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Stress tensor of cosmic and laboratory type-II superconductors*

Ian Easson
Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

C. J. Pethick
Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801 and NORDITA, Blegdamsvej 17, DK-2100 Copenhagen, Denmark

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It has recently been pointed out by P. B. Jones that for $B \ll H_{c1}$ the anisotropic part of the stress tensor associated with the flux lines in a type-II superconductor could be considerably larger than $B^2/4\pi$, and that consequently the magnetic distortions of a neutron star could be appreciably larger than previously supposed. We calculate the stress tensor of a type-II superconductor using a thermodynamic approach, and find that the magnitude of the anisotropic term is just $BH/4\pi$, which agrees in order of magnitude, but not in detail, with Jones's result. The stress tensor derived here is then compared with that found by Josephson. We find additional terms arising from the strain dependence of the magnetic free energy.

I. INTRODUCTION

The protons in the interior of a neutron star may well be a type-II superconductor.[1,2] When magnetic flux threads such a superconductor, it does so in the form of quantized flux lines, each containing a flux $\phi_0 = hc/2e \simeq 2\times10^{-7}$ G cm^2 which is confined within a distance of the order of the superconducting penetration depth, $\lambda_p \simeq (m_p c^2/4\pi n_p e^2)^{1/2} \sim 10^{-11}$ cm from the axis of the flux line. Here n_p is the proton number density, and m_p is the proton mass. Each flux line has a core with a radius of the order of the proton superconducting coherence length, ξ_p, within which superconductivity tends to be suppressed. For the matter to be a type-II superconductor the condition $\xi_p/\lambda_p < \sqrt{2}$ must hold.

Recently Jones[3] calculated the stress tensor of a type-II superconductor using Ginzburg-Landau theory and came to the important conclusion that its anisotropic part could be several orders of magnitude greater than the corresponding part of the Maxwell stress tensor for a uniform distribution of the same magnetic flux. He further argued that as a consequence of this, static distortions of magnetized neutron stars could be considerably larger than one would estimate using the Maxwell stress tensor. He then explored some of the implications for pulsars of this conclusion. We note in passing that the form of the stress tensor has important consequences not only for the static deformation of neutron stars, but also for the dynamics of their interiors.

The stress tensor of type-II superconductors is also of interest for laboratory superconductors, especially in the context of flux flow and vortex pinning. A careful discussion of the stress tensor is given by Josephson,[4] and a review of work in this area is given by Kim and Stephen.[5]

In this paper we give a thermodynamic derivation of the stress tensor which is valid for arbitrary temperatures and field strengths. This method of derivation, which was previously used by Josephson,[4] has the advantage of giving results independent of any detailed model. We then evaluate the stress tensor using Ginzburg-Landau theory and find that there are indeed anisotropies in the stress tensor of the same order of magnitude as those found by Jones.[3] Our result differs somewhat from that obtained by Jones, and we trace the origin of the discrepancy to an implicit assumption in Ref. 3 about the core of the flux lines. Our result also differs from Josephson's[4]; we find additional terms which arise from the strain dependence of the flux-line free energy.

The paper is organized as follows. Section II contains the basic derivation of the stress tensor for a material which is isotropic in the absence of flux lines. The more general case of an anisotropic material is considered in the Appendix. In Sec. III the results are compared with those of Jones[3] and of Josephson.[4] Section IV contains some concluding remarks.

II. DERIVATION OF THE STRESS TENSOR

For simplicity let us consider a medium which in the absence of a magnetic field is isotropic. This is the case of interest for neutron stars, since the matter there is a fluid. In general, laboratory superconductors are anisotropic due to the presence of the crystalline lattice. The stress tensor for such an intrinsically anisotropic medium is derived in the Appendix. We use a thermodynamic approach so that our results are independent of any detailed microscopic model. In fact they apply to any magnetic material for which an average magnetic field is well defined. We shall present the rather simple calculations in some detail in order to bring out clearly the physical origin of the effects we consider.

Imagine a rectangular parallelepiped with sides of length L_x, L_y, and L_z. We choose the coordinate axes to be parallel to the sides of the parallelepiped, which is oriented such that the magnetic flux lines are parallel to the z axis. We assume that there are a large number of flux lines threading the volume, but that the volume is sufficiently small that spatial variations in the matter density and the number of flux lines per unit area may be neglected. We calculate the stress tensor by considering how the Helmholtz free energy changes under virtual deformations of the parallelepiped at constant temperature.

There are two assumptions implicit in this procedure. The first is that the matter is in local thermodynamic equilibrium. This is certainly a good approximation if changes occur on time scales long compared with thermal equilibration times. The stress tensor thus calculated will be a good approximation to the true one at low temperatures even if this assumption is invalid, since for highly degenerate matter the thermal contribution to the free energy is negligible. The second assumption is that we neglect dissipative processes associated with the motion of flux lines. Because of the very high electrical conductivity, diffusion of magnetic flux in the interiors of neutron stars occurs only on time scales of the order of the age of the universe even if the matter is not superconducting.[6] Thus it is a good approximation to neglect the dissipation associated with diffusion of magnetic flux in calculating the stress tensor. If the matter is superconducting it is an even better approximation.

The Helmholtz free energy of the parallelepiped is given by VF. Here $V=L_x L_y L_z$ is the volume of the parallelepiped and F is the Helmholtz free energy density, whose natural thermodynamic variables are the temperature, T, the matter density, ρ, and the spatially averaged magnetic field, B. In terms of the average number of flux lines per unit area, n_ϕ, B is just $n_\phi \phi_0$. From our assumption of isotropy, F is independent of the direction of B. For the coordinate system chosen, the off-diagonal components of the stress tensor σ_{ij} vanish by symmetry, and σ_{xx} and σ_{yy} are equal.

Consider first a small virtual change in L_z. This changes the density ρ but leaves B unaffected. Thus

$$\sigma_{zz}=\frac{1}{L_x L_y}\frac{d}{dL_z}(VF)=F(\rho,B)-\rho\frac{\partial F}{\partial\rho}(\rho,B)\,. \quad (1)$$

Now consider a small virtual change in L_x or L_y. This changes not only ρ, but also B, since the flux lines move with the matter as in the case of magnetic flux in highly conducting normal matter. Thus, since the total number of flux lines N_ϕ in the volume remains unchanged during the deformation, B is $N_\phi \phi_0 (L_x L_y)^{-1}$, and consequently

$$\sigma_{xx}=\sigma_{yy}=\frac{1}{L_y L_z}\frac{d}{dL_x}(VF)=\sigma_{zz}-B\left(\frac{\partial F}{\partial B}\right)_{\rho,T}=\sigma_{zz}-\frac{BH}{4\pi}\,. \quad (2)$$

Here

$$H=4\pi\left(\frac{\partial F}{\partial B}\right)_{\rho,T}\,. \quad (3)$$

In tensor notation our result may be written

$$\sigma_{ij}=\left[F-\rho\left(\frac{\partial F}{\partial\rho}\right)_{\vec{B},T}-\frac{\vec{H}\cdot\vec{B}}{4\pi}\right]\delta_{ij}+\frac{H_i B_j}{4\pi}\,, \quad (4)$$

where $\vec{H}$ is a vector of magnitude H in the direction of $\vec{B}$. It may be seen from (4) that the isotropic part depends on B, and that the anisotropic part has only a zz component, of magnitude $BH/4\pi$. Physically this corresponds to a tension in the z direction of size $H/4\pi$ per unit flux.

To bring out clearly the contribution of the flux lines to the stress tensor it is convenient to write the free energy density in the form

$$F=F^{\text{matter}}(\rho)+F^{\text{mag}}(\rho,B)\,, \quad (5)$$

where $F^{\text{matter}}(\rho)$ is the free energy density for $B=0$, including the superconducting condensation energy, and the remainder, F^{mag}, is that associated with the flux lines. One should note that F^{mag} in general depends on the matter density as well as on B. With the help of (5), we may rewrite the stress tensor in the form

$$\sigma_{ij}=-P^{\text{matter}}\delta_{ij}+\sigma_{ij}^{\text{mag}}\,, \quad (6)$$

where

$$P^{\text{matter}}=\rho\left(\frac{\partial F^{\text{matter}}}{\partial\rho}\right)_T-F^{\text{matter}} \quad (7)$$

is the pressure of the matter in the absence of the field, and

$$\sigma_{ij}^{\text{mag}}=\left[F^{\text{mag}}-\rho\left(\frac{\partial F^{\text{mag}}}{\partial\rho}\right)_{\vec{B},T}-\frac{\vec{H}\cdot\vec{B}}{4\pi}\right]\delta_{ij}+\frac{H_i B_j}{4\pi}\,. \quad (8)$$

To appreciate the physical origin of the anisotropic term, consider the case when the spacing between flux lines is large compared with the penetration depth λ_p. In this discussion we shall neglect the dependence of the free energy on the matter density because this affects only the isotropic part of the stress tensor, as can be seen from Eq. (8). The free energy of the flux lines is then proportional to their total length and is given by $N_\phi \mathcal{E} L_z$.

Here $\mathscr{E}$ is the free energy per unit length of an *isolated* flux line, since the interaction energy of the flux lines may be neglected in this limit. When the material volume is stretched in the z direction by an amount δL_z the free energy of the volume increases by an amount $N_\phi \mathscr{E} \delta L_z$, which corresponds to a tension $n_\phi \mathscr{E}$ per unit area. On the other hand, when the material is stretched perpendicular to the z axis there is no such change in the energy since the total length of flux lines remains unaltered. These effects give rise to an anisotropic contribution to the stress tensor of magnitude

$$\sigma_{zz} - \sigma_{xx} = \sigma_{zz} - \sigma_{yy} = n_\phi \mathscr{E} . \tag{9}$$

To make contact with our earlier result (2) we note that in the low-flux-density limit

$$H = 4\pi \mathscr{E} / \phi_0 \equiv H_{c1} . \tag{10}$$

Equation (10) follows directly from the definition of H, Eq. (3), the relation $B = n_\phi \phi_0$, and the fact that the free energy density associated with the flux lines is $n_\phi \mathscr{E}$. Thus from Eqs. (9) and (10) $\sigma_{zz} - \sigma_{xx} = BH_{c1}/4\pi$, in agreement with Eq. (2).

In the low-flux-density limit the anisotropic part of the stress tensor is H_{c1}/B times larger than the corresponding part of the Maxwell stress tensor. H_{c1} is given in order of magnitude by $(\phi_0/4\pi\lambda_p^2)\ln(\lambda_p/\xi_p)$,[7] and therefore the condition that the spacing between flux lines is large compared with λ_p may be expressed as $B \ll H_{c1}$. For neutron stars, one expects H_{c1} to be of order 10^{15} G, and therefore if $B \sim 10^{12}$ G, an often-quoted value, the low-flux-density limit is appropriate, and the enhancement of the anisotropic part of the stress tensor is of order 10^3, in agreement with the conclusions of Ref. 3.

Detailed estimates of the enhancement factor require estimates of H_{c1}, but unfortunately there exist no detailed calculations for the conditions of interest in neutron stars, namely temperatures T small compared with the superconducting transition temperature T_c. Calculations of H_{c1} have been made for T close to T_c using Ginzburg-Landau theory and its microscopic extensions, and for arbitrary temperatures in limit of large $\kappa \equiv \lambda_p/\xi_p$.[8] For the latter case the result is

$$H_{c1} = \frac{\phi_0}{4\pi\lambda_p^2}(\ln\kappa + 0.08) . \tag{11}$$

For the former case the results are fitted very well by

$$H_{c1} = \frac{\phi_0}{4\pi\lambda_p^2}\{[1 - w(\kappa)]1.16\kappa^{0.42} + w(\kappa)(\ln\kappa + 0.08)\} , \tag{12}$$

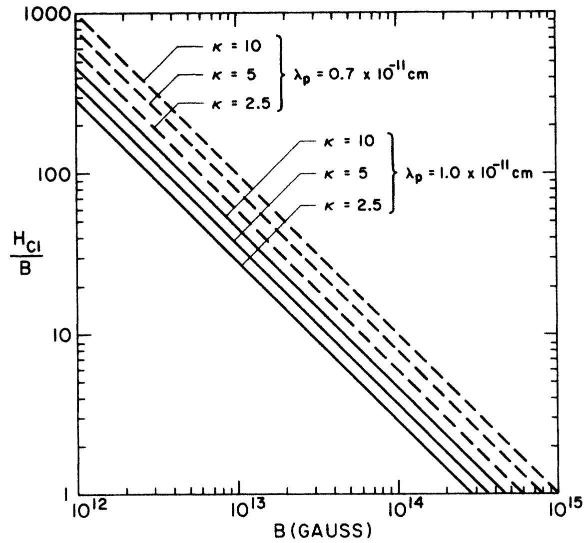

FIG. 1. The factor H_{c1}/B by which the anisotropic part of the stress tensor is enhanced in the low-flux-density limit for a variety of values of κ and λ_p. H_{c1} is taken from Eq. (12). (When $B \gtrsim H_{c1}$ the factor by which the stress tensor is enhanced will be somewhat greater than H_{c1}/B since, as a result of interactions between flux lines, H will be greater than H_{c1}.)

where the weighting function $w(\kappa)$ is

$$w(\kappa) = \begin{cases} 0, & \kappa < 4 \\ 0.31\ln\kappa - 0.43, & 4 \lesssim \kappa \lesssim 100 \\ 1, & \kappa > 100 . \end{cases} \tag{13}$$

To make estimates we assume that H_{c1} is given by (12), even for $T \ll T_c$, provided λ_p and ξ_p in (12) are identified with their actual temperature-dependent values rather than their Ginzberg-Landau values. Calculations by Neumann and Tewordt[9] for clean superconductors such as one expects in a neutron star show that as T decreases from T_c, $H_{c1}/(\phi_0/4\pi\lambda_p^2)$ increases. If this trend continues, Eq. (12) will underestimate H_{c1} for $T \ll T_c$. For $T \ll T_c$, λ_p is given by the London expression $(m_p c^2/4\pi n_p e^2)^{1/2}$ and $\xi_p \simeq \hbar v_{Fp}/\pi\Delta_p$, where v_{Fp} is proton Fermi velocity and Δ_p is the proton superconducting energy gap. Thus, for a proton density $\simeq 5\times10^{36}$ cm^{-3}, $\kappa = \lambda_p/\xi_p \simeq 14.6\,\Delta_p$ (MeV). The enhancement factor is plotted in Fig. 1 for several values of κ and λ_p. In the next section we shall make a detailed comparison of our results with those of Ref. 3.

III. COMPARISON WITH PREVIOUS WORK

In this section we compare our results with those of Jones[3] and of Josephson.[4] First we note that the contribution $-\rho(\partial F^{\text{mag}}/\partial\rho)_{\vec{B},T}\delta_{ij}$ in Eq. (8) was neglected in Ref. 3, and in Ref. 4, which was for the general case, the corresponding term $\partial F^{\rm mag}/\partial u_{i,j}$ was neglected [see Eq. (A10)]. Apart from this, the stress tensor given by Josephson is correct.

Even if we neglect the $\partial F^{\rm mag}/\partial\rho$ term in the stress tensor Eq. (8), the result still disagrees with that of Jones. To understand the origin of this discrepancy we again consider the case where $B \ll H_{c1}$. With the density derivative term neglected, the stress tensor of the flux lines is simply

$$\sigma_{ij}^{\rm mag} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{BH_{c1}}{4\pi} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_\phi \mathcal{E} \end{pmatrix}, \qquad (14)$$

since in this limit $H = H_{c1}$ and $F^{\rm mag} = BH_{c1}/4\pi$. By comparison the result of Ref. 3 is[10]

$$\sigma_{ij}^{\rm mag} = \begin{pmatrix} -\langle T^M\rangle & 0 & 0 \\ 0 & -\langle T^M\rangle & 0 \\ 0 & 0 & \langle T^M\rangle + \langle T^S\rangle \end{pmatrix}. \qquad (15)$$

Here $\langle T^M\rangle$ is the spatially averaged magnetic energy density and $\langle T^S\rangle$ is the spatially averaged kinetic energy density of the circulating supercurrents associated with the flux lines. $\langle T^S\rangle$ is greater than $\langle T^M\rangle$ by a factor of order $(2-3)\ln(\lambda_p/\xi_p) \gtrsim 3-5$ for values of λ_p/ξ_p appropriate for neutron stars.

The zz component of (15) agrees with that of (14) since the energy per unit length of a flux line is just $(\langle T^M\rangle + \langle T^S\rangle)/n_\phi$, but the xx and yy components do not. In Ref. 3 the stress tensor was calculated by taking the expression for the microscopic canonical stress tensor derived from Ginzburg-Landau theory, and then averaging it over all space, excluding the cores of the flux lines of radius ξ_p. This procedure implicitly assumes that when a volume containing many flux lines is stretched in a direction perpendicular to the flux lines, the cores of the flux lines are similarly stretched. In general the cores will not behave in this fashion, and one finds that the macroscopic stress tensor is that found in Ref. 3 plus additional terms coming from the surfaces of the flux-line cores. These extra surface terms are just those required to make $\sigma_{xx}^{\rm mag}$ and $\sigma_{yy}^{\rm mag}$ vanish.

The enhancement factor of the anisotropic part of the stress tensor differs from that of Ref. 3 in two respects. First the basic expression for the stress tensors differ as we saw above. Taken by itself this would imply that in Ref. 3 the enhancement of the anisotropic part of the stress tensor is overestimated by a factor of $1 + \langle T^M\rangle/(\langle T^M\rangle + \langle T^S\rangle)$, which is approximately equal to 1 plus a number of order $1/\ln\kappa$ for $\kappa \gg 1$. A second difference is that the calculation in Ref. 3 is restricted to the high-κ limit, and is equivalent to the use of Eq. (11) for H_{c1}. On the other hand we have used Eq. (12) which is valid for a wider range of κ. H_{c1} calculated from Eq. (12) is larger than that calculated from Eq. (11). For example for $\kappa = 2.5$, one of the values for which calculations were made in Ref. 3, the estimates of H_{c1} from the two equations differ by a factor ~2. These two differences tend to compensate each other, and our final estimates are not so different from those of Jones.

The isotropic contribution to the magnetic part of the stress tensor, denoted by $-P^{\rm mag}\delta_{ij}$, is also enhanced. From Eq. (8) this is given by

$$-P^{\rm mag}\delta_{ij} = \left[-\rho^2 \frac{\partial}{\partial\rho}\left(\frac{F^{\rm mag}}{\rho}\right) - \frac{\vec{H}\cdot\vec{B}}{4\pi}\right]\delta_{ij}. \qquad (16)$$

In the low-flux-density limit ($B \ll H_{c1}$), $F^{\rm mag} = BH_{c1}/4\pi$, and thus in this limit

$$P^{\rm mag} = \frac{B}{4\pi}\left[\rho^2 \frac{\partial}{\partial\rho}\left(\frac{H_{c1}}{\rho}\right) + H_{c1}\right] = \frac{B}{4\pi}\rho\frac{\partial H_{c1}}{\partial\rho}. \qquad (17)$$

The calculation of $\partial H_{c1}/\partial\rho$ is complicated since it involves, through ξ_p, density derivatives of the superconducting energy gap, which cannot be estimated reliably. However, in the extreme type-II limit ($\kappa \gg 1$) the ξ_p dependence of H_{c1} is unimportant, since it enters only through the κ in the logarithm [Eq. (11)], and therefore, neglecting variations of the logarithm, one has

$$\frac{\partial H_{c1}}{\partial\rho} = \frac{H_{c1}}{\rho} \qquad (18)$$

and

$$P^{\rm mag} = \frac{BH_{c1}}{4\pi}. \qquad (19)$$

This is greater than the corresponding term in the Maxwell stress term by a factor $2H_{c1}/B$, which is twice the factor by which the anisotropic part of the stress tensor is enhanced. For the values of κ expected in neutron stars, the result (19) should be a not unreasonable estimate.

IV. CONCLUDING REMARKS

We have calculated the stress tensor from a thermodynamic approach. We find, first, that the stress tensor contains terms arising from the dependence of the free energy of flux lines on the matter density (or more generally on strains) which appear not to have been previously considered. However, the quantity of interest in studies of flux flow in laboratory superconductors is the force on the flux lines and the additional terms in σ_{ij} do not contribute to this force, which in equilibrium is $(\nabla\times\vec{H})\times\vec{B}/4\pi$. In the fluid interiors of neutron stars the additional terms are isotropic, and are of the same order of magnitude as the anisotropic term. If the magnetic field in a neutron star varies spatially, these terms can give rise to distortions of the star comparable with those produced by the anisotropic part of the stress tensor. It should be noted, however, that the distortions of a superconducting star cannot be calculated simply by scaling from the distortions of a star containing normal protons with the same B field configuration. This follows from the fact that the ratio of the isotropic and anisotropic contributions to the magnetic part of the stress tensor in a superconductor differs from the corresponding ratio for the Maxwell stress tensor.

Second, we have confirmed Jones's important conclusion that there can be significant anisotropies in the stress tensor. We find that the anisotropic part of the stress tensor associated with the flux lines is simply $HB/4\pi$. Under typical conditions in neutron stars one expects the anisotropic part of the stress tensor to be the order of magnitude Jones estimated.

Note added in proof. In a paper which appeared after the present article was submitted, Kogan[11] has independently pointed to the importance of contributions to the stress tensor arising from the strain dependence of the magnetic part of the free energy.

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APPENDIX: THE STRESS TENSOR OF AN INTRINSICALLY ANISOTROPIC MEDIUM

We derive here the stress tensor for a material, such as a laboratory superconductor, which is anisotropic even in the absence of a magnetic field. The Helmholtz free energy density F is then a function of the temperature, the spatially averaged magnetic field B_i, and the tensor $u_{i,j} \equiv \partial u_i/\partial x_j$, where u_i is the displacement vector of the medium.

We consider a thin slab of material of thickness h. As before, we assume that the temperature, matter density, and $\vec{B}$ are uniform, and that the field lines are frozen into the material. We now imagine that the upper surface of the slab is subjected to a virtual translation over an infinitesimal distance ξ_i whose direction is not necessarily that of the unit normal n_i. The force per unit area is $\sigma_{ij}n_j$, and therefore by equating the work done by the force to the increase in the free energy hF per unit area we find

$$\sigma_{ij}n_j\xi_i = \delta(hF) = h\delta F + F\delta h\ . \qquad \text{(A1)}$$

Since the deformation is assumed to be homogeneous and isothermal,

$$\delta F = \frac{\partial F}{\partial u_{i,j}}\delta u_{i,j} + \frac{1}{4\pi}H_i\delta B_i\ , \qquad \text{(A2)}$$

$$\delta u_{i,j} = \frac{\xi_i n_j}{h}\ , \qquad \text{(A3)}$$

and

$$\delta h = \xi_i n_i\ . \qquad \text{(A4)}$$

In addition, since we are assuming that flux is conserved,

$$\delta B_i = [\vec{\nabla}\times(\delta\vec{u}\times\vec{B})]_i = \frac{(B_j n_j)\xi_i}{h} - \frac{(\xi_j n_j)B_i}{h}\ . \qquad \text{(A5)}$$

Substituting (A2)–(A5) into (A1) we get the following expression for the total stress tensor of the matter and the flux lines:

$$\sigma_{ij} = \left(F - \frac{\vec{H}\cdot\vec{B}}{4\pi}\right)\delta_{ij} + \left(\frac{\partial F}{\partial u_{i,j}}\right)_{T,\vec{B}} + \frac{H_iB_j}{4\pi}\ . \qquad \text{(A6)}$$

If F depends on $u_{i,j}$ only through the trace $u_{i,i}$, then $F\delta_{ij} + \partial F/\partial u_{i,j}$ becomes $(F - \rho\partial F/\partial\rho)\delta_{ij}$, and the expression for σ_{ij} reduces to that derived in Sec. II.

We now split F into a nonmagnetic part and a magnetic part, as in Eq. (5):

$$F(u_{i,j}, B_i) = F^{\text{matter}}(u_{i,j}) + F^{\text{mag}}(u_{i,j}, B_i)\ . \qquad \text{(A7)}$$

This allows us to write the stress tensor as

$$\sigma_{ij} = \sigma_{ij}^{\text{matter}} + \sigma_{ij}^{\text{mag}}\ , \qquad \text{(A8)}$$

where

$$\sigma_{ij}^{\text{matter}} = F^{\text{matter}}\delta_{ij} + \left(\frac{\partial F^{\text{matter}}}{\partial u_{i,j}}\right)_{T,\vec{B}}\ , \qquad \text{(A9)}$$

and

$$\sigma_{ij}^{\text{mag}} = \left(F^{\text{mag}} - \frac{\vec{H}\cdot\vec{B}}{4\pi}\right)\delta_{ij} + \left(\frac{\partial F^{\text{mag}}}{\partial u_{i,j}}\right)_{T,\vec{B}} + \frac{H_iB_j}{4\pi}\ . \qquad \text{(A10)}$$

Equations (A8)–(A10) are the generalizations to an anisotropic medium of Eqs. (6)–(8), which hold for an isotropic medium.

If one is interested in situations where the flux lines can move relative to the matter, as in the case in flux flow in laboratory superconductors, then it is useful to know the forces on the matter and on the flux lines separately. Including pinning forces $\vec{P}$ on the flux lines, we find that in equilibrium the forces on the matter satisfy

$$\frac{\partial}{\partial x_j}\left[\left(\frac{\partial F^{\text{matter}}}{\partial u_{i,j}}\right)_{T,\vec{B}} + \left(\frac{\partial F^{\text{mag}}}{\partial u_{i,j}}\right)_{T,\vec{B}}\right] - P_i = 0, \tag{A11}$$

while the condition for balance of forces on the flux lines is

$$\frac{1}{4\pi}(\vec{\nabla}\times\vec{H})\times\vec{B} + \vec{P} = \vec{0}. \tag{A12}$$

We note that the term $\partial F^{\text{mag}}/\partial u_{i,j}$ in the total stress tensor enters the condition for the balance of forces on the matter, but not that for the balance of forces on the flux lines.

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