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Gauge fixing and mass renormalization in the lattice gauge theory*†

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The lattice gauge theory proposed by Wilson is discussed. Gauge fixing is defined for the lattice theory, and it is shown that gauge fixing is done in this theory solely for calculational purposes. The gauge-fixing method is used to study the mass renormalization of the gauge field quantum. An explicit calculation is done to lowest order which shows that there is no mass renormalization. This same result is proved to all orders in perturbation theory using the Slavnov identity.

I. THE LATTICE GAUGE THEORY

The lattice gauge theory has been introduced by Wilson[1] to explain the dynamics of strongly interacting elementary particles. The non-Abelian gauge field has many well-known and remarkable properties. In particular, it is a nonlinear field which couples to itself (and, of course, to anything else which carries the requisite quantum number). In this sense it is similar to the gravitational field. The gauge field also exhibits asymptotic freedom (that is, the strength of the coupling goes to zero for zero-distance interaction); and, when coupled to the quark field, the coupled quark-gluon theory shows quark confinement in the strong-coupling limit. The gauge field quantum is an elementary particle. For the case of strong interaction, this quantum is called the gluon. The quantum of the Abelian gauge field is the photon and its properties are fairly well understood.

Wilson[1,2] has given an action functional formulation of quantum field theory using the Feynman path integral. In particular, the lattice gauge field is quantized on a discrete lattice embedded in a four-dimensional Euclidean spacetime. The reason for going to a lattice is twofold. Firstly, the lattice provides an ultraviolet cutoff, and hence there are no ultraviolet divergences in the theory. We will sometimes work with a finite-size lattice, and this will provide an infrared cutoff. The problem of renormalization has to be solved to go to the continuum limit, i.e., to let the lattice spacing go to zero. Secondly, using the lattice as a cutoff allows one to formulate the cutoff theory so that we have exact *local gauge invariance* for the lattice gauge field. Any other conventional way of defining the cutoff theory usually destroys local gauge invariance. Local gauge invariance is the single most important property of the gauge field, and the lattice gauge field is a more accurate representation of it than, say, would be a theory which preserves Lorentz invariance but gives up local gauge invariance.

We work in Euclidean spacetime as this allows us to rigorously define the Feynman path integral. Analytically continuing to physical time is necessary for computing physical quantities.

Consider a finite lattice of N^4 lattice sites and with periodic boundary conditions. Let n specify the lattice site and μ the directions on the lattice. The local gauge degrees of freedom are the finite group elements $U_{n\mu}$ belonging to the gauge group G, which for definiteness is taken to be SU(n).

The gauge field action functional is defined by[1,2]

$$A=\frac{1}{2g_0^2}\sum_n\sum_{\mu\neq\nu}\mathrm{Tr}(W_{n\mu\nu}), \tag{1.1}$$

where g_0 is the bare coupling constant (Tr signifies trace). Note that

$$W_{n\mu\nu}=U_{n\mu}U_{n+\hat{\mu},\nu}U^\dagger_{n+\hat{\nu},\mu}U^\dagger_{n\nu}. \tag{1.2}$$

The gauge field theory is quantized by integrating e^A over all possible values for $U_{n\mu}$, i.e.,

$$Z(g_0^2)=\prod_n\prod_\mu\int dU_{n\mu}e^A, \tag{1.3}$$

where $dU_{n\mu}$ is the invariant measure.

Note that A is invariant under local gauge transformations, which for the lattice is defined by

$$U_{n\mu}\to V_nU_{n\mu}V^\dagger_{n+\hat{\mu}}, \tag{1.4}$$

where V_n is also an element of the gauge group.

Let $\{X^a\}$ be the generators of the group. Then

$$[X^a,X^b]=iC^{abc}X^c, \tag{1.5}$$

$$\mathrm{Tr}(X^aX^b)=\delta^{ab}/s^2; \tag{1.6}$$

for the fundamental representation, $s^2=2$.[3] Let $B^a_{n\mu}$ be the local lattice spacetime gauge field, ϕ^a_n be a local scalar field, and let $f^a_{n\mu\nu}$ be the local gauge field tensor. Then

$$W_{n\mu\nu}=e^{if^a_{n\mu\nu}X^a}, \tag{1.7}$$

$$U_{n\mu}=e^{iB^a_{n\mu}X^a}, \tag{1.8}$$

$$V_n=e^{i\phi^a_nX^a}, \tag{1.9}$$

$B^a_{n\mu}$ and $f^a_{n\mu\nu}$ are bounded variables which take values in the compact parameter space of SU(n). We consider the case when $B^a_{n\mu} \ll 1$. Using the equation $e^A e^B = e^{A+B+[A,B]/2+\cdots}$ we find from (1.8)

$$f^a_{n\mu\nu} = \Delta_\mu B^a_{n\mu} - \Delta_\nu B^a_{n\mu} - \tfrac{1}{2}C^{abc}(B^b_{n+\hat\nu,} B^c_{n+\hat\mu,\nu} + B^b_{n+\hat\nu,\mu} B^c_{n\nu} + B^b_{n\mu} B^c_{n+\hat\mu,\nu} - B^b_{n\mu} B^c_{n\nu} - B^b_{n\mu} B^c_{n+\hat\nu,\mu} - B^b_{n+\hat\mu,\nu} B^c_{n\nu}) + O(B^3) = -f^a_{n\nu\mu}, \tag{1.10}$$

where repeated indices are summed over $\Delta_\mu h_n \equiv h_{n+\hat\mu} - h_n$ is the finite lattice derivative. In general, $f^a_{n\mu\nu}$ is an infinite power series of the $\{B^a_{n\mu}, B^a_{n+\hat\mu\nu}, B^a_{n+\hat\nu\mu}, B^a_{n\nu}\}$ variables. That $f^a_{n\mu\nu}$ is an analytic function of these variables is a consequence of the group multiplication law. We also determine the effect of the gauge transformation on the $B^a_{n\mu}$ variables. Let $\phi^a_n \ll 1$; then, from (1.4),

$$\exp(i\tilde{B}^a_{n\mu}X^a) = \exp(i\,\phi^a_n X^a)\exp(iB^a_{n\mu}X^a)\exp(-i\,\phi^a_{n+\hat\mu}X^a)$$

giving

$$\tilde{B}^a_{n\mu} = B^a_{n\mu} - \Delta_\mu \phi^a_n - \tfrac{1}{2}C^{abc}(\phi^b_n + \phi^b_{n+\hat\mu})B^c_{n\mu} + \tfrac{1}{2}C^{abc}\,\phi^b_n\,\phi^c_{n+\hat\mu} + O(\phi^3). \tag{1.11}$$

We will return to these equations in Sec. II. [In Sec. IIB, we use $B^a_{n\mu}(\phi)$ to denote $\tilde{B}^a_{n\mu}$.]

II. THE WEAK-COUPLING APPROXIMATION

The lattice gauge theory is studied for its weak-coupling behavior. It will be shown that a gauge-fixing term is necessary in this limit solely for the purpose of calculations. A counterterm has to be introduced into the action to cancel the gauge-invariant effects of the gauge-fixing term. The counterterm will be evaluated in the weak-coupling approximation, and the result is seen to be significantly different from the results of the conventional continuum non-Abelian gauge fields. We attribute these differences to the lattice cutoff that is built into the theory. The main purpose of the gauge-fixing/counterterm formalism is to reduce the lattice theory, in the weak-coupling approximation, to conventional field theory on a lattice. This, in essence, means that *all* the field variables $\{B^a_{n\mu}\}$ take values over an infinite range (i.e., over the real line R) rather than over the compact parameter space. Having all the variables $B^a_{n\mu}$ range over R will allow us to define Feynman perturbation theory for the lattice gauge field. In this section, we will basically discuss under what conditions the above-mentioned reduction is possible. The gauge fixing/counterterm formalism will be introduced to make this reduction possible; we will also discuss why, without this formalism, we have a well-defined theory which is, however, unsuitable for calculations. We will first discuss, for pedagogical reasons, the theory without the gauge-fixing term, and then show the necessity for introducing it. The necessity for the counterterm arises as follows: (a) The gauge fixing breaks local gauge invariance of the theory. This is necessary, since it is local gauge invariance which is the obstacle to setting up a Feynman perturbation expansion for the original action. (b) The counterterm is introduced to cancel the gauge-invariant effects generated by the gauge-fixing term. The resultant theory gives the same gauge-invariant vacuum expectation values as the original theory.

A. Gauge fixing

We will discuss gauge fixing from the weak-coupling point of view, although the basic results are valid for arbitrary coupling. The reason for this is that the usefulness of this approach is obvious for the weak-coupling limit. By the weak-coupling limit we mean the behavior of the lattice gauge field when we let $g_0 \to 0$. The properties of the gauge field can then be computed as an expansion in g_0. We will look at the $O(g_0{}^2)$ behavior of the field.

We will first study the behavior of the theory without any gauge fixing. To do so, we have to make a change of variables such that all the variables in the path integral that have no coupling to the gauge-invariant sector are factored out of the path integral. This change of variables is called choosing a gauge for the gauge field. We choose the generalized axial gauge as defined in Ref. 1 for the Abelian lattice theory; the non-Abelian case is essentially the same as the Abelian case except for some not so minor complications. The choice of a specific gauge will help clarify the role of the gauge-fixing term.

To choose the axial gauge, we have to partition the finite lattice into disjoint domains. On each domain will be defined a distinct change of variables. The domains are defined as follows. We consider a finite lattice $1 \le n_\mu \le N$ with periodic boundary conditions. We partition the lattice sites into the following disjoint domains:

$D^{(0)} = \{n \mid 1 \le n_0 \le N-1, 1 \le n_i \le N\}$,

4-dimensional hypervolume;

$D^{(1)} = \{n \mid n_0 = N,\ 1 \le n_1 \le N-1, 1 \le n_2, n_3 \le N\}$,

3-dimensional volume;

$D^{(2)}=\{n\mid n_0=n_1=N,\ 1\le n_2\le N-1, 1\le n_3\le N\}$,

2-dimensional surface;

$D^{(3)}=\{n\mid n_0=n_1=n_2=N,\ 1\le n_3\le N-1\}$,

1-dimensional line;

$D^{(4)}=\{N\equiv(N,N,N,N)\}$, single lattice point.

Do the following gauge transformation:

$$U_{n\mu}\to\tilde U_{n\mu}=V_n U_{n\mu}V^\dagger_{n+\hat\mu}\,. \tag{2.1}$$

The axial gauge is defined by the following change of variables:

$$\begin{aligned}&n\in D^{(\nu)},\quad \mu\neq\nu\\ &dU_{n\nu}=d\tilde U_{n\mu},\quad \tilde U_{n\nu}=1\end{aligned} \tag{2.2}$$

and

$$dU_{n\nu}=dV_n\ .$$

For the single lattice site N we have

$$\begin{aligned}&n=N:\ V_N=1\\ &\text{and}\\ &dU_{N\mu}=d\tilde U_{N\mu}=d\tilde U_{N\mu}\ \text{for all }\mu.\end{aligned} \tag{2.3}$$

Note that $V_N=1$ is the only choice for V_N which is gauge invariant. The only difference, in the choice of gauge, between the Abelian and the non-Abelian case is in (2.3), the reason being that the Abelian case has a higher symmetry than the non-Abelian case, which allows one to eliminate the variables $\{U_{N\mu}\}_{\mu=0}^3$ from the action. This is no longer possible for the non-Abelian case and causes some complications. We will return to this point in Sec. III.

For concreteness, we examine the effect of the gauge transformation on the path integral of the action functional. Firstly, not that gauge invariance implies that the action is invariant under this transformation, that is,

$$A[W]=A[\tilde W],\ \text{independent of the }\{V_n\}\text{ variables.}$$

Hence

$$Z(g_0)=\prod_n\prod_\mu\int dU_{n\mu}e^{A[U]}$$

$$=\Big(\prod_{n\neq N}\int dV_n\Big)\Big(\prod_{\nu=0}^3\prod_{n\in D^{(\nu)}}\prod_{\mu(\neq\nu)}\int d\tilde U_{n\mu}\prod_\mu d\tilde U_{N\mu}e^{A[\tilde U]}\Big) \tag{2.4a}$$

$$=\prod_{\nu=0}^3\prod_{n\in D^{(\nu)}}\prod_{\mu(\neq\nu)}\int d\tilde U_{n\mu}d\tilde U_{N\mu}e^{A[\tilde U]}\,. \tag{2.4b}$$

Let

$$\prod_{n,\mu}{}'\equiv\prod_\nu\prod_{n\in D^{(\nu)}}\prod_{\mu(\neq\nu)}$$

then we show in Appendix A that

$$Z(g_0)\simeq\prod_{n,\mu}{}'\prod_a\int_{-\infty}^{+\infty}dB^a_{n\mu}\,\mu(B_{n\mu})\prod_\alpha\int_G d\tilde U_{N\alpha}e^A<\infty\,. \tag{2.5}$$

In other words, Z can be represented by a convergent multiple integral where all the variables $\{B^a_{n\mu}\}$ (except at the lattice site N) range over an infinite range. Note also from (2.4a) that the redundant variables $\{V_n\}$ have been factorized in the path integral from the gauge-invariant sector. Note, however, that the variables $\{\tilde B^a_{n\mu}\}$ are nonzero on very complicated domains, and this makes any tractable Fourier transform to k space virtually impossible. Hence (2.5) is not suited for perturbation theory, although it is well defined.

The gauge-fixing term is introduced to control the divergence due to the $\{\phi^a_n\}$ variables. This means, in terms of the original variables $\{B^a_{n\mu}\}$, that the action has added to it a term which necessarily breaks gauge invariance. To leave invariant the gauge-invariant sector, we further add the counterterm. The counterterm is a gauge-invariant functional of the gauge field and is evaluated from the gauge-fixing term via a path integral. (We will relax the property of gauge invariance later on.)

Let A_α be the gauge-fixing term and A_c be the counterterm. The modified action is defined as

$$A'=A+A_\alpha+A_c\,. \tag{2.6}$$

The actions A and A' give the same gauge-invariant physics. (We will prove this later.) One has a wide choice as to what functional of the field variables A_α should be. The only necessary condition is that

$$Z'=\prod_{n\neq N}\prod_{\mu,a}\int_{-\infty}^{+\infty}dB^a_{n\mu}\int_G\prod_\mu dU_{N\mu}e^{A'}\mu(B^a_{n\mu})<\infty\,. \tag{2.7}$$

(We will make a specific choice for A_α in Sec. II B.) To define A_c, we introduce the following notation:

$$dV=\prod_{n\neq N}dV_n\,, \tag{2.8}$$

$$dU=\prod_n\prod_\mu dU_{n\mu}\,, \tag{2.9}$$

$$U^{(V)}_{n\mu}=V_nU_{n\mu}V^\dagger_{n+\hat\mu}\,. \tag{2.10}$$

Define A_c by

$$e^{A_c[U]}=1\Big/\int dV e^{A_\alpha[U^{(V)}]},\quad\text{gauge invariant.} \tag{2.11}$$

Note the identity

$$1=\int dVe^{A_\alpha[U^{(V)}]}\Big/\int dV' e^{A_\alpha[U^{(V')}]}. \tag{2.12}$$

Let $K[U]$ be an arbitrary gauge-invariant function. Then

$$\int dUK[U]=\int dUK[U]\frac{\int dV\,e^{A_\alpha[U^{(V)}]}}{\int dV'e^{A_\alpha[U^{(V')}]}}$$
$$=\int dV\int dU\,K[U]e^{A_\alpha[U^{(V)}]}e^{A_c[U]}\,. \tag{2.13}$$

Perform the gauge transformation on $\{U_{n\mu}\}$ variables such that

$$U'_{n\mu}=V_nU_{n\mu}V^\dagger_{n+\hat\mu}\,,\qquad dU'=dU\,, \tag{2.14}$$

$$K[U']e^{A_c[U']}=K[U]e^{A_c[U]}. \tag{2.15}$$

We then have

$$\int dUK[U]=\left(\int dV\right)\left\{\int dU'K[U']e^{A_c[U']+A_\alpha[U']}\right\}.$$
$$=\int dUK[U]e^{A_c[U]+A_\alpha[U]}\,. \tag{2.16}$$

We thus see that $e^{A_c+A_\alpha}$ leaves the gauge-invariant sector unchanged. Hence, in particular,

$$Z(g_0)=\int dU\,e^{A[U]}=\int dU\,e^{A+A_\alpha+A_c}\,. \tag{2.17}$$

Note that the result (2.16) is valid exactly for the lattice theory. This formulation reduces to the Faddeev-Popov[4] formulation in the weak-coupling approximation. We now choose a specific A_α and calculate A_c for it.

B. Evaluation of the counterterm

Choose the gauge-fixing term[4] to be

$$e^{A_\alpha[B]}=\prod_{n,a}{}'\,\delta(s^a_n-t^a_n)\,, \tag{2.18}$$

where $\{t^a_n\}$ are fixed numbers, $\Pi'_n\equiv\Pi_{n\neq N}$, and

$$s^a_n=\sum_\mu\Delta_\mu B^a_{n-\hat\mu,\mu}\,, \tag{2.19}$$

Define $B^a_{n\mu}(\phi)$ by

$$\exp[iB^a_{n\mu}(\phi)X^a]=V_nU_{n\mu}V^\dagger_{n+\hat\mu} \tag{2.20}$$

and

$$s^a_n(\phi)=\sum_\mu\Delta_\mu B^a_{n-\hat\mu,\mu}(\phi)\,. \tag{2.21}$$

Note that $\sum_n s^a_n=\sum_n s^a_n(\phi)=0$; hence there are only (N^4-1)-independent variables for the s^a_n. Let $\sum'_n\equiv\sum_{n\neq N}$. Then from (2.12)

$$e^{A_\alpha+A_c}=\prod_{n,a}{}'\,\delta(s^a_n-t^a_n)\Big/\int dV\prod_{n,a}{}'\,\delta(s^a_n(\phi)-t^a_n)$$
$$=\prod_{n,a}{}'\,\delta(s^a_n-t^a_n)\Big/\int dV\prod_{n,a}{}'\,\delta(s^a_n(\phi)-s^a_n)\,. \tag{2.22}$$

Note that in taking the step to (2.22) we have lost gauge invariance for e^{A_c}, since it now depends on gauge transformations through the variable s^a_n. However, the *combined* effect of $e^{A_\alpha+A_c}$ is to leave the gauge-invariant sector unchanged. (We will return to this point later.)

From (2.22) we see that e^{A_c} is independent of $\{t^a_n\}$. Recall from (2.17) that

$$Z(g_0)=\int dU\,e^{A+A_\alpha+A_c}=\int dUe^A\,, \tag{2.23}$$

i.e., $Z(g_0)$ is independent of $\{t^a_n\}$. Therefore[4]

$$Z(g_0)=(\text{const})\cdot\prod_{n,a}{}'\int_{-\infty}^{+\infty}dt^a_n\exp\left[-\frac{\alpha}{2}(t^a_n)^2\right]Z(g_0)$$
$$=\int dU\,e^{A+A_c}\prod_{n,a}{}'\int_{-\infty}^{+\infty}dt^a_n\exp\left[-\frac{\alpha}{2}(t^a_n)^2\right]\delta(s^a_n-t^a_n)$$
$$=\int dU\,e^A\exp\left[-\frac{\alpha}{2}\sum_{n,a}{}'(s^a_n)^2\right]\Big/\int dV\prod_{n,a}{}'\,\delta(s^a_n(\phi)-s^a_n)\,. \tag{2.24}$$

Equation (2.24) is the final form for A_α and A_c which we will use for computations. We show in Appendix B that the combined effect of $e^{A_\alpha + A_c}$ in fact leaves the gauge-invariant sector unchanged. e^{A_c} is no longer gauge invariant, but $e^{A_\alpha+A_c}$ has a lower symmetry, which is the Slavnov symmetry (see Sec. II C).

Let $\alpha = O(1/g_0^2)$; then the modified action $A' = A + A_\alpha + A_c$ restricts *all* the variables (except $B^a_{N\mu}$) to be $O(g_0)$. We look only at regions for which $B^a_{N\mu} = O(g_0)$ and hence have for all n, μ

$$B^a_{N\mu} = O(g_0). \tag{2.25}$$

What we mean by (2.25) is that in evaluating the path integral of $e^{A'}$, only those regions of the phase space contribute to the path integral for which $B^a_{n\mu} = O(g_0)$. In other words, in this gauge the path integral is evaluated over those points of Ω which are a distance $\leq g_0$ from the origin. Equation (2.25) can be derived from the results of Sec. III.

In summary, from (2.24) we have

$$A' = A - \frac{\alpha}{2}\sum_{n,a}{}'(s^a_n)^2 + A_c\,, \tag{2.26}$$

where

$$e^{-A_c} = \int dV \prod_{n,a}{}' \delta(s^a_n(\phi) - s^a_n)\,. \tag{2.27}$$

We now evaluate $A_c[B]$ to $O(g_0^2)$. For this, we need $B^a_{n\mu}(\phi)$ up to terms linear in ϕ^a_n and quadratic in $B^a_{n\mu}$. We computed $B^a_{n\mu}(\phi)$ to $O(g_0^2)$ in (1.13); the only two terms missing there are of order $B_{n\mu}{}^2\phi_n$ and $B_{n\mu}{}^2\phi_{n+\hat\mu}$. Since there is no mixing of ϕ_n and $\phi_{n+\hat\mu}$, we can set one of them equal to zero and compute for the other. Using the equation

$$\exp(A)\exp(B) = \exp\{A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}[[A,B],B] - \tfrac{1}{12}[[A,B],A] + \cdots\}, \tag{2.28}$$

we have, for $\phi^a_{n+\hat\mu} = 0$,

$$B^a_{n\mu}(\phi) = \text{lower order} + \tfrac{1}{12}C^{abe}C^{cde}B^b_{n\mu}B^c_{n\mu}\phi^d_n\,.$$

We can similarly do the calculation setting $\phi^a_n = 0$, and from these results and (1.11) we have

$$B^a_{n\mu}(\phi) = B^a_{n\mu} - \Delta_\mu\phi^a_n - \tfrac{1}{2}C^{abc}(\phi^b_n + \phi^b_{n+\hat\mu})B^c_{n\mu} + \tfrac{1}{12}C^{abe}C^{cde}B^b_{n\mu}B^c_{n\mu}(\phi^d_n - \phi^d_{n+\hat\mu}) + O(\phi^2, B^3\phi)\,. \tag{2.29}$$

Define $u^a_n = u^a_n(\phi)$ by

$$s^a_n(\phi) = u^a_n + s^a_n\,. \tag{2.30}$$

Then from (2.27)

$$e^{-A_c[B]} = \int dV \prod_{n,a}{}' \delta(s^a_n(\phi) - s^a_n) = \int dV \prod_{n,a}{}' \delta(u^a_n)\,. \tag{2.31}$$

We will now make a change of variable from $\{\phi^a_n\}$ to $\{u^a_n\}$ to evaluate (2.31). The δ functions make $u^a_n(\phi) = 0$; this in turn implies $\phi^a_n = 0$ as the unique solution for which $u^a_n = 0$ (as long as $B^a_{n\mu} \ll 1$). We analyze the variable $u^a_n = u^a_n(\phi)$. To do this, we define the Fourier transform of the variables. Let h_n by any arbitrary function of n. Owing to the torus structure of the lattice, we have $h_{n+N\hat\mu} = h_n$: periodic in all the coordinates with period N. Hence h_n can be expanded in terms of the basis functions $\{e^{ik_\mu n_\mu}\}, k_\mu = 0, 2\pi/N, \ldots, (2\pi/N)(N-1)$. That is,

$$h_n = \sum_n e^{ikn}h_k \equiv \frac{1}{N^4}\left(\prod_\mu \sum_{k_\mu=0}^{2\pi(N-1)/N} e^{ik_\mu n_\mu}\right)h_k\,,$$

$$h_k = \sum_n e^{-ikn}h_n\,, \quad \delta_{k,q} \equiv N^4\prod_{i=0}^{3}\delta_{k_i,q_i}\,.$$

Let

$$u^a_n = \sum_k e^{ikn}u^a_k\,, \quad B^a_{n\mu} = \sum_k e^{ikn}B^a_{k\mu}\,,$$

$$\phi^a_n = \sum_k e^{ikn}\phi^a_k\,.$$

Then, from (2.29) and (2.30),

$$u^a_k = \sum_n e^{-ik\cdot n}u^a_n\,.$$

Using (2.29) gives

$$u^a_k = \sum_\mu |1 - e^{ik_\mu}|^2\phi^a_k + \tfrac{1}{2}C^{abc}\sum_{q,\mu}(1 - e^{-ik_\mu})(1 + e^{iq_\mu})B^b_{k-q,\mu}\phi^c_q + \tfrac{1}{12}C^{abe}C^{cde}\sum_{k,q}\sum_\mu(1 - e^{-ik_\mu})(1 - e^{iq_\mu})B^b_{k-k'-q,\mu}B^c_{k',\mu}\phi^d_q\,. \tag{2.32}$$

Note from (2.32) that $u^a_{k=0} = 0$; i.e., it is not coupled to the ϕ^a_n. We can hence redefine $u^a_{k=0}$ to be

$$u^a_{k=0} = \phi^a_{k=0}\,. \tag{2.33}$$

Then, from (2.32) and (2.33),

$$u^a_k = d_k\phi^a_k + \sum_q (M^{ad}(k,q) + L^{ad}(k,q))\phi^d_q\,, \tag{2.34}$$

where

$$d_k=\begin{cases}1 & \text{if } k=0\,,\\ \sum_\mu |1-e^{ik_\mu}|^2 & \text{if } k\neq 0\,,\end{cases} \tag{2.35}$$

$$M^{ad}(k,q)=\tfrac{1}{2}C^{abd}\sum_\mu (1-e^{-ik_\mu})(1+e^{iq_\mu})B^b_{k-q,\mu}\,, \tag{2.36}$$

$$L^{ad}(k,q)=\tfrac{1}{12}C^{abe}C^{cde}\sum_{k'}\sum_\mu (1-e^{-ik_\mu})(1-e^{iq_\mu})\times B^b_{k-k'-q,\mu}B^c_{k',\mu}\,. \tag{2.37}$$

Let

$$T^{ab}_{k,q}=d_k\delta^{ab}\delta_{k,q}+M^{ab}(k,q)+L^{ab}(k,q) \tag{2.38}$$

$$=d_k\left[\delta^{ab}\delta_{k,q}+\frac{1}{d_k}M^{ab}(k,q)+\frac{1}{d_k}L^{ab}(k,q)\right]. \tag{2.39}$$

From (2.34), making a change of variable from $\{\phi^a_k\}$ to $\{u^a_k\}$ gives

$$du^a_k=\sum_{q,b}T^{ab}_{k,q}\,d\phi^b_q \tag{2.40}$$

and

$$\prod_k\prod_a du^a_k=\det(T)\prod_{q,a}d\phi^b_q\,. \tag{2.41}$$

Hence, from (2.31),

$$\begin{aligned}e^{-A_c}&=\int dV\prod_{n,a}{}'\,\delta(u^a_n)\\ &=\prod_{n\neq N}\int dV_n\prod_{n,a}{}'\,\delta(u^a_n)\\ &=\prod_n\int dV_n\prod_a\delta(\phi^a_N)\prod_{n,a}{}'\,\delta(u^a_n)\\ &=\prod_{n,a}\int d\phi^a_n\mu(\phi^a_n)\prod_a\delta(\phi^a_N)\prod_{n,a}{}'\,\delta(u^a_n)\,.\end{aligned} \tag{2.42}$$

The integrand fixes $\phi^a_n=0$, and $\mu(\phi^a_n=0)=\text{const}$. Also,

$$\prod_{n,a}{}'\,\delta(u^a_n)=\prod_{k\neq 0}\prod_a\delta(u^a_k)\,,$$

giving

$$\begin{aligned}e^{-A_c}&=\prod_k\prod_a\int\frac{1}{\det T}du^a_k\prod_{k\neq 0}\prod_a\delta(u^a_k)\prod_a\delta\Big(\sum_k\phi^a_k\Big)\\ &=\frac{1}{\det T}\Big[\prod_{k\neq 0}\prod_a\int du^a_k\delta(u^a_k)\Big]\\ &\quad\times\Big[\int\prod_a du^a_{k=0}\delta\Big(u^a_{k=0}+\sum_k{}'\,\phi^a_k\Big)\Big]\\ &=\frac{1}{\det T}\,.\end{aligned} \tag{2.43}$$

Therefore,

$$e^{A_c[B]}=\det T=\det\left(d\left(1+\frac{1}{d}M+\frac{1}{d}L\right)\right), \tag{2.44}$$

where we have used (2.39) to obtain (2.44) and we are using simplified notation. Using property $\det(AB)=\det A\det B$, and that d_k is independent of the gauge field $\{B^a_{n\mu}\}$ gives

$$\begin{aligned}e^{A_c[B]}&=(\text{const})\times\det\left(1+\frac{1}{d}M+\frac{1}{d}L\right)\\ &=\exp\left[\operatorname{Tr}\ln\left(1+\frac{1}{d}M+\frac{1}{d}L\right)\right]\\ &=(\text{const})\times\det\left(d\left(1+\frac{1}{d}M\right)\right)\exp\left[\operatorname{Tr}\left(\frac{1}{d}L\right)\right],\end{aligned} \tag{2.45}$$

where the overall constant is independent of the gauge field. We evaluate

$$\begin{aligned}\operatorname{Tr}\left(\frac{1}{d}L\right)&=\sum_a\sum_k\frac{1}{d_k}L^{aa}(k,k)\\ &=-\frac{n}{12}\sum_k\sum_a\frac{1}{d_k}\sum_\mu|1-e^{ik_\mu}|^2\times\sum_{k'}B^a_{-k'\mu}B^a_{k'\mu}\,,\end{aligned} \tag{2.46}$$

where we have used

$$C^{abc}C^{\alpha bc}=n\delta^{a\alpha}\,. \tag{2.47}$$

Using the fact that

$$\sum_k\frac{1}{d_k}|1-e^{ik_\mu}|^2=\tfrac{1}{4} \tag{2.48}$$

gives

$$\begin{aligned}\operatorname{Tr}\left(\frac{1}{d}L\right)&=-\frac{n}{48}\sum_k\sum_{a,\mu}B^a_{-k\mu}B^a_{k\mu}\\ &=-\frac{n}{48}\sum_n\sum_{\mu,a}(B^a_{n\mu})^2\,.\end{aligned} \tag{2.49}$$

Note that $\operatorname{Tr}[(1/d)L]$ is completely local. Hence, we conclude from (2.45) and (2.49) that

$$e^{A_c[B]}=\det(d_k\delta^{ab}\delta_{k,q}+M^{ab}(k,q))\times\exp\left[-\frac{n}{48}\sum_{n\mu a}(B^a_{n\mu})^2\right]+O(g_0^3)\,. \tag{2.50}$$

This is the final answer. The determinant in the expression for $A_c[B]$ can be represented by a fermion integration; it is this term which is called the Faddeev-Popov ghost term. However, the extra local term $\sum_{n\mu a}(B^a_{n\mu})^2$ is absent in the continuum formulation. This term is quadratically divergent (we will show this in Sec. III) and plays an important role in ensuring that there is no mass renormalization necessary for the lattice gauge field. We will return to (2.50) in Sec. III.

We note in passing that choosing the axial gauge and using a gauge-fixing term are both ways of choosing a gauge for the gauge field. The only difference is that in choosing the axial gauge there is no counterterm, whereas using A_α for gauge fixing introduces a nontrivial counterterm. However, from a practical point of view, the two ways of choosing a gauge are vastly different. In contrast to the axial gauge, gauge fixing using A_α allows us to treat *all* the field variables on an equal footing, and hence allows the systematic use of perturbation theory.

C. Slavnov identity

Recall that in the last section we proved that

$$Z=\int dU\,e^{A}=\int dU\,e^{A+A_\alpha+A_c}\,.$$

We also had computed $e^{A_c}=\det(T^{ab}(k,q))$ to $O(g_0^{\,2})$. Note that A_α of necessity breaks gauge invariance; also, our definition of e^{A_c} is not gauge invariant. However, the term $A_\alpha+A_c$ is invariant under the Slavnov transformation,[5] which we will define in this section. This invariance is more restricted than gauge invariance, but its usefulness lies in that it holds for the gauge theory in the presence of gauge fixing.

To define the Slavnov transformation, we first rewrite e^{A_c} in a more formal way. From (2.27) (for an infinite-size lattice)

$$e^{-A_c}=\int dV\prod_{n,a}\delta(s_n^a(\phi)-s_n^a)\,. \tag{2.51}$$

The value of $\{\phi_n^a\}$ for which the δ functions are satisfied is $\phi_n^a=0$. For an infinite lattice

$$dV=\prod_{n,a}d\phi_n^a\,\mu(\phi_n)\to\prod_n\mu(\phi_n=0)\prod_{n,a}d\phi_n^a\,. \tag{2.52}$$

Therefore

$$e^{-A_c}=(\text{const})\prod_{n,a}\int d\phi_n^a\,\delta(s_n^a(\phi)-s_n^a)\,. \tag{2.53}$$

We make the change of variable from $\{\phi_n^a\}$ to $\{\Phi_n^a\}$ defined by

$$\Phi_n^a=s_n^a(\phi)-s_n^a\,. \tag{2.54}$$

In evaluating the Jacobian of the transformation, the δ functions make us evaluate this at $\phi_n^a=0$, i.e.,

$$d\Phi_n^a=\left.\frac{\partial s_n^a(\phi)}{\partial\phi_m^b}\right|_{\phi=0}d\phi_m^b \tag{2.55}$$

$$\equiv\frac{\partial s_n^a}{\partial\phi_m^b}\,d\phi_m^b \tag{2.56}$$

(all repeated indices to be summed over). Hence

$$\prod_{n,a}d\Phi_n^a=\det\left(\frac{\partial s_n^a}{\partial\phi_m^b}\right)\prod_{n,a}d\phi_n^a \tag{2.57}$$

and

$$\begin{aligned}e^{A_c}&=1\Big/\int\prod_{n,a}d\phi_n^a\delta(s_n^a(\phi)-s_n^a)\\&=\det\left(\frac{\partial s_n^a}{\partial\phi_m^b}\right)\Big/\prod_{n,a}\int d\Phi_n^a\,\delta(\Phi_n^a)\\&=\det(\partial s_n^a/\partial\phi_m^b)\,.\end{aligned} \tag{2.58}$$

To define the Slavnov transformation, we have to represent the determinant e^{A_c} using fermion integration (this is discussed in Refs. 4 and 7). Let c_n^a, $c_n^{\dagger a}$ be scalar fermion fields, and let $\langle\;\rangle$ denote

$$\prod_{n,a}\int dc_n^a\,dc_n^{\dagger a}\,.$$

Then

$$e^{A_c}=\det\left(\frac{\partial s_m^b}{\partial\phi_n^a}\right)=\left\langle\exp\left(c_n^{\dagger a}\frac{\partial s_m^b}{\partial\phi_n^a}c_m^b\right)\right\rangle. \tag{2.59}$$

Hence we have

$$A_\alpha=-\tfrac{1}{2}\alpha s_n^a s_n^a\,, \tag{2.60}$$

$$A_c=c_n^{\dagger a}(\partial s_m^b/\partial\phi_n^a)c_m^b\,. \tag{2.61}$$

Let λ be a spacetime-independent fermion variable which anticommutes with other fermion variables and commutes with bosons. We adopt the notation that $(\partial h_n^a(\phi)/\partial\phi_m^b|_{\phi=0}\equiv\partial h_n^a/\partial\phi_m^b$; let c^{abc} be the structure constants. Then the Slavnov transformation[5] is defined by

$$B_{n\mu}^a\to B_{n\mu}^a+\lambda\frac{\partial B_{n\mu}^a}{\partial\phi_m^b}c_m^{\dagger b}\,, \tag{2.62}$$

$$c_n^a\to c_n^a-\alpha\lambda s_n^a\,, \tag{2.63}$$

$$c_n^{\dagger a}\to c_n^{\dagger a}+\tfrac{1}{2}\lambda c^{abc}c_n^{\dagger b}c_n^{\dagger c}\,. \tag{2.64}$$

From (2.62) we have

$$s_n^a\to s_n^a+\lambda\frac{\partial s_n^a}{\partial\phi_m^b}c_m^{\dagger b}\,, \tag{2.65}$$

$$\partial s_n^a/\partial\phi_m^b\to\partial s_n^a/\partial\phi_m^b+\lambda(\partial^2 s_n^a/\partial\phi_m^b\partial\phi_l^\gamma)c_l^{\dagger\gamma}\,. \tag{2.66}$$

We now examine the effect of this transformation. The gauge field action A is left unchanged since it is gauge invariant, and (2.62) is a linearized gauge transformation. For A_α we have

$$A_\alpha \to -\frac{\alpha}{2}\left(s_n^a + \lambda \frac{\partial s_n^a}{\partial \phi_m^b} c_m^{\dagger b}\right)^2$$
$$= A_\alpha - \alpha\lambda s_n^a (\partial s_n^a/\partial \phi_m^b) c_m^{\dagger b} \tag{2.67}$$

and for the counterterm

$$A_c \to \left(c_n^{\dagger a} + \frac{\lambda}{2} c^{a\beta\gamma} c_n^{\dagger \beta} c_n^{\dagger \gamma}\right)\left(\frac{\partial s_m^b}{\partial \phi_n^a} + \lambda \frac{\partial^2 s_m^b}{\partial \phi_n^a \partial \phi_l^c} c_l^{\dagger c}\right)$$
$$\times (c_m^b - \alpha\lambda s_m^b).$$

After simplifications using anticommutation of fermion variables, we have

$$A_c \to A_c + \alpha\lambda c_n^{\dagger a} \frac{\partial s_m^b}{\partial \phi_n^a} s_m^b + \frac{\lambda}{2} c^{a\beta\gamma} \frac{\partial s_m^b}{\partial \phi_n^a} c_n^{\dagger \beta} c_n^{\dagger\gamma} c_m^b$$
$$- \lambda \frac{\partial^2 s_m^b}{\partial \phi_n^a \partial \phi^c} c_n^{\dagger a} c_l^{\dagger c} c_m^b. \tag{2.68}$$

Therefore, from (2.67) and (2.68), we have

$$A_\alpha + A_c \to A_\alpha + A_c + \lambda\left(\tfrac{1}{2} c^{a\beta\gamma} \frac{\partial s_m^b}{\partial \phi_n^a} c_n^{\dagger\beta} c_n^{\dagger\gamma} c_m^b - \frac{\partial^2 s_m^b}{\partial \phi_n^a \partial \phi_l^c} c_n^{\dagger a} c_l^{\dagger c} c_m^b\right). \tag{2.69}$$

Note that the term in parentheses is zero since

$$\frac{\partial^2 s_m^b}{\partial \phi_n^a \partial \phi_l^c} c_n^{\dagger a} c_l^{\dagger c} c_m^b = \frac{1}{2}\frac{\partial^2 s_m^b}{\partial \phi_n^a \partial \phi_l^c} c_n^{\dagger a} c_l^{\dagger c} + \frac{\partial^2 s_m^b}{\partial \phi_l^c \partial \phi_n^a} c_l^{\dagger c} c_n^{\dagger a} c_m^b$$
$$= \frac{1}{2}\left[\frac{\partial}{\partial \phi_n^a}, \frac{\partial}{\partial \phi_l^c}\right] s_m^b c_n^{\dagger a} c_l^{\dagger c} c_m^b \tag{2.70}$$

and[5]

$$\left[\frac{\partial}{\partial \phi_n^a}, \frac{\partial}{\partial \phi_m^b}\right] = \delta_{n,m} c^{abc} \frac{\partial}{\partial \phi_n^c}. \tag{2.71}$$

Therefore, the term in (2.70) cancels the other term in parentheses of (2.69), giving

$$A_\alpha + A_c \to A_\alpha + A_c: \text{ invariant.} \tag{2.72}$$

Hence we have proved that $A + A_\alpha + A_c$ is invariant under the Slavnov transformation. In the next section we will use the invariance to show formally that the gluon has zero mass renormalization.

III. MASS RENORMALIZATION

We know from general considerations that mass renormalization for the gauge field is incompatible with local gauge invariance—since any mass counterterms in the Lagrangian would violate gauge invariance. Hence, for the renormalized theory to be gauge invariant, all the quadratic mass divergences in the theory must exactly cancel. From asymptotic freedom, we know that we have to study the lattice theory for $g_0 \to 0$ to ascertain the high-momentum behavior of the quantum theory, i.e., the behavior for a (lattice spacing) $\to 0$ (see Ref. 6).

In particular, we will study the $B_{n\mu}^a$ field propagator in the weak-coupling limit, and we will show by calculation that to lowest order the proper self-energy of the gauge field quantum for zero momentum is zero. This will show that there is no mass renormalization for it. We will then prove this same result more formally by making use of the Slavnov identity.

Owing to the infrared instability of the non-Abelian gauge field, it is in general not possible to compute the behavior of the zeroth mode without solving the large-distance strong-coupling problem. The same is true for the lattice theory *provided* that there is no quadratic divergence arising from a nonzero mass renormalization term. However, if there is a quadratic divergence in the theory, then this would destroy asymptotic freedom; the divergence would completely dominate the e^{-1/g_0^2} effects arising from the high-momentum modes due to coupling-constant renormalization, etc.; and we could compute this divergence using the weak-coupling approximation for the zeroth mode propagator. Hence, we *assume* that there is a quadratic divergence, and compute it using weak coupling for the zeroth mode. We will then show that the divergence is in fact absent. The calculation is self-consistent, since if there were a quadratic divergence our calculation would determine it.

We now discuss the main features of the calculation before going into the details. Define the (global) color-singlet propagator

$$D_{n\mu\nu} = \int dU\, B_{n\mu}^a B_{0\nu}^a e^{A+A_\alpha+A_c}/Z, \tag{3.1}$$

$$D_{k\mu\nu} = \sum_n e^{-ik\cdot n} D_{n\mu\nu}. \tag{3.2}$$

Using translational invariance (due to the periodic lattice) gives

$$D_{k\mu\nu} = \frac{1}{N^4}\int dU\, B_{-k\mu}^a B_{k\nu}^a e^{A'}/Z. \tag{3.3}$$

Let $D_{k\mu\nu}^{(0)}$ be the bare propagator defined by the quadratic part of A'; let $\Pi_{\mu\nu}(k)$ be the proper self-energy. Then, in matrix notation, Dyson's equation states that

$$D_k = D_k^{(0)} + D_k^{(0)} \Pi(k) D_k. \tag{3.4}$$

Recall from (1.6b) that $B_{n\mu}^\alpha = a g_0 s A_{n\mu}^\alpha$ is dimensionless, making $\Pi(k)$ dimensionless in (3.4). Hence, the continuum self-energy, which has the dimension of

$(\text{mass})^2$, is given by dimensional analysis. Since the only dimensional quantity in the entire theory is the lattice spacing a, we have

$$\Pi^{\text{phy}}(p) = \frac{1}{a^2}\Pi(k=pa) \tag{3.5}$$

$$= \frac{1}{a^2}\{\Pi(0) + [\Pi(pa) - \Pi(0)]\}. \tag{3.6}$$

It can be shown using perturbation theory that $(p \neq 0)$

$$\lim_{a\to 0}\frac{1}{a^2}[\Pi(pa) - \Pi(0)] \sim \text{logarithmic divergences in } a. \tag{3.7}$$

Hence, in the $a \to 0$ limit,

$$\Pi^{\text{phy}}(p) = \frac{1}{a^2}\Pi(0) + \text{logarithmic divergences in } a. \tag{3.8}$$

We conclude that for there to be no mass renormalization, the quadratic divergence $(1/a^2)\Pi(0)$ must be zero, i.e.,

$$\Pi(0) = 0. \tag{3.9}$$

The logarithmic divergences in a are taken care of by wave-function renormalization.

When $g_0 \to 0$, we have an expansion

$$\Pi(0) = \Pi_0 + \Pi_1 g_0 + \Pi_2 g_0{}^2 + \cdots \tag{3.10}$$

In our lowest-order calculation, we will show that $\Pi_0 = 0$. The general result that $\Pi(0) = 0$ is proved by the Slavnov identity. From (3.4)

$$D_k = \frac{1}{D_k^{(0)-1} - \Pi(k)}. \tag{3.11}$$

It will be true that, for $N \to \infty$, $k \to 0$, $D_{k=0}^{(0)-1} \to 0$; hence

$$D_{k=0} = -\frac{1}{\Pi(0)}. \tag{3.12}$$

In order to evaluate $\Pi(0)$, we will evaluate

$$D \equiv D_{k=0,\mu\mu} = \frac{1}{N^4}\int dU\, B^a_{k=0,\mu} B^a_{k=0,\mu} e^{A'}/Z. \tag{3.13}$$

To calculate D, we will first perform an integration over all $\{B^a_{k\mu}, k \neq 0\}$ in the path integral; this will leave us with an effective action involving only the $B^a_{k=0,\mu}$ variables, and will provide us with $\Pi(0)$. In the following we will analyze $Z(g_0)$ and then show how to extract $\Pi(0)$.

A. The weak-coupling action

Recall from (2.9) that with a change of notation $B^a_{n\mu} \to \tilde{B}^a_{n\mu}$

$$Z = \prod_{n\neq N}\prod_{\mu,a}\int_{-\infty}^{+\infty} d\tilde{B}^a_{n\mu}\,\mu(\tilde{B}^a_{n\mu})\prod_\alpha\int d\tilde{U}_{N\alpha} e^{A+A_\alpha+A_c}. \tag{3.14}$$

As we discuss in Appendix B, $d\tilde{U}_{N\alpha}$ could not be treated like the other variables since there is no Gaussian factor for it in the integrand. When we Fourier transform the $\{\tilde{B}^a_{n\mu}\}$, we see that the role of $U_{N\alpha}$ is taken by the variable $B^a_{k=0,\mu}$, since there is no Gaussian factor for it either. (This can be easily seen later.) We also have to isolate this zeroth mode in the action, since we are interested in integrating out all the other modes. We do this as follows (the original variables are $\tilde{B}^a_{n\mu} = \sum_k e^{ikn}B^a_{k\mu}$):

$$\tilde{B}^a_{n\mu} = B^a_{k=0,\mu}/N^4 + \sum_k{}' e^{ikn}B^a_{k\mu}\left(\sum_k{}' \equiv \sum_{k\neq 0}{}'\right). \tag{3.15}$$

Define

$$\theta^a_\mu = B^a_{k=0,\mu}/N^4, \tag{3.16}$$

$$B^a_{n\mu} = \sum_k{}' e^{ikn}B^a_{k\mu}. \tag{3.17}$$

In the presence of $A_\alpha + A_c$, we have (for $\alpha = 1/2g_0{}^2$)

$$B^a_{n\mu} = O(g_0), \tag{3.18}$$

$$\theta^a_\mu = O(1). \tag{3.19}$$

Therefore, since $B^a_{N\mu} = \theta^a_\mu + \sum_k' B^a_{k\mu} = \theta^a_\mu + O(g_0)$,

$$\tilde{U}_{N\mu} = e^{i\tilde{B}^a_{N\mu}X^a} = e^{i\theta^a_\mu X^a} + O(g_0) \tag{3.20}$$

and

$$d\tilde{U}_{N\mu} = dU_\mu + O(g_0), \tag{3.21}$$

where $U_\mu = e^{i\theta^a_\mu X^a}$.

Also,

$$U_{n\mu} = \exp[i(B^a_{n\mu} + \theta^a_\mu)X^a] \equiv U_\mu(1 + A_{n\mu}). \tag{3.22}$$

$A_{n\mu}$ is a matrix of $O(g_0)$. Let

$$A_{n\mu} = A^{(0)}_{n\mu}1 + iA^a_{n\mu}X^a \tag{3.23}$$

with

$$\text{Tr}(X^aX^b) = \tfrac{1}{2}\delta^{ab}, \quad \text{Tr}(1) = n. \tag{3.24}$$

[The gauge group is SU(n).] Then

$$A = \frac{1}{2g_0{}^2}\sum_n\sum_{\mu\nu}{}' \text{Tr}(\tilde{U}_{n\mu}\tilde{U}_{n+\hat{\mu},\nu}\tilde{U}^\dagger_{n+\hat{\nu},\mu}\tilde{U}^\dagger_{n\nu}),$$

and, using (3.22), (3.23) gives a complicated expression involving the U_μ and $A_{n\mu}$. As already discussed, the entire calculation is based on the assumption that there is a quadratic mass divergence. This means that we are interested only in the coefficient of the $\theta_\mu^a\theta_\mu^a$ term; if there is a quadratic divergence, then all the higher powers of θ_μ^a will be negligibly small. Hence, in the action, we keep only the terms for θ_μ^a which are at most quadratic. Secondly, we are doing the calculation to lowest order in g_0, i.e., to $O(1)$; this means that we will keep at most terms which are quadratic in the $B_{n\mu}^a$. We will show that $A_{n\mu}^{(0)}$ is of $O(g_0^2)$ and $A_{n\mu}^a$ is of $O(g_0)$; hence we keep terms linear in $A_{n\mu}^{(0)}$ and quadratic in $A_{n\mu}^a$. Note also that if the above approximations are consistently used for the action, then a simple bookkeeping rule is to consider all the U_μ as commuting. (If one goes to higher order in θ_μ^a or $B_{n\mu}^a$, then this bookkeeping method is no longer valid.) To summarize, we use

$$[U_\mu, U_\nu]=0+O(\theta^3)\,, \tag{3.25}$$

where $O(\theta^3)$ means the order of the terms generated in the action by the approximation in (3.25). Hence, the action is

$$A=\frac{1}{2g_0^2}\sum_{n\mu\nu}{}'\,\mathrm{Tr}[(U_\mu U_\nu+U_\mu A_{n\mu}U_\nu+U_\mu U_\nu A_{n+\hat\mu,\nu}+U_\mu A_{n\mu}U_\nu A_{n+\hat\mu,\nu})$$
$$\times(U_\mu^\dagger U_\nu^\dagger+U_\mu^\dagger A_{n\nu}^\dagger U_\nu^\dagger+A_{n+\hat\nu,\mu}^\dagger U_\mu^\dagger U_\nu^\dagger+A_{n+\hat\nu,\mu}^\dagger U_\mu^\dagger A_{n\nu}^\dagger U_\nu^\dagger)] \tag{3.26}$$

$$=\frac{N^4}{2g_0^2}\sum_{\mu\nu}{}'\,\mathrm{Tr}(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)+\frac{1}{2g_0^2}\sum_n\sum_{\mu\nu}{}'\,[4nA_{n\mu}^{(0)}+2A_{n+\nu,\mu}^a A_{n\mu}^b\,\mathrm{Tr}(U_\nu X^a U_\nu^\dagger X^b)+A_{n\mu}^a A_{n\nu}^b\,\mathrm{Tr}(U_\nu^\dagger U_\mu X^a U_\nu U_\mu^\dagger X^b)$$
$$+A_{n+\hat\mu,\nu}^a A_{n+\hat\nu,\mu}^b\,\mathrm{Tr}(X^a X^b)-2A_{n+\hat\nu,\mu}^a A_{n\nu}^b\,\mathrm{Tr}(U_\mu X^a U_\mu^\dagger X^b)]+O(\theta^3,B^3)\,. \tag{3.27}$$

In studying the above action, we will consider it as a polynomial in θ_μ^a and $B_{n\mu}^a$ and, as already pointed out, keep at most terms of $O(\theta^2 B^2)$. We use the notation

$$a\circ b=a^\alpha b^\alpha\,,\quad (a\times b)^\alpha=c^{\alpha\beta\gamma}a^\beta b^\gamma\,.$$

Then, from (3.22) and (3.23),

$$nA_{n\mu}^{(0)}=-\tfrac14[B_{n\mu}{}^2-\tfrac{1}{12}(\theta_\mu\times B_{n\mu})^2]+O(g_0{}^3)\,, \tag{3.28}$$

$$A_{n\mu}=B_{n\mu}+\tfrac12\theta_\mu\times B_{n\mu}+\tfrac16\,\theta_\mu\times(\theta_\mu\times B_{n\mu})\,. \tag{3.29}$$

Therefore

$$A_{n\mu}\circ A_{m\mu}=B_{n\mu}\cdot B_{m\mu}-\tfrac{1}{12}(B_{n\mu}\times\theta_\mu)\cdot(B_{m\mu}\times\theta_\mu)\,, \tag{3.30}$$

$$A_{n\mu}\circ A_{m\nu}=B_{n\mu}\cdot B_{m\nu}+\tfrac12(B_{n\mu}\times B_{m\nu})\cdot(\theta_\mu-\theta_\nu)\,, \tag{3.31}$$

$$A_{n\mu}\times A_{m\nu}=B_{n\mu}\times B_{m\nu}+\tfrac12(\theta_\mu\times B_{n\mu})\times B_{m\nu}+\tfrac12 B_{n\mu}\times(\theta_\nu\times B_{m\nu})\,. \tag{3.32}$$

We need the matrix

$$G_\mu^{ab}=\mathrm{Tr}(U_\mu X^a U_\mu^\dagger X^b)=\tfrac12\delta^{ab}-\tfrac12 C^{ab\alpha}\theta_\mu^\alpha-\tfrac14 C^{a\alpha\gamma}C^{b\beta\gamma}\theta_\mu^\alpha\theta_\mu^\beta\,. \tag{3.33}$$

Therefore

$$A_{n\nu}^a G_\mu^{ab}A_{m\pi}^b=\tfrac12[A_{n\nu}\cdot A_{m\pi}-(A_{n\nu}\times A_{m\pi})\circ\theta_\mu-\tfrac12(A_{n\nu}\times\theta_\mu)\circ(A_{m\pi}\times\theta_\mu)]\,. \tag{3.34}$$

Collecting (3.28)–(3.34) and simplifying the action gives, from (3.25) and (3.27),

$$A=A_U+\frac{1}{2g_0^2}\sum_n\sum_{\mu\nu}{}'\{4(-\tfrac14)[B_{n\mu}{}^2-\tfrac1{12}(\theta_\mu\times B_{n\mu})^2]+2(\tfrac12)[A_{n\mu}\cdot A_{n+\hat\nu,\mu}+(A_{n\mu}\times A_{n+\hat\nu,\mu})\cdot\theta_\nu-\tfrac12(A_{n\mu}\times\theta_\nu)\cdot(A_{n+\hat\nu,\mu}\times\theta_\nu)]$$
$$+\tfrac12[A_{n\mu}\cdot A_{n\nu}-(A_{n\mu}\times A_{n\nu})\cdot(\theta_\mu-\theta_\nu)-\tfrac12(A_{n\mu}\times(\theta_\mu-\theta_\nu))\cdot(A_{n\nu}\times(\theta_\mu-\theta_\nu))]$$
$$+\tfrac12 A_{n+\hat\mu,\nu}\circ A_{n+\hat\nu,\mu}-2(\tfrac12)[A_{n+\hat\nu,\mu}\cdot A_{n\nu}-(A_{n+\hat\nu,\mu}\times A_{n\nu})\cdot\theta_\mu-\tfrac12(A_{n+\hat\nu,\mu}\times\theta_\mu)\cdot(A_{n\nu}\times\theta_\nu)]\}\,. \tag{3.35}$$

We break up the action as a polynomial in θ_μ^a and write

$$A=A_U+A_0+A_I^{(1)}+A_I^{(2)}\,, \tag{3.36}$$

where, after considerable simplification,

$$A_0=-\frac{1}{4g_0^2}\sum_n\sum_{\mu\nu}[(\Delta_\nu B_{n\mu})^2-\Delta_\mu B_{n\nu}\Delta_\nu B_{n\mu}]\,, \tag{3.37}$$

$$A_I^{(1)} = \frac{C^{abc}}{4g_0^2}\sum_n\sum_{\mu\nu}\theta_\nu^a(\Delta_\mu B_{n\nu}^b - \Delta_\nu B_{n\mu}^b)(B_{n+\hat\nu,\mu}^c + B_{n\mu}^c). \tag{3.38}$$

(We have replaced $\sum_{\mu\nu}'$ by $\sum_{\mu\nu}$, using antisymmetry of the summand.) Note that the general structure of $A_I^{(2)}$ is

$$A_I^{(2)} = \frac{1}{4g_0^2}\sum_{m,n}\sum_{\mu\nu}\sum_{ab} B_{n\mu}^a M_{\mu\nu}^{ab} B_{m\nu}^b. \tag{3.39}$$

In the final calculation, we will keep only terms of $O(\theta^2)$ in performing the $B_{n\mu}^a$ integrations. Since $M_{\mu\nu}^{ab} = O(\theta^2)$, we will, owing to integration over the $B_{n\mu}^a$, end up evaluating its trace, and hence need only the diagonal elements $M_{\mu\mu}^{aa}$ of the matrix $M_{\mu\nu}^{ab}$. Hence

$$A_I^{(2)} = \frac{1}{2g_0^2}\sum_n\sum_{\mu\nu}{}'\left[\tfrac{1}{12}(\theta_\mu\times B_{n\mu})^2 - \tfrac{1}{12}(B_{n\mu}\times\theta_\mu)\cdot(B_{n+\hat\nu,\mu}\times\theta_\mu) - \tfrac{1}{2}(B_{n\mu}\times\theta_\mu)\cdot(B_{n+\hat\nu,\mu}\times\theta_\nu)\right] + \text{off diagonal in } \binom{ab}{\mu\nu}. \tag{3.40}$$

We will work in the Feynman gauge, i.e.,

$$A_\alpha = -\frac{1}{4g_0^2}\sum_n\left(\sum_\mu \Delta_\mu B_{n-\hat\mu,\mu}\right)^2. \tag{3.41}$$

Using the definition of $B_{n\mu}^a = \sum_k' e^{ikn} B_{k\mu}^a$, we have

$$A_0 + A_\alpha = -\frac{1}{4g_0^2}\sum_k{}'\sum_\mu\left(\sum_\sigma|1-e^{ik_\sigma}|^2\right)B_{-k\mu}^a B_{k\mu}^a = -\frac{1}{4g_0^2}\sum_k{}'\sum_\mu d_k B_{-k\mu}^a B_{k\mu}^a, \tag{3.42}$$

$$A_I^{(1)} = -\frac{C^{abc}}{4g_0^2}\sum_k{}'\sum_{\mu\nu}\left[\theta_\nu^a(e^{ik_\mu}-1)(e^{-ik_\nu}+1) + \delta_{\mu\nu}\sum_\sigma\theta_\sigma^a(e^{-ik_\sigma}-1)(e^{-ik_\sigma}+1)\right]B_{-k\mu}^b B_{k\nu}^c$$

$$= -\frac{C^{abc}}{4g_0^2}\,\tfrac{1}{2}\sum_k{}'\sum_{\mu\nu}\left[\theta^a(e^{ik_\mu}-1)(e^{-ik_\nu}+1) - \theta_\mu^a(e^{-ik_\nu}-1)(e^{ik_\mu}+1) - 4i\delta_{\mu\nu}\sum_\sigma\theta_\sigma^a\sin k_\sigma\right]B_{-k\mu}^b B_{k\nu}^c, \tag{3.43}$$

$$A_I^{(1)} \equiv -\frac{1}{4g_0^2}\sum_k{}'\sum_{\mu\nu} B_{-k\mu}^b N_{\mu\nu}^{bc}(k) B_{k\nu}^c, \tag{3.44}$$

and finally

$$A_I^{(2)} = \frac{C^{abc}C^{a\beta\gamma}}{2g_0^2}\sum_k{}'\sum_\mu\left(\tfrac{1}{4}\theta_\mu^b\theta_\mu^\beta - \tfrac{1}{12}\theta_\mu^b\theta_\mu^\beta\sum_{\nu\neq\mu}e^{ik_\nu} - \tfrac{1}{2}\sum_{\nu\neq\mu}\theta_\nu^b\theta_\nu^\beta e^{ik_\nu}\right)B_{-k\mu}^c B_{k\mu}^\gamma, \tag{3.45}$$

$$A_I^{(2)} = \frac{C^{abc}C^a}{2g_0^2}\sum_k{}'\sum_\mu\left[\tfrac{1}{4}\theta_\mu^b\theta_\mu^\beta - \tfrac{1}{24}\theta_\mu^b\theta_\mu^\beta\sum_{\nu\neq\mu}(e^{ik_\nu}+e^{-ik_\nu}) - \tfrac{1}{4}\sum_{\nu\neq\mu}\theta_\nu^b\theta_\nu^\beta(e^{ik_\nu}+e^{-ik_\nu})\right]B_{-k\mu}^c B_{k\mu}^\gamma, \tag{3.46}$$

$$A_I^{(2)} \equiv -\frac{1}{4g_0^2}\sum_k\sum_\mu B_{-k\mu}^c M_{\mu\mu}^c B_{k\mu}^\gamma. \tag{3.47}$$

Collecting Eqs. (3.42), (3.44), and (3.47) gives

$$A + A_\alpha = -\frac{1}{4g_0^2}\sum_k{}'\sum_{\mu\nu} d_k\left(\delta^{ab}\delta_{\mu\nu} + \frac{1}{d_k}N_{\mu\nu}^{ab} + \frac{1}{d_k}M_{\mu\nu}^{ab}\right)B_{-k\mu}^a B_{k\nu}^b, \tag{3.48}$$

where both $N_{\mu\nu}^{ab}$ and $M_{\mu\nu}^{ab}$ have been made explicitly Hermitian. Let

$$L = 1 + \frac{1}{d}N + \frac{1}{d}M. \tag{3.49}$$

The gauge-fixing term A has no dependence on θ_μ^a; however, the A_c term is a function of θ_μ^a. In performing the $\{B_{k\mu}^a, k\neq 0\}$ integrations, we can ignore the coupling of θ_μ^a to $B_{k\mu}^a$ coming from the A_c term, as this will produce $O(g_0)$ terms multiplying θ_μ^a, which we are ignoring anyway. The same is true for the measure $\mu(B_{n\mu}^a)$, for which we have, from Ref. 7,

$$\mu = \prod_{n\mu}\mu(\tilde B_{n\mu}^a) = \exp\left[-\frac{n}{24}\sum_{n\mu a}(B_{n\mu}^a + \theta_\mu^a)^2\right] + O(g_0)$$

or

$$\mu(\theta) = \exp\left(-\frac{n}{24}N^4\sum_\mu\theta_\mu^2\right) + O(g_0). \tag{3.50}$$

Then, collecting all the results, we have

$$A=\prod_\mu\int dU_\mu e^{A_U}\mu(\theta)e^{A_c(\theta)}\prod_{k\neq 0}\prod_{\mu,a}\int_{-\infty}^{+\infty}dB^a_{k\mu}e^{A+A_\alpha}$$
$$=\prod_\mu\int dU_\mu e^{A_U}\mu(\theta)e^{A_c(\theta)}\frac{(\text{const})}{(\det L)^{1/2}}\,. \tag{3.51}$$

B. Self-energy calculation

We evaluate the lowest-order contribution to the proper self-energy. This will consist of calculating the integrand of (3.51), i.e., of

$$Z(g_0)=\prod_\mu\int dU_\mu\, e^{A_U}e^{A_c(\theta)}\frac{\mu(\theta)}{[\det L(\theta)]^{1/2}}\,. \tag{3.51'}$$

To do so, we calculate $\det L$ and $e^{A_c(\theta)}$. We will make use of our results from Sec. II to evaluate $e^{A_c(\theta)}$. Since we are considering θ^a_μ to be small, we will expand exponential functions of θ^a_μ in a power series. We will then consistently use the identity

$$\prod_\mu\int dU_\mu e^{A_U}\theta_\tau\cdot\theta_\sigma=\delta_{\tau\sigma}\prod_\mu\int dU_\mu e^{A_U}\theta_\tau^{\,2} \tag{3.52a}$$
$$=\delta_{\tau\sigma}\times(\text{constant})\,. \tag{3.52b}$$

We will signify the use of (3.52) by an arrow ($\rightarrow$). We will also use [for SU(n)]

$$C^{ab\alpha}C^{ab\beta}=n\delta^{ab}\,. \tag{3.53}$$

Therefore

$$\det L=\det\left(1+\frac{1}{d}N+\frac{1}{d}M\right)$$
$$\simeq\exp\left[\mathrm{Tr}\left(\frac{1}{d}N+\frac{1}{d}M\right)-\tfrac{1}{2}\mathrm{Tr}\left(\frac{1}{d}N\frac{1}{d}M\right)\right]$$
$$=\exp\left[\mathrm{Tr}\left(\frac{1}{d}M\right)-\tfrac{1}{2}\mathrm{Tr}\left(\frac{1}{d}N\frac{1}{d}N\right)\right]\,. \tag{3.54}$$

Let

$$\phi=nN^4\tfrac{1}{4}\sum_\mu\theta_\mu^{\,2}\,. \tag{3.55}$$

Then

$$\mathrm{Tr}\left(\frac{1}{d}M\right)\rightarrow\phi\sum_k{}'\frac{4}{d_k}\left(-3+\tfrac{7}{16}d_k\right)\,. \tag{3.56}$$

Define

$$J=\sum_k{}'\frac{1}{d_k}\,. \tag{3.57}$$

Then

$$\mathrm{Tr}\left(\frac{1}{d}M\right)=12\phi J-\tfrac{7}{4}\phi\,. \tag{3.58}$$

Also,

$$\mathrm{Tr}\left(\frac{1}{d}N\frac{1}{d}N\right)\rightarrow\left(+nN^4\tfrac{1}{4}\sum_\mu\theta_\mu^{\,2}\right)12\sum_k{}'\left(\frac{|e^{ik_1}-e^{-ik_1}|^2}{d_k^{\,2}}+2\frac{1-\cos k_1\cos k_2}{d_k^{\,2}}\right)\,. \tag{3.59}$$

Define

$$I=\sum_k{}'\frac{|e^{ik_1}-e^{-ik_1}|^2}{d_k^{\,2}}\,, \tag{3.60}$$

$$K=\sum_k{}'(1-\cos k_1\cos k_2)/d_k^{\,2}\,. \tag{3.61}$$

Then

$$\mathrm{Tr}\left(\frac{1}{d}N\frac{1}{d}N\right)=12\phi(I+2K)\,. \tag{3.62}$$

Therefore

$$\det L\simeq e^{(12J-7/4-6I-12K)\phi}\,. \tag{3.63}$$

We now evaluate e^{A_c}. Recall from (2.50) that

$$e^{A_c}=\exp\left[-\frac{n}{48}\sum_{n\mu}\sum_a(B^a_{n\mu})^2\right]$$
$$\times\det(d_k\delta^{ab}\delta_{k,q}+M_{ab}(k,q))\,, \tag{3.64}$$

where

$$M^{ab}(k,q)=-\tfrac{1}{2}C^{abc}\sum_\mu(1-e^{-ik_\mu})(1+e^{iq_\mu})B^c_{k-q,\mu}\,, \tag{3.65}$$

$$M^{ab}(k,q)=-\tfrac{1}{2}C^{abc}\sum_\mu(e^{ik_\mu}-e^{ik_\mu})\theta^c_\mu\delta_{k,q}+O(g_0)\,, \tag{3.66}$$

$$M^{ab}(k,q)\equiv n^{ab}(k)\delta_{k,q}+O(g_0)\,. \tag{3.67}$$

Therefore

$$e^{A_c}=\exp\left(-\frac{N^4n}{48}\sum_\mu\theta_\mu^{\,2}\right)(\text{const})\times\det\left(1+\frac{1}{d}n\right)$$
$$+O(g_0)$$
$$\simeq\exp\left(-\tfrac{1}{12}\phi\right)\exp\left[-\tfrac{1}{2}\mathrm{Tr}\left(\frac{1}{d}n\frac{1}{d}n\right)\right]\,. \tag{3.68}$$

But

$$\mathrm{Tr}\left(\frac{1}{d}n\frac{1}{d}n\right)\rightarrow\phi I\,. \tag{3.69}$$

Therefore

$$e^{A_c} = \exp[-\phi(\tfrac{1}{12} + \tfrac{1}{2} I)] + O(g_0) . \tag{3.70}$$

Also,

$$\mu(\theta) = \exp\left(-\frac{n}{24} N^4 \sum_\mu \theta_\mu{}^2\right) = \exp(-\tfrac{1}{6}\phi) . \tag{3.71}$$

Hence, from (3,63), (3.70), and (3.71) we have, using (3.51),

$$Z = \prod_\mu \int dU\, e^{A_U} e^{A_c} \frac{\mu(\theta)}{(\det L)^{1/2}}$$

$$= \prod_\mu \int dU_\mu e^{A_U} \exp(\tfrac{5}{8} + \tfrac{5}{2} I + 6K - 6J)\phi . \tag{3.72}$$

Let

$$\Delta = \tfrac{5}{8} + \tfrac{5}{2} I + 6K - 6J . \tag{3.73}$$

From the identity $\sum'_k (d_k^2/d_k^2) = 1$, we have

$$I = 4J - 12K - \tfrac{1}{4} , \tag{3.74}$$

giving

$$\Delta = 4(J - 6K) . \tag{3.75}$$

It can be shown that[7]

$$J = 6K + O(e^{-N}) , \tag{3.76}$$

giving

$$\Delta = 0 + O(e^{-N}) , \tag{3.77}$$

$$\Delta = -ce^{-N} , \tag{3.78}$$

where c is a constant and is $O(1)$.

Therefore

$$Z = \prod_\mu \int dU_\mu \exp(A_U) \exp(-ce^{-N}\phi)$$

$$= \prod_\mu \int dU_\mu \exp(A_U) \exp\left[-N^4 (cne^{-N}/4) \sum_\mu \theta_\mu{}^2\right] . \tag{3.79}$$

Let

$$\Pi_0 = \tfrac{1}{4}\, cne^{-N} . \tag{3.80}$$

We discuss our results in the next section, and show how, if $\Pi_0 \neq 0$ and $N \to \infty$, we would have a quadratic divergence.

C. Discussion

The main result of the last section, from (3.79) and (3.80), is

$$Z = \prod_\mu \int dU_\mu e^{A_U} \exp\left(-N^4 \Pi_0 \sum_\mu \theta_\mu{}^2\right) . \tag{3.81}$$

We now show how a finite Π_0, in the $N \to \infty$ limit, would lead to a quadratic divergence. Let

$$\langle\ \rangle = \prod_\mu \int dU_\mu e^{A_U} . \tag{3.82}$$

The propagator was defined by

$$D = \frac{1}{N^4} \left\langle B^a_{k=0,\mu} B^a_{k=0,\mu} \exp\left(-N^4 \Pi_0 \sum_\mu \theta_\mu{}^2\right)\right\rangle \Big/ Z$$

$$= N^4 \left\langle \theta_\mu{}^2 \exp\left(-N^4 \Pi_0 \sum_\mu \theta_\mu{}^2\right)\right\rangle \Big/ Z . \tag{3.83}$$

Suppose $\Pi_0 > 0$; then we can extend the range of θ_μ integrations to infinity, giving

$$D \sim N^4 \frac{1}{N^4 \Pi_0} = \frac{1}{\Pi_0} . \tag{3.84}$$

Hence we see that, if $\Pi_0 \neq 0$ as $N \to \infty$, we get a quadratic divergence $\sim (1/a^2)\Pi_0$. However, since $\Pi_0 \sim e^{-N}$, we have

$$D \sim e^N \to \infty \quad \text{as } N \to \infty . \tag{3.85}$$

Note that the bare propagator $D^{(0)}$ also diverges as $N \to \infty$ since

$$D^{(0)} = N^4 \langle \theta_\mu{}^2 \rangle / Z \to \infty \quad \text{as } N \to \infty . \tag{3.86}$$

We therefore conclude that, in the $N \to \infty$ limit, the lattice gauge theory has no mass renormalization. The continuum theory also shows zero mass renormalization, and we conclude that discretizing spacetime does not violate this property since the lattice gauge theory was defined to exactly preserve gauge invariance.

On the finite-size periodic lattice, our calculation shows $m_{\text{quantum}}{}^2 \sim e^{-N}$; however, for the infinite lattice we have no information about the mass of the gauge field quantum, since the absence of mass renormalization means that the large-distance problem has to be solved for determining $m_{\text{quantum}}{}^2$.

The cubic and higher-order terms in θ^a_μ cannot affect the divergence of D for $N \to \infty$; that is why they can be ignored. All arguments we used apply equally well for $\Pi(0)$, and we see that the coefficient of the quadratic term $\sum_\mu \theta_\mu{}^2$ in the action contains *all* the information regarding mass renormalization. The calculation we performed for Π_0 can be done using Feynman diagrams. The external lines are $B^a_{k=0,\mu}$; the propagator for the internal gluon lines is $\delta_{\mu\nu}/d_k$ and for the internal ghost lines is $1/d_k$. The vertices are rather complicated and can be read off from the action. The graphs used are shown in Fig. 1. Note that, since the θ^a_μ variables were held fixed when performing the $\{B^a_{k\mu}, k \neq 0\}$ integrations in the path integral, the proper self-energy is equal to the complete self-energy for the gauge field quantum.

We now give a general proof that $\Pi(0)$ is zero to all order in perturbation theory using the Slavnov transformation. We will obtain an identity involving $D_{k\mu\nu}$ and this will give us the desired result.

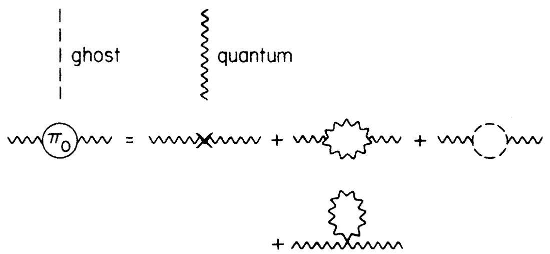

FIG. 1. Feynman diagrams for the computation of the lowest-order self-energy of the gauge field quantum.

Recall from (2.63) and (2.65) that

$$c_n^a \to c_n^a - \alpha\lambda s_n^a\,, \qquad (3.87)$$

$$s_n^a \to s_n^a + \lambda \frac{\partial s_n^a}{\partial \phi_m^b} c_m^{\dagger b} = s_n^a - \lambda \frac{\delta A_c}{\delta c_n^a}\,. \qquad (3.88)$$

In obtaining (3.88) we have used

$$A_c = c_n^{\dagger a} \frac{\partial s_m^b}{\partial \phi_n^a} c_m^b \qquad (3.89)$$

and the fact that $\delta/\delta c_n^a$ anticommutes with all fermion variables. In particular, we are using

$$s_n^a = \sum_\mu \Delta_\mu B^a_{n-\hat\mu,\mu}\,. \qquad (3.89')$$

Therefore, from (3.87) and (3.88),

$$c_l^a \Delta_\sigma B^b_{n-\hat\sigma,\sigma} \to c_l^a \Delta_\sigma B^b_{n-\hat\sigma,\sigma} + \lambda c_l^a \frac{\delta A_c}{\delta c_n^b} - \alpha\lambda \Delta_\mu B^a_{l-\hat\mu,\mu} \Delta_\sigma B^b_{n-\hat\sigma,\sigma}\,. \qquad (3.90)$$

Let

$$\langle\ \rangle \equiv \prod_n \prod_{\mu,a} \int dU_{n\mu} dc_n^a dc_n^{\dagger a} e^A\,.$$

Then

$$\langle c_l^a \Delta_\sigma B^b_{n-\hat\sigma,\sigma} e^{A_\alpha + A_c}\rangle = \langle c_l^a \Delta_\sigma B^b_{n-\hat\sigma,\sigma} e^{A_\alpha + A_c}\rangle + \lambda \left\langle c_l^a \frac{\delta A_c}{\delta c_n^b} e^{A_\alpha + A_c}\right\rangle - \alpha\lambda \langle \Delta_\mu B^a_{l-\hat\mu,\mu} \Delta_\sigma B^b_{n-\hat\sigma,\sigma} e^{A_\alpha + A_c}\rangle\,, \qquad (3.91)$$

$$\langle \Delta_\mu B^a_{l-\hat\mu,\mu} \Delta_\sigma B^b_{n-\hat\sigma,\sigma} e^{A_\alpha + A_c}\rangle = \frac{1}{\alpha}\left\langle c_l^a \frac{\delta e^{A_c}}{\delta c_n^b} e^{A_\alpha}\right\rangle\,. \qquad (3.92)$$

To perform integration by parts for the fermion variables, note that

$$\left\langle \frac{\delta}{\delta c_l^b}(c_n^a e^{A_c})\right\rangle_{c,c^\dagger} = 0 = \delta^{ab}\delta_{n,l}\langle e^{A_c}\rangle_{c,c^\dagger} - \left\langle c_n^a \frac{\delta e^{A_c}}{\delta c_l^b}\right\rangle_{c,c^\dagger}\,. \qquad (3.93)$$

Therefore

$$\langle \Delta_\mu B^a_{l-\hat\mu,\mu} \Delta_\sigma B^b_{n-\hat\sigma,\sigma} e^{A_\alpha + A_c}\rangle / Z = \frac{1}{\alpha}\delta^{ab}\delta_{n,l}\,. \qquad (3.94)$$

Fourier transforming the above equation and using translational invariance gives

$$(1-e^{ik_\mu})(1-e^{-ik_\nu})D^{ab}_{k\mu\nu} = \frac{1}{\alpha}\delta^{ab}\,. \qquad (3.95)$$

To determine the behavior of $\Pi(0)$, we need only the $k\simeq 0$ behavior of the propagator. From (3.95), we have that $D_k \sim 1/k^2$ for $k \simeq 0$. Hence we conclude that $\Pi(0)=0$, and there is no mass renormalization for the gauge field.

One might be tempted to conclude from the above result that the gauge field quanta are massless for the exact theory. However, this conclusion cannot be made for the lattice theory. In the strongly coupled region for the lattice theory, the degrees of freedom are no longer $B^a_{n\mu}$, but instead are $U_{n\mu} = \exp(iB^a_{n\mu}X^a)$. If the s_n^a are written directly in terms of the $U_{n\mu}$ [such that (3.89) is recovered in the weak-coupling limit], then one finds that the expression for e^{A_c} is no longer a pure determinant, but instead e^{-A_c} is a *sum* of (determinants)$^{-1}$ due to the fact that the $\prod_{n,a}\delta(s_n^a(\phi) - s_n^a)$ now no longer has a unique solution for the ϕ_n^a at $\phi_n^a = 0$. (This fact has also been recently recognized for the continuum theory by Gribov[8] and leads to nontrivial modifications of the continuum Yang-Mills theory.) This in turn means that the Slavnov identity no longer holds, and hence the identity for the propagator is lost when we arrive at a strongly coupled theory. We hence cannot conclude that the gauge field quanta are massless for the exact lattice theory. This question can be resolved by studying the behavior of the lattice gauge field under the renormalization-group transformation.

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APPENDIX A: WEAK COUPLING

From (2.4) we have

$$Z(g_0) = \prod_{\nu=0}^{3} \prod_{n\in D^{(\nu)}} \prod_{\mu\neq\nu} \int d\tilde U_{n\mu} d\tilde U_{N\mu} e^{A[\tilde U]} \qquad (A1)$$

$$\equiv \prod_{n,\mu}{}' \int d\tilde U_{n\mu} d\tilde U_{N\mu} e^{A[\tilde U]}\,. \qquad (A2)$$

We are interested in $g_0 \to 0$. In this limit, the action has a sharp maxima about $\tilde{W}_{n\mu\nu}=1$, and expanding about this gives

$$A=\frac{1}{2g_0^2}\sum_{n\mu\nu}\mathrm{Tr}(\tilde{W}_{n\mu\nu})$$
$$=\frac{1}{2g_0^2}\sum_{n\mu\nu}\mathrm{Tr}[\exp(i\tilde{f}^a_{n\mu\nu}X^a)]$$
$$\simeq -\frac{s^{-2}}{4g_0^2}\sum_{n\mu\nu}\sum_{a}(\tilde{f}^a_{n\mu\nu})^2. \tag{A3}$$

Recall that we are in the axial gauge. For every domain except $n=N$, we have three independent $\tilde{f}^a_{n\mu\nu}$. More precisely, we have[1]

$$\nu=0,1,2,3,\quad n\in D^{(\nu)} \tag{A4}$$
$$\tilde{f}^a_{n\mu\nu}\ \text{independent variables.}$$

However, at $n=N$, *all* the $\tilde{f}^a_{N\mu\nu}$ are dependent variables.

Hence we see that in each domain except $n=N$, e^A provides a Gaussian factor for the three independent variables $\tilde{B}^a_{n\alpha}$ (the non-Abelian index is irrelevant here) through the three independent variables $\tilde{f}^a_{n\mu\nu}$. Hence, from the action, we see that $\tilde{B}^a_{n\alpha}=O(g_0)$, and we can extend their range to infinity. However, $\tilde{B}^a_{N\mu}$ has no Gaussian factor and remains $O(1)$. Hence its range has to be kept over the compact space. This special behavior of $\tilde{B}^a_{N\mu}$ is not without consequence, since it is connected to mass renormalization.

Collecting our results, we have

$$Z(g_0)\simeq {\prod_{n\mu a}}'\int_{-\infty}^{+\infty}d\tilde{B}^a_{n\mu}\,\mu(\tilde{B}_{n\mu})\prod_{\mu}\int_G d\tilde{U}_{N\mu}e^{A[\tilde{U}]}$$
$$<\infty. \tag{A5}$$

APPENDIX B: GAUGE-FIXING AND COUNTERTERM

Recall that from (2.26) and (2.27)

$$Z=\int dU\, e^{A+A_\alpha+A_c}, \tag{B1}$$

where (for an infinite-size lattice)

$$A_\alpha=-\frac{\alpha}{2}\sum_{n,a}(s^a_n)^2, \tag{B2}$$

$$e^{-A_c}=\int dV\prod_{n,a}\delta(s^a_n(\phi)-s^a_n). \tag{B3}$$

We show that with this form for A_α and A_c (where A_c is not gauge invariant), we still have

$$Z=\int dU\, e^{A+A_\alpha+A_c}$$
$$=\int(\text{const})\times dU\, e^{A}.$$

Perform the gauge transformation

$$U_{n\mu}\to\tilde{V}_nU_{n\mu}\tilde{V}^\dagger_{n+\hat{\mu}}, \tag{B4}$$

$$dU_{n\mu}\to dU_{n\mu}. \tag{B5}$$

Then

$$Z=\int dU\, e^{A[U]}\,\frac{\exp[-\frac{1}{2}\alpha\sum_{n,a}(s^a_n(\tilde{\phi}))^2]}{\int dV\prod_{n,a}\delta(s^a_n(\phi\tilde{\phi})-s^a_n(\tilde{\phi}))}$$
$$=\int dU\, e^{A[U]}\,\frac{\exp[-\frac{1}{2}\alpha\sum_{n,a}(s^a_n(\tilde{\phi}))^2]}{\int dV\prod_{n,a}\delta(s^a_n(\phi)-s^a_n(\tilde{\phi}))}, \tag{B6}$$

where, in taking the last step, we have used $d(\tilde{V}_nV_n)=dV_n$. Note that e^{A_c} is now a function of $\tilde{\phi}$, i.e., not gauge invariant. Since Z is independent of $\tilde{V}_n$, we can trivially integrate it over all $\tilde{V}_n$, i.e.,

$$Z=\int d\tilde{V}\,Z$$
$$=\int dU\, e^{A[U]}\int d\tilde{V}\,\frac{\exp[-\frac{1}{2}\alpha\sum_{n,a}(s^a_n(\tilde{\phi}))^2]}{\int dV\prod_{n,a}\delta(s^a_n(\phi)-s^a_n(\tilde{\phi}))}. \tag{B7}$$

Define a change of variables from $\tilde{\phi}^a_n$ to Φ^a_n by

$$\Phi^a_n=s^a_n(\tilde{\phi}), \tag{B8}$$

$$\prod_{n,a}d\Phi^a_n=\det\left(\frac{\partial s^a_n(\tilde{\phi})}{\partial\tilde{\phi}^b_m}\bigg|_{s^a_n(\phi)=\Phi^a_n}\right)\prod_{n,a}d\tilde{\phi}^a_n. \tag{B9}$$

Let

$$J[B,\Phi]=\det(\partial s^a_n(\tilde{\phi})/\partial\tilde{\phi}^b_m)|_{s^a_m(\phi)=\Phi^a_n}. \tag{B10}$$

Then

$$d\tilde{V}=\prod_n d\tilde{V}_n=\prod_{n,a}d\tilde{\phi}^a_n\prod_n\mu(\tilde{\phi}^a_n)$$
$$=\frac{1}{J[B,\Phi]}\,\mu[B,\Phi]\prod_{n,a}d\Phi^a_n, \tag{B11}$$

where

$$\mu[B,\Phi]=\prod_n\mu(\phi_n[B,\Phi]). \tag{B12}$$

We now evaluate e^{A_c}.

$$e^{-A_c}=\int dV\prod_{n,a}\delta(s^a_n(\phi)-s^a_n(\tilde{\phi}))$$
$$=\prod_{n,a}\int d\phi^a_n\,\mu(\phi_n)\prod_{n,a}\delta(s^a_n(\phi)-\Phi^a_n). \tag{B13}$$

Define a change of variable from ϕ^a_n to u^a_n by

$$u^a_n=s^a_n(\phi)-\Phi^a_n. \tag{B14}$$

The δ functions in (B13) force us to evaluate the Jacobian of the transformation (B14) at the value of ϕ^a_n for which $s^a_n(\phi)=\Phi^a_n$. Therefore

$$\prod_{n,a} du_n^a = \left(\det \frac{\partial s_n^a(\phi)}{\partial \phi_m^b}\bigg|_{s_n^a(\phi)=\Phi_n^a}\right)\prod_{n,a} d\phi_n^a$$

$$= \frac{1}{J[B,\Phi]}\prod_{n,a} d\phi_n^a . \tag{B15}$$

Similarly, the value of the measure $\mu(\phi_n)$ is fixed by the δ functions giving

$$\prod_n \mu(\phi_n[B,\Phi]) = \mu[B,\Phi] . \tag{B16}$$

Therefore

$$e^{-A_c} = \mu[B,\Phi]/J[B,\Phi] . \tag{B17}$$

Collecting Eqs. (B.7), (B.11), and (B.17), we have

$$Z = \int dU\, e^{A[U]}\prod_{n,a}\int d\Phi_n^a \frac{\mu[B,\Phi]}{J[B,\Phi]} \times \exp\left[-\frac{\alpha}{2}(\Phi_n^a)^2\right]\frac{J[B,\Phi]}{\mu[B,\Phi]}$$

$$= \int dU\, e^{A[U]}\prod_{n,a}\int d\Phi_n^a \exp[-\tfrac{1}{2}\alpha(\Phi_n^a)^2]$$

$$= (\text{const})\times \int dU\, e^{A[U]} . \tag{B18}$$

Hence, we have proved the desired result. Note that the result is exact and valid for any value of g_0. I thank M. Peskin for a discussion on this topic.

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