)$:

$$\rho = \frac{1}{S}\sum L_{ABCD} I(ABCD) T(ABCD), \tag{B5}$$

where

$$L_{ABCD} = (2L_A+1)(2L_B+1)(2L_C+1)(2L_D+1).$$

We shall describe the implementation of our strategy for the case in which one of the amplitudes $\mathfrak{M}_{cd,ab}$ is dominant. For clarity and simplicity, we shall give details for the case in which the spin components of the dominant amplitude are *all* maximal. This latter situation does involve some specialization, but the approach may be extended to other cases. (We see no point in presenting a general analysis, unless it is dictated by physical considerations.) For the case in question, the dominant density matrix has the form

$$\rho_d = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{B6}$$

the only nonvanishing element corresponding to the maximal spin state. Because the matrices $T(ABCD)$ are mutually independent, and since this density matrix is diagonal, the sum in (B5) goes only over the S diagonal T matrices:

$$\rho_d = \frac{1}{S}\sum L_{ABCD} I(L_A 0, L_B 0, L_C 0, L_D 0) T^0_{L_A} \otimes T^0_{L_B} \otimes T^0_{L_C} \otimes T^0_{L_D}. \tag{B7}$$

One may use (B4) to show that the observables in (B7) are

$$I(L_A 0, L_B 0, L_C 0, L_D 0) = (T^0_{L_A})_{s_a s_a} (T^0_{L_B})_{s_b s_b} (T^0_{L_C})_{s_c s_c} (T^0_{L_D})_{s_d s_d}, \tag{B8}$$

with the individual maximal matrix elements being given by the formula (A14).

The "short list" of measurements is chosen from the set of S measurements in (B7), and must be sufficient to *guarantee* that the density matrix has the form (B6). While this list is not unique, a convenient list consists of the measurements

$$I(10\,00\,00\,00),\quad I(00\,10\,00\,00),\quad I(00\,00\,10\,00),\quad I(00\,00\,00\,10). \tag{B9}$$

When these observables take on their maximal values,

$$\left(\frac{s_a}{s_a+1}\right)^{1/2}, \left(\frac{s_b}{s_b+1}\right)^{1/2}, \left(\frac{s_c}{s_c+1}\right)^{1/2}, \left(\frac{s_d}{s_d+1}\right)^{1/2}, \tag{B10}$$

respectively, the density matrix has the form (B6). Consequently, when the four observables (B9) take on their maximal values (B10), the S observables that correspond to diagonal T matrices *all* have their expected values, given in (B8).

The long list of observables involves all independent interference terms of the dominant spin amplitude. Because the dominant density matrix ρ_d has the simple form (B6), such interference terms correspond only to the first row (or first column) of the T-matrix elements. By choosing T-matrix elements with mutually independent first rows, we obtain a suitable short list. Let us recall that the matrix $T_L^{\pm M}$ has nonvanishing elements *only* on the Mth diagonal above (below) the main diagonal, and that *none* of those elements vanish. As a consequence of that feature, one can show that a suitable short list of observables is

$$I(L_a M_a, L_b M_b, L_c M_c, L_d M_d), \tag{B11}$$

where there is *one and only one* observable for *each* different 4-tuple (M_a, M_b, M_c, M_d), subject to the constraints $0 \le M_a, M_b, M_c, M_d$ and $M_a + M_b + M_c + M_d > 0$. The short list consists of $(S-1)$ observables. Since the T matrices corresponding to these observables are non-Hermitian, the numbers in (B11) are intrinsically complex. Their real and imaginary parts, which are $2S-2$ real numbers, permit the determination of the interferences.

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