Properties of the solutions of cold ultradense configurations in the Brans-Dicke theory

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(Received 28 February 1977)

We have studied the properties of cold ultrahigh-density static configurations in the Brans-Dicke theory. We used a perfect-fluid model with a simple equation of state, $p = \epsilon\rho$, for matter. An exact solution with infinite central density is obtained, and the properties of solutions with finite central density are also examined.

I. INTRODUCTION

The properties of cold superdense stars, with densities of the order of or larger than nuclear density, have been studied extensively in recent years.[1,2,3] A major source of uncertainty in these investigations is the form of the equation of state at very high densities. Consequently, there has been particular interest in obtaining the properties of ultrahigh-density stars that are relatively insensitive to the choice of the equation of state. Steps in this direction have been taken in general relativity by Misner and Zapolsky,[1] and by Harrison.[2] In this paper we make an initial investigation of the corresponding problem in the Brans-Dicke-Jordan theory of gravitation.

The Brans-Dicke-Jordan scalar-tensor theories were first investigated by Jordan,[4] and later, in connection with Mach's principle, they were studied extensively by Brans and Dicke.[5] The theory can be expressed in units in which the local value of the Newtonian "gravitational constant" is a function of a scalar field which is in turn determined by the trace of the energy-momentum tensor of all other nongravitational fields. The field equations are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R^{\alpha}_{\alpha} = \frac{8\pi T_{\mu\nu}}{\phi} + \frac{\omega}{\phi^2}(\phi_{;\mu}\phi_{;\nu} - \tfrac{1}{2} g_{\mu\nu}\phi^{;\alpha}\phi_{;\alpha}) + \frac{1}{\phi}(\phi_{;\mu\nu} - g_{\mu\nu}\phi^{;\alpha}_{;\alpha}), \tag{1}$$

$$\Box^2\phi \equiv \phi^{;\alpha}_{;\alpha} = \frac{(\sqrt{-g}\,\phi^{;\alpha})_{,\alpha}}{\sqrt{-g}} = \frac{8\pi T^{\alpha}_{\alpha}}{2\omega+3} \equiv \frac{8\pi T}{2\omega+3}, \tag{2}$$

where

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu}, \tag{3}$$

$$\Gamma^{\alpha}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}), \tag{4}$$

$$g \equiv \det g_{\mu\nu}\,. \tag{5}$$

Commas denote partial derivatives with respect to the coordinates x^{μ} ($\mu = 0, \ldots, 3$). Semicolons mean covariant derivatives. $T_{\mu\nu}$ is the energy-momentum tensor of matter and all nongravitational fields. ω is a positive constant.[5,6] The field equation can also be written as

$$R_{\mu\nu} = \frac{8\pi}{\phi}\left(T_{\mu\nu} - g_{\mu\nu}\frac{\omega+1}{2\omega+3}T\right) + \frac{\omega\phi_{;\mu}\phi_{;\nu}}{\phi^2} + \frac{\phi_{;\mu\nu}}{\phi}. \tag{6}$$

In Sec. II we discuss the general assumptions made in this paper; spacetime is taken to be static and spherically symmetric, and the energy-momentum tensor is that of a perfect fluid. The equation of state chosen is what is generally believed to be the asymptotic form of the proper pressure-density relation for very high density. In the next section we derive the relation between the scalar field ϕ and g_{00}. The equation of equilibrium is derived in Sec. IV, where a second-order nonlinear differential equation is obtained for the density distribution. In Sec. V we show how solutions to the equation of equilibrium are generated from a given solution by a homology transformation. All solutions with finite nonzero central density are generated in this way. In Sec. VI an exact solution of the field equation is found corresponding to a density distribution that is infinite at the origin. It is shown in the next section that all solutions of the equation of equilibrium approach this exact solution in an oscillatory fashion as the central density increases. Finally, in the last section, we suggest some possible applications of this result.

II. GENERAL ASSUMPTIONS

We assume that spacetime is static; the metric and the scalar field can be chosen such that

$$g_{\mu\nu,0} \equiv \frac{\partial g_{\mu\nu}}{\partial t} = \phi_{,0} = 0, \tag{7}$$

$$g_{0i}=0, \quad i=1-3. \tag{8}$$

If the configuration is spherically symmetric, further simplification of the metric is possible; the line element can be put in the Schwarzschild form

$$ds^2=-e^{2\psi}(r)dt^2+e^{2\Lambda}(r)dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2). \tag{9}$$

The energy-momentum tensor $T_{\mu\nu}$ is specialized to that of a perfect fluid

$$T_{\mu\nu}=(p+\rho)U_\mu U_\nu+p g_{\mu\nu}, \tag{10}$$

where p and ρ are the proper pressure and energy density, respectively. U_μ are the components of the fluid four-velocity. For a static fluid $U_i=0$, and since $U^\mu U_\mu=-1$, it follows that

$$U^0U_0=-1, \tag{11}$$

and therefore

$$T^0_0=-\rho, \tag{12}$$

$$T_{ij}=pg_{ij}. \tag{13}$$

For the further specification of the problem an equation of state is needed. We are interested in densities greater than nuclear ($\rho>10^{14}$ g/cm^3). In view of the large experimental and theoretical uncertainties concerning the equation of state at ultrahigh density it has been common[1,2] to make the simplest choice that keeps the velocity of sound less than the velocity of light, and to use

$$p=\epsilon\rho, \tag{14}$$

where ϵ is a constant. Since

$$s^2=\frac{dp}{d\rho}, \tag{15}$$

where s is the velocity of sound, then in units where the velocity of light is unity, we must have $0\le\epsilon\le 1$. For a static configuration we further require that $\epsilon>0$. In this paper we consider configurations with Eq. (14) as the equation of state.

III. THE RELATION BETWEEN ϕ AND g_{00}

If the metric satisfies Eqs. (7) and (8), Eq. (3) gives

$$R^0_0\equiv-\frac{1}{2\sqrt{-g}}[\sqrt{-g}(\ln g_{00})^{,\kappa}]_{,\kappa}. \tag{16}$$

From Eq. (6) we obtain

$$R^0_0=\frac{8\pi}{\phi}\left(T^0_0-\frac{\omega+1}{2\omega+3}T\right)+\frac{\phi^{;0}_{;0}}{\phi}$$
$$=\frac{8\pi}{\phi}\left(T^0_0-\frac{\omega+1}{2\omega+3}T\right)+\frac{1}{2}\frac{\phi^{,\kappa}}{\phi}\frac{(g_{00})_{,\kappa}}{g_{00}}. \tag{17}$$

Combining Eqs. (16) and (17), we get

$$-\tfrac{1}{2}[\sqrt{-g}\phi(\ln g_{00})^{,\kappa}]_{,\kappa}=8\pi\left(T^0_0-\frac{\omega+1}{2\omega+3}T\right)\sqrt{-g}. \tag{18}$$

Furthermore in the static case Eq. (2) simplifies to

$$(\sqrt{-g}\phi^{,\kappa})_{,\kappa}=[\sqrt{-g}\phi(\ln\phi)^{,\kappa}]_{,\kappa}=\frac{8\pi T\sqrt{-g}}{2\omega+3}. \tag{19}$$

Substituting Eq. (12) and (13) into Eqs. (18) and (19) and making use of Eq. (14), we obtain

$$-\tfrac{1}{2}[\sqrt{-g}\phi(\ln g_{00})^{,\kappa}]_{,\kappa}=-8\pi\left[\rho+\frac{\omega+1}{2\omega+3}(3p-\rho)\right]\sqrt{-g}$$
$$=-8\pi\left[1+\frac{\omega+1}{2\omega+3}(3\epsilon-1)\right]\rho\sqrt{-g}, \tag{20}$$

$$[\sqrt{-g}\phi(\ln\phi)^{,\kappa}]_{,\kappa}=\frac{8\pi(3p-\rho)\sqrt{-g}}{2\omega+3}$$
$$=\frac{8\pi(3\epsilon-1)\rho\sqrt{-g}}{2\omega+3}. \tag{21}$$

Combining Eqs. (20) and (21), we get

$$[\sqrt{-g}\phi(\ln\phi)^{,\kappa}]_{,\kappa}-\tfrac{1}{2}c[\sqrt{-g}\phi(\ln g_{00})^{,\kappa}]_{,\kappa}=0, \tag{22}$$

or

$$(\sqrt{-g}\phi\{\ln[\phi/(g_{00})^{c/2}]\}^{,\kappa})_{,\kappa}=0, \tag{23}$$

where

$$c\equiv\frac{3\epsilon-1}{(2\omega+3)+(\omega+1)(3\epsilon-1)}. \tag{24}$$

We integrate Eq. (23) and use Gauss's theorem to obtain

$$\int(\sqrt{-g}\phi\{\ln[\phi/(g_{00})^{c/2}]\}^{,\kappa})_{,\kappa}d^3x$$
$$=\int\sqrt{-g}\,\phi\{\ln[\phi/(g_{00})^{c/2}]\}^{,\kappa}d^2S_\kappa=0. \tag{25}$$

Assuming that spacetime is spherically symmetric, and that the metric is chosen in the Schwarzschild form (9), Eq. (25) becomes

$$\int\sqrt{-g}\phi(r)\{\ln[\phi/(g_{00})^{c/2}]\}^{,\kappa}d^2S_\kappa$$
$$=\int r^2\sin\theta\, e^{\psi+\Lambda}\phi[\ln(\phi/e^{c\psi})]^{,r}d\theta d\varphi$$
$$=4\pi r^2e^{\psi+\Lambda}\phi[\ln(\phi/e^{c\psi})]^{,r}=0, \tag{26}$$

therefore

$$[\ln(\phi/e^{c\psi})]^{,r}=e^{-2\Lambda}[\ln(\phi/e^{c\psi})]_{,r}=0, \tag{27}$$

and hence

$$\phi = \text{const} \times e^{c\psi}. \tag{28}$$

We see from Eqs. (20) and (21) that Eq. (28) is also valid outside matter with c an arbitrary constant. If we choose $c \neq 0$, and if we require $\phi(r)$ to be finite and different from zero, then g_{00} is finite and different from zero; consequently, this type of vacuum solution does not have a Schwarzschild horizon. This assumption, which is satisfied in the vacuum Brans solutions,[5] is quite reasonable since $\phi(r)$ can be identified with the inverse of the local measurable value of the gravitational constant.

For simplicity we choose units such that

$$\phi = e^{c\psi}. \tag{29}$$

In the limits $\omega \to \infty$, or $\epsilon \to \frac{1}{3}$ it follows from Eq. (24) that $c \to 0$; thus, from Eq. (29), $\phi \to 1$. Therefore in these cases the Brans-Dicke theory reduces to general relativity.

IV. THE EQUATION OF EQUILIBRIUM

Under the assumptions of static spherically symmetric perfect fluid with the metric in the Schwarzschild form, Eq. (9), we find from Eq. (6) that

$$\begin{aligned} R_{rr} &= -[\psi'' + (\psi')^2 - \Lambda'\psi' - 2\Lambda'/r] \\ &= \frac{8\pi}{\phi}\left[p - \frac{\omega+1}{2\omega+3}(3p-\rho)\right]e^{2\Lambda} + (\omega+1)[(\ln\phi)']^2 + (\ln\phi)'' - \Lambda'(\ln\phi)', \end{aligned} \tag{30}$$

$$R_{\theta\theta} = R_{\varphi\varphi} = 1 - re^{-2\Lambda}(\psi' - \Lambda' + 1/r) = \frac{8\pi}{\phi}\left[p - \frac{\omega+1}{2\omega+3}(3p-\rho)\right]r^2 + re^{-2\Lambda}(\ln\phi)', \tag{31}$$

$$R_{tt} = e^{2(\psi-\Lambda)}[\psi'' + (\psi')^2 - \Lambda'\psi' + 2\psi'/r] = \frac{8\pi}{\phi}\left[\rho + \frac{\omega+1}{2\omega+3}(3p-\rho)\right]e^{2\psi} - \psi' e^{2(\psi-\Lambda)}(\ln\phi)', \tag{32}$$

where the primes denote derivatives with respect to r. Making use of Eqs. (14) and (29) in Eqs. (30)–(32), they simplify to

$$(c+1)\psi'' + [(\omega+1)c^2+1](\psi')^2 - (c+1)\Lambda'\psi' - 2\Lambda'/r = -\frac{8\pi}{e^{c\psi}}\left[\epsilon - \frac{\omega+1}{2\omega+3}(3\epsilon-1)\right]\rho e^{2\Lambda}, \tag{33}$$

$$1 - re^{-2\Lambda}[(c+1)\psi' - \Lambda' + 1/r] = \frac{8\pi}{e^{c\psi}}\left[\epsilon - \frac{\omega+1}{2\omega+3}(3\epsilon+1)\right]\rho r^2, \tag{34}$$

$$e^{2(\psi-\Lambda)}[\psi'' + (c+1)(\psi')^2 - \Lambda'\psi' + 2\psi'/r] = \frac{8\pi}{e^{c\psi}}\left[1 + \frac{\omega+1}{2\omega+3}(3\epsilon-1)\right]\rho e^{2\psi}. \tag{35}$$

The conditions of energy-momentum conservation, $T^{\mu\nu}_{;\nu} = 0$, that follow from the identity $G^{\mu\nu}_{;\nu} = 0$, imply[7] that for a static spherically symmetric perfect fluid

$$\frac{dp}{dr} = -(p+\rho)\frac{d\ln(-g_{00})^{1/2}}{dr} = -(p+\rho)\frac{d\psi}{dr}. \tag{36}$$

With $p = \epsilon\rho$, we find

$$\frac{d(\ln\rho)}{dr} = -\frac{\epsilon+1}{\epsilon}\frac{d\psi}{dr}, \tag{37}$$

or

$$\rho = \rho_0 e^{-[(\epsilon+1)/\epsilon](\psi-\psi_0)} \equiv E e^{-[(\epsilon+1)/\epsilon]\psi}, \tag{38}$$

where ρ_0 and ψ_0 are the values of ρ and ψ at some arbitrary point $r = r_0$.

Making use of Eqs. (33)–(35), and (38), we eliminate Λ', $e^{-2\Lambda}$, and ρ from Eq. (35) thus obtaining

$$\frac{d^2\psi}{dx^2} + a_1\left(\frac{d\psi}{dx}\right)^3 + a_2\left(\frac{d\psi}{dx}\right)^2 + \frac{d\psi}{dx} + e^{-2\psi/D+2x}\left[\epsilon\frac{d^2\psi}{dx^2} + a_3\left(\frac{d\psi}{dx}\right)^3 + a_4\left(\frac{d\psi}{dx}\right)^2 + a_5\frac{d\psi}{dx} + a_6\right] = 0, \tag{39}$$

where

$$D \equiv \frac{2\epsilon}{\epsilon(c+1)+1}, \tag{40}$$

$$x \equiv \ln[(8\pi E)^{1/2} r]$$

and

$$a_1 \equiv -(\tfrac{1}{2}\omega c^2 - c), \tag{41}$$

$$a_2 \equiv 2(c+1), \tag{42}$$

$$a_3 \equiv \frac{[1-(\epsilon-1)\omega](\frac{1}{2}\omega c^2 - c)}{2\omega+3}, \tag{43}$$

$$a_4 \equiv \frac{-2[1-(\epsilon-1)\omega](c+1)+[2\omega+3+(3\epsilon-1)(\omega+1)](\frac{1}{2}\omega c^2-c)}{2\omega+3}, \tag{44}$$

$$a_5 \equiv \frac{-[1-(\epsilon-1)\omega]-2[2\omega+3+(3\epsilon-1)(\omega+1)](c+1)}{2\omega+3}, \tag{45}$$

$$a_6 \equiv \frac{-(2\omega+3)-(3\epsilon-1)(\omega+1)}{2\omega+3}. \tag{46}$$

Since $\rho = Ee^{-[(\epsilon+1)/\epsilon]\psi}$, Eq. (39) governs the density distribution for a static spherically symmetric perfect fluid in which an equation of state of the form $p = \epsilon\rho$ is valid.

V. THE HOMOLOGY TRANSFORMATION

A general homologous transformation is one in which the density and the radius at each point are multiplied by constant factors in order to obtain another equilibrium configuration. We shall show that the field equation (39) admits a transformation of this type. Specifically we prove the following:

Let $\psi(x)$ be a solution of Eq. (39), then

$$\Omega(x) \equiv \psi(x+\ln A) - \frac{2\epsilon \ln A}{\epsilon(c+1)+1} = \psi(x+\ln A) - D\ln A, \tag{47}$$

where A is an arbitrary constant, is also a solution. Furthermore

$$\frac{\rho_\Omega(x)}{\phi_\Omega(x)} = A^2\frac{\rho(x+\ln A)}{\phi(x+\ln A)}, \tag{48}$$

where ρ_Ω/ϕ_Ω and ρ/ϕ are the ratio of proper density to the scalar field for the Ω and the ψ solutions, respectively.

To establish this result let

$$\bar{x} \equiv x - \ln A, \tag{49}$$

then

$$\psi(x) = \Omega(\bar{x}) + D\ln A. \tag{50}$$

Substituting in Eq. (39), we obtain

$$\frac{d^2\Omega(\bar{x})}{d\bar{x}^2} + a_1\left[\frac{d\Omega(\bar{x})}{d\bar{x}}\right]^3 + a_2\left[\frac{d\Omega(\bar{x})}{d\bar{x}}\right]^2 + \frac{d\Omega(\bar{x})}{d\bar{x}} + e^{-2\Omega(\bar{x})/D+2\bar{x}}\left\{\frac{\epsilon\, d^2\Omega(\bar{x})}{d\bar{x}^2} + a_3\left[\frac{d\Omega(\bar{x})}{d\bar{x}}\right]^3 + a_4\left[\frac{d\Omega(\bar{x})}{d\bar{x}}\right]^2 + a_5\frac{d\Omega(\bar{x})}{d\bar{x}} + a_6\right\} = 0. \tag{51}$$

Dropping the bars, we see that $\Omega(x)$ is also a solution to Eq. (39). Moreover we have

$$\frac{\rho_\Omega(x)}{\phi_\Omega(x)} = Ee^{-(2/D)\Omega(x)} = A^2Ee^{-(2/D)\psi(x+\ln A)} = A^2\frac{\rho(x+\ln A)}{\phi(x+\ln A)}, \tag{52}$$

or in terms of r

$$\frac{\rho_\Omega(r)}{\phi_\Omega(r)} = A^2\frac{\rho(Ar)}{\phi(Ar)}. \tag{53}$$

In addition we have

$$\phi_\Omega(r) = \phi_\Omega(r_0)e^{c[\Omega(r)-\Omega(r_0)]} = \phi_\Omega(r_0)e^{c[\psi(Ar)-\psi(Ar_0)]} = \frac{\phi_\Omega(r_0)}{\phi(Ar_0)}\phi(Ar), \tag{54}$$

and

$$\rho_\Omega(r) = \frac{\rho_\Omega(r_0)}{\rho(Ar_0)}\rho(Ar). \tag{55}$$

Therefore we can write

$$\frac{\rho_\Omega(r)}{\phi_\Omega(r_0)} = A^2\frac{\rho(Ar)}{\phi(Ar_0)}. \tag{56}$$

We have seen that from one solution of the field equation (39) a continuous family of solutions can be derived by the homology transformation, Eq. (56). We shall show in the Appendix that all solutions ψ of Eq. (39) that are finite at the origin have $d\psi/dr = 0$ (or equivalently $d\rho/dr = 0$) at $r = 0$, and therefore the transformation (47) relates all solutions that are finite at the origin. The corresponding property for Newtonian polytropes had been worked out by Chandrasekhar,[8] and in general relativity by Bondi.[9]

VI. AN EXACT SOLUTION

We can verify that Eq. (39) is satisfied by the following singular solution:

$$\psi_s = \frac{\epsilon(2x+\ln B)}{\epsilon(c+1)+1} = \frac{D(2x+\ln B)}{2}, \tag{57}$$

where

$$B \equiv \frac{\epsilon^2+6\epsilon+1+(3\epsilon-1)^2/(2\omega+3)}{4\epsilon}. \tag{58}$$

We have then

$$\frac{\rho}{\phi} = Ee^{-2/D\psi_s} = EB^{-1}e^{-2x}, \tag{59}$$

or, using $x = \ln[(8\pi E)^{1/2} r]$,

$$\frac{\rho}{\phi} = \frac{B^{-1}}{8\pi r^2} = \frac{\epsilon}{2\pi[\epsilon^2+6\epsilon+1+(3\epsilon-1)^2/(2\omega+3)]r^2}. \tag{60}$$

Also

$$g_{00} = e^{2\psi_s} = B^D e^{2Dx} = (8\pi EB)^D r^{2D}. \tag{61}$$

Finally, using Eqs. (57) and (59) in Eqs. (33)–(35), and solving for $e^{2\Lambda}$, we get

$$g_{rr} \equiv e^{2\Lambda_s} = \frac{\left[1+\frac{(3\epsilon-1)^2}{(2\omega+3)(3\epsilon+1)^2}\right]\left[\epsilon^2+6\epsilon+1+\frac{(3\epsilon-1)^2}{2\omega+3}\right]}{\left[1+\epsilon+\frac{(3\epsilon-1)(\epsilon-1)}{(2\omega+3)(3\epsilon+1)}\right]^2}. \tag{62}$$

Note that in the limits $\omega\to\infty$ or $\epsilon\to\frac{1}{3}$, we have

$$c = \frac{3\epsilon-1}{(2\omega+3)+(\omega+1)(3\epsilon-1)} \to 0, \tag{63}$$

and therefore

$$\phi = e^{c\psi_s} \to 1, \tag{64}$$

$$\frac{\rho}{\phi} \to \rho \to \frac{\epsilon}{2\pi(\epsilon^2+6\epsilon+1)r^2}, \tag{65}$$

$$g_{00} \to \text{const} \times r^{4\epsilon/(\epsilon+1)}, \tag{66}$$

$$g_{rr} \to \frac{\epsilon^2+6\epsilon+1}{(1+\epsilon)^2}. \tag{67}$$

These results agree with the corresponding solution in general relativity.[1] The Newtonian approximation is obtained from (65)–(67) in the limit as $\epsilon = p/\rho \to 0$. We have in this case[8]

$$\rho \sim \frac{\epsilon}{2\pi r^2}, \tag{68}$$

$$g_{00} \sim \text{const} \times (1+4\epsilon \ln r), \tag{69}$$

$$g_{rr} \sim 1+6\epsilon. \tag{70}$$

In the next section we will show that the singular solution (57) is the limiting form of all solutions when the central density increases indefinitely.

VII. APPROXIMATE SOLUTION FOR ULTRADENSE STARS

In Sec. VI we found an exact solution of Eq. (39) corresponding to a configuration with a central ratio ρ/ϕ that is infinite. On physical grounds, it is reasonable to expect that this solution is the limiting form of all finite solutions when $\rho(0)/\phi(0)\to\infty$. This is indeed the case in general relativity where all solutions approach the singular solution, Eq. (65), in an oscillatory manner as the central density increases.[3] Presently we will see that this is also the case in the Brans-Dicke theory.

We consider a solution $\psi(x)$ that is near $\psi_s(x)$ for some range of x. The domain in which such a solution can be found is contained later in this section; there we will also explicitly obtain the function $\psi(x)$ under consideration. We thus define

$$z \equiv \psi - \psi_s = \psi - \frac{D}{2}(2x+\ln B), \tag{71}$$

$$\frac{dz}{dx} = \frac{d\psi}{dx} - D, \tag{72}$$

and write Eq. (39) in terms of z, then linearize the equation by assuming

$$z \ll 1, \tag{73}$$

$$\frac{dz}{dx} \ll 1. \tag{74}$$

In this way we find the linear system

$$b_0\frac{d^2z}{dx^2} + b_1\frac{dz}{dx} + b_2 z = 0, \tag{75}$$

where

$$b_0 \equiv B+\epsilon = \frac{1}{4\epsilon}\left[5\epsilon^2+6\epsilon+1+\frac{(3\epsilon-1)^2}{2\omega+3}\right], \tag{76}$$

$$b_1 \equiv -\frac{a_6}{D}(a_1D^2+a_2D+1)$$

$$= \frac{3\epsilon+1}{\epsilon+1}\,\frac{\left[1+\frac{(3\epsilon-1)^2}{(2\omega+3)(3\epsilon+1)^2}\right]b_0}{\left[1+\frac{(3\epsilon-1)(\epsilon-1)}{(2\omega+3)(3\epsilon+1)(\epsilon+1)}\right]}, \tag{77}$$

$$b_2 \equiv 2B(a_1D^2+a_2D+1)=2e^{2\Lambda_s}b_0, \tag{78}$$

where $e^{2\Lambda_s}$, given by Eq. (62), is the g_{rr} metric element for the singular solution. The most general solution to Eq. (75) is

$$z=F_1e^{m_1x}+F_2e^{m_2x}, \tag{79}$$

where F_1 and F_2 are constants, and

$$m_1\equiv\frac{-b_1+(b_1^{\,2}-4b_0b_2)^{1/2}}{2b_0}, \tag{80}$$

$$m_2\equiv\frac{-b_1-(b_1^{\,2}-4b_0b_2)^{1/2}}{2b_0}. \tag{81}$$

Let us define

$$\eta\equiv\frac{b_1}{2b_0}, \tag{82}$$

$$\xi\equiv\frac{(4b_0b_2-b_1^{\,2})^{1/2}}{2b_0}, \tag{83}$$

or, using Eqs. (76)–(78),

$$\eta=\frac{3\epsilon+1}{2(\epsilon+1)}\,\frac{1+\dfrac{(3\epsilon-1)^2}{(2\omega+3)(3\epsilon+1)^2}}{1+\dfrac{(3\epsilon-1)(\epsilon-1)}{(2\omega+3)(3\epsilon+1)(\epsilon+1)}}, \tag{84}$$

$$\xi^2=2e^{\Lambda_s}\left[1-\frac{1}{8}\,\frac{(3\epsilon+1)^2+\dfrac{(3\epsilon-1)^2}{2\omega+3}}{\epsilon^2+6\epsilon+1+\dfrac{(3\epsilon-1)^2}{2\omega+3}}\right]. \tag{85}$$

It can be easily shown that for $0<\epsilon\le1$, and $\omega>0$, we have $\eta>0$, $\xi^2>0$. We write Eq. (79) in the form

$$z=e^{-\eta x}(F_1e^{i\xi x}+F_2e^{-i\xi x}). \tag{86}$$

We see that the condition $z\ll1$ is achieved when $x\to\infty$. The real part of Eq. (86) can be put in the form

$$z=Fe^{-\eta x}\cos(\xi x+\alpha), \tag{87}$$

where F and α are constants. Using

$$x=\ln[(8\pi E)^{1/2}r]\equiv\ln y \tag{88}$$

in Eq. (87), we find

$$z=Fy^{-\eta}\cos(\xi\ln y+\alpha). \tag{89}$$

Substituting for z from Eq. (89) in Eq. (71), we obtain

$$\psi=\psi_s+Fy^{-\eta}\cos(\xi\ln y+\alpha). \tag{90}$$

Finally, since

$$\rho=Ee^{-[(\epsilon+1)/\epsilon]\psi}, \tag{91}$$

we have for the density distribution

$$\rho=Ee^{-[(\epsilon+1)/\epsilon]\psi_s}e^{-[(\epsilon+1)/\epsilon]Fy^{-\eta}\cos(\xi\ln y+\alpha)}$$
$$=\rho_s e^{-[(\epsilon+1)/\epsilon]Fy^{-\eta}\cos(\xi\ln y+\alpha)}, \tag{92}$$

where

$$\rho_s\equiv Ee^{-[(\epsilon+1)/\epsilon]\psi_s}. \tag{93}$$

Since $z=Fy^{-\eta}\cos(\xi\ln y+\alpha)$ is small, we can further expand the exponential, and retain the first two terms. In this way we find that

$$\rho=\rho_s\left[1-\left(\frac{\epsilon+1}{\epsilon}\right)Fy^{-\eta}\cos(\xi\ln y+\alpha)\right]. \tag{94}$$

From Eq. (94) it follows that as $y\to\infty$ the density distribution approaches that corresponding to the singular solution. If the solution are finite at the origin we can choose E to be the central density (i.e., choose $\psi=0$ at $r=0$). Since $y=(8\pi E)^{1/2}r$, we have shown that $\rho\to\rho_s$ as the central density increases (also $\rho\to\rho_s$ as $r\to\infty$; however, in this limit we do not expect the equation of state $p=\epsilon\rho$ to remain valid). The solution (94) intersects the singular solution ρ_s at points

$$y=\exp\left[\frac{(2n+1)(\pi/2)-\alpha}{\xi}\right], \tag{95}$$

where n is an integer. The amplitude of the oscillations $\rho-\rho_s$, fall off by the factor $e^{-(\eta/\xi)\pi}$ at successive peaks.

VIII. CONCLUSIONS

We have shown that the existence of a singular solution, corresponding to the limit of infinite central density, is a common feature of Newtonian and relativistic gravitation as well as of the Brans-Dicke theory. Misner and Zapolsky[1] have made use of this fact in general relativity to show that, provided that the equation of state at high densities is representable by Eq. (14), there is a maximum mass for cold matter beyond which equilibrium cannot be achieved. Harrison *et al.*[3] extended this result and showed that the mass and radius of ultrahigh-density stars approach finite values in an oscillatory way as the central density increases. Since numerical model calculations in the Brans-Dicke theory[10] similarly show that the mass of a static spherically symmetric cold star is bounded from above, our result may be used to establish more generally that, under certain reasonable assumptions about the equation of state, the mass of an equilibrium configuration has a damped oscillatory character as the central density increases.

The close resemblance of the Brans-Dicke theory to that of Einstein gravitation for a static ultradense perfect fluid suggests the possibility that a systematic way to go from a solution in one to a solution in the other could be found. In a future paper we will show that this is indeed the case. For static vacuum solutions this has already been given by Buchdahl.[11]

APPENDIX

In this appendix we show that all solutions ψ of Eq. (39) that are finite at the origin necessarily have $d\psi/dr=0$ at $r=0$.

In terms of

$$W\equiv\frac{d\psi}{dx}, \quad \text{(A1)}$$

Eq. (39) takes the form

$$\left(\frac{dW}{dx}+a_1W^3+a_2W^2+W\right)+e^{-2\psi/D+2x}\left(\epsilon\frac{dW}{dx}+a_3W^3+a_4W^2+a_5W+a_6\right)=0, \quad \text{(A2)}$$

where

$$x\equiv\ln[(8\pi E)^{1/2}r]\equiv\ln y. \quad \text{(A3)}$$

The origin ($r=0$) corresponds to $x=-\infty$. Hence we will study solutions with ψ finite in the limit as $x\to-\infty$. We first show that W remains finite as $x\to-\infty$. The proof proceeds by showing that the contrary leads to an inconsistency. Thus, assume that $|W|\to\infty$ as $x\to-\infty$. We have the following possibilities:

(i) $a_1\equiv-(\frac{1}{2}\omega c^2-c)\neq0$. For $x\to-\infty$, the terms within the second set of parentheses in Eq. (A2) can be ignored in comparison to the first two terms within the first set of parentheses, hence we obtain

$$\frac{dW}{dx}\approx-a_1W^3, \quad \text{(A4)}$$

or

$$W^2\approx\frac{1}{2a_1x+\text{const}}\to0, \quad \text{(A5)}$$

which contradicts the assumption $|W|\to\infty$ as $x\to-\infty$.

(ii) $a_1=0$. In this case, it is easily shown that for $\omega>0$, and $0<\epsilon\leq1$, we must have

$$c=\frac{3\epsilon-1}{(2\omega+3)+(3\epsilon-1)(\omega+1)}=0. \quad \text{(A6)}$$

Since, from Eq. (41)–(43)

$$a_3=-\frac{(\epsilon-1)(\omega-1)}{2\omega+3}a_1=0, \quad \text{(A7)}$$

$$a_2=2(c+1)=2, \quad \text{(A8)}$$

then we have as $x\to-\infty$

$$\frac{dW}{dx}\approx-2W^2, \quad \text{(A9)}$$

or

$$W\approx\frac{1}{2x+\text{const}}\to0. \quad \text{(A10)}$$

We have shown, for arbitrary a_1, that the assumption $|W|\to\infty$ for $x\to-\infty$ leads to a contradiction; we conclude that W remains finite in this limit.

Hardy[12] has shown that every solution of an algebraic equation of the form

$$\frac{dU}{dx}=\frac{Q(U,x)}{R(U,x)}, \quad \text{(A11)}$$

where Q and R are polynomials in U and x, is ultimately monotonic as $|x|\to\infty$. Equation (A2) becomes an algebraic equation of this type when $x\to-\infty$, and therefore W must approach a limit monotonically as $x\to-\infty$. Thus we have either

$$W\to\text{const},\quad x\to-\infty \quad \text{(A12)}$$

or

$$W\to0,\quad x\to-\infty. \quad \text{(A13)}$$

From (A12)

$$\psi-\psi_0=\int W\,dx\approx\text{const}\times x,\quad x\to-\infty. \quad \text{(A14)}$$

But this type of solution is not bounded as $x\to-\infty$. Therefore, we are left with the case $W\to0$ as $x\to-\infty$.

We are now going to show that condition (A13) and Eq. (A2) imply that We^{-2x} remains finite when $x\to-\infty$. The proof is by contradiction. Thus we assumed that $|We^{-2x}|\to\infty$, and $W\to0$ as $x\to-\infty$. Equation (A2) takes the limiting form

$$\frac{dW}{dx}\approx-W, \quad \text{(A15)}$$

from which it follows that

$$W\approx\text{const}\times e^{-x}\to\infty, \quad \text{(A16)}$$

and we have a contradiction. Therefore We^{-2x} is finite at $x=-\infty$.

Using Eq. (A3), we obtain the relation

$$\frac{d\psi}{dy}=\frac{d\psi}{de^x}=e^{-x}\frac{d\psi}{dx}=e^{-x}W=e^xWe^{-2x}. \quad \text{(A17)}$$

Since We^{-2x} remains finite as $x\to-\infty$, it follows from Eq. (A17) that

$$\frac{d\psi}{dy}\sim\text{const}\times e^x\to0,\quad x\to-\infty. \quad \text{(A18)}$$

Finally, since

$$\frac{d\psi}{dy}=\frac{1}{(8\pi E)^{1/2}}\frac{d\psi}{dr}, \quad \text{(A19)}$$

we have

$$\frac{d\psi}{dr}\to0,\quad x\to-\infty,\quad r\to0.\quad \text{Q.E.D.} \quad \text{(A20)}$$

*Research supported by the Ford Foundation Graduate Fellowships Program for Puerto Ricans.

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