Equations of motion of a Yang-Mills particle

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(Received 24 January 1977)

We discuss the problem of the extent to which the equations of motion for a Yang-Mills particle, in the presence of an external non-Abelian gauge field, can be derived from the field equations. Following an argument due to Dresden and Chen we prove that the equations of motion of a Yang-Mills particle with spin follow from the field equation and the conservation of the energy-momentum tensor. Conservation of angular momentum is explicitly discussed and the pole-dipole character of the Yang-Mills particle, in the sense of Papapetrou, is elucidated.

I. INTRODUCTION

A characteristic feature of classical theories of particles *and* fields is that it is always necessary to postulate two sets of equations. On the one hand, equations for the fields under consideration are postulated, and on the other hand the equations of motion of a test particle in a given external field are given. It is a remarkable fact that nonlinear field equations can, in some cases, be used to derive the equations of motion of a test particle (i.e., as a particle whose field is neglected in the equations of motion for the particle) in a given external field. In the theory of general relativity this is a well-known fact.[1] In some remarkable works by Einstein and co-workers,[2] it was also shown that in a certain sense the equations of motion for particles, which also are sources of the fields, can be derived. This "great Einstein theorem" has recently been discussed by Dresden and Chen.[3] In the same reference it was suggested that Wong's equations,[4] describing a test particle interacting with a non-Abelian external gauge field, can be derived from the field equations. The essential ingredients in the derivation will now be discussed.[5] The field equations can be written in the form

$$D_\mu^{\alpha\beta} F^{\beta\mu\nu} = J^{\alpha\nu} \ , \tag{1}$$

where $F^{\beta\mu\nu}$ is the field tensor of the non-Abelian gauge field A_μ^α, and $D_\mu^{\alpha\beta}$ is a gauge-covariant derivative

$$D_\mu^{\alpha\beta} = \delta^{\alpha\beta}\partial_\mu - g f^{\alpha\gamma\beta} A_\mu^\gamma \ . \tag{2}$$

The particle current density is a minor generalization of the current for a point particle in classical electrodynamics,

$$J^{\alpha\nu}(x) = g \int_{-\infty}^{\infty} d\tau\, I^\alpha(\tau)\dot{\xi}^\nu(\tau)\,\delta^4(x - \xi(\tau)) \ , \tag{3}$$

where τ is an arbitrary parametrization of the world line of the particle and the overdot denotes differentiation with respect to τ. I^α is the classical analog of the non-Abelian charge. We will call it "isospin." We define the following energy-momentum tensor for the particle:

$$T^{\mu\nu}_{(m)}(x) = m \int_{-\infty}^{\infty} d\tau\, \dot{\xi}^\mu \dot{\xi}^\nu \delta^4(x - \xi(\tau)) \ . \tag{4}$$

Neglecting gravitational forces one has the conservation law

$$\partial_\mu (T^{\mu\nu}_{(m)} + T^{\mu\nu}_{(f)}) = 0 \ , \tag{5}$$

where the energy-momentum tensor for the external gauge field, $T^{\mu\nu}$, can be calculated by standard methods. A few, elementary steps then give rise to the Wong equations of motion

$$m\ddot{\xi}_\mu(\tau) = g I^\alpha F^\alpha_{\mu\nu}(\xi(\tau))\dot{\xi}^\nu(\tau) \ , \tag{6}$$

$$\dot{I}^\alpha(\tau) = g f_{\alpha\beta\gamma}\dot{\xi}^\mu A^\beta_\mu(\xi(\tau)) I^\gamma(\tau) \ . \tag{7}$$

II. A SPINNING YANG-MILLS PARTICLE

Recently the Wong equations have been generalized to the case of a spinning particle interacting with a non-Abelian gauge field.[6] The spin of the "pseudoclassical" particle is described by anticommuting variables ψ^μ (Grassmann parameters). Equations (6) and (7) go over into the following equations:

$$m\ddot{\xi}_\mu(\tau) = g I^\alpha(\tau)\left[F^\alpha_{\mu\nu}(\xi(\tau))\dot{\xi}^\nu - \frac{i}{2m}\psi^\rho(\tau) D_\mu^{\alpha\beta} F^\beta_{\rho\sigma}(\xi(\tau))\psi^\sigma(\tau)\right] \tag{8}$$

and

$$\dot{I}^\alpha(\tau) = g f_{\alpha\beta\gamma}\left[A^\beta_\mu(\xi(\tau))\dot{\xi}^\mu(\tau) - \frac{i}{2m}\psi^\mu(\tau) F^\beta_{\mu\nu}(\xi(\tau))\psi^\nu(\tau)\right] I^\gamma(\tau). \tag{9}$$

The spin variables ψ^μ must satisfy the constraint

$$\dot{\xi}_\mu \psi^\mu = 0, \tag{10}$$

which is the classical version of the Dirac wave equation.

These equations of motion can be derived from a supersymmetric Lagrangian, and when quantized, in a suitable manner, they describe a spin-$\frac{1}{2}$ particle interacting with a non-Abelian (external) gauge field.[6]

One can now ask the question whether Eqs. (8) and (9) can also be derived from the field equations. We consider it somewhat remarkable that a minor change of the current (3) is sufficient in order to carry through the same discussion as that by Dresden and Chen and obtain Eqs. (8) and (9). The current density describing the particle with spin and isospin used is

$$J^{\alpha\nu}(x) = g\int_{-\infty}^{\infty} d\tau \left[I^\alpha(\tau)\dot{\xi}^\nu(\tau) + \frac{i}{m}\psi^\mu(\tau)\psi^\nu(\tau) I^\alpha(\tau)\partial_\mu - \frac{ig}{m}\psi^\mu(\tau)\psi^\nu(\tau) f_{\alpha\beta\gamma} A^\beta_\mu(\xi(\tau)) I^\gamma(\tau)\right] \delta^4(x - \xi(\tau)). \tag{11}$$

The second term in the current corresponds, physically, to the effective current due to a magnetic dipole density (see, e.g., Ref. 7) and the last term in (11) is needed to make the current gauge covariant. At a given point, $\xi(\tau)$, it can always be transformed away by a suitable gauge transformation. Under such conditions it is straightforward to show that

$$\int d\sigma_\nu J^{\alpha\nu}(x) = gI^\alpha(\tau), \tag{12}$$

where $d\sigma^\nu$ is a spacelike surface perpendicular to ξ. The isospin is conserved in the sense that

$$\frac{d}{d\tau}(I^\alpha(\tau) I^\alpha(\tau)) = 0, \tag{13}$$

as can be shown from the equations of motion for I^α.

The derivation of the equations of motion (8) and (9) from the field equations will show that the argument due to Dresden and Chen is not completely general; the result depends on the form of the current, which is not *a priori* given.

Analogous to the discussion in Ref. 3 we derive the following equation from the conservation of the total energy-momentum tensor:

$$\int_\Omega d^4x\, J^{\alpha\rho}(x) F^\alpha_{\mu\rho}(x) = \int_{\tau_a}^{\tau_b} d\tau\, \ddot{\xi}_\mu(\tau), \tag{14}$$

where Ω is the finite space-time volume chosen by Dresden and Cnen. Now consider the left-hand side of Eq. (14). The volume integral can easily be evaluated because of the δ functions in the current (11). After some routine manipulations we obtain

$$\int_\Omega d^4x\, J^{\alpha\rho}(x) F^\alpha_{\mu\rho}(x) = \int_{\tau_a}^{\tau_b} d\tau \left[gI^\alpha F^\alpha_{\mu\rho}(\xi)\dot{\xi}^\rho - i\frac{g}{m} I^\alpha \psi^\nu D^{\alpha\beta}_\nu F^\beta_{\mu\rho}(\xi)\psi^\rho \right]. \tag{15}$$

Because of condition (10), a partial integration in (14) has been done without any surface contribution. In (15) we suppress the τ dependence. The following identity now holds:

$$D^{\alpha\beta}_\nu F^\beta_{\mu\rho} + D^{\alpha\beta}_\rho F^\beta_{\nu\mu} + D^{\alpha\beta}_\mu F^\beta_{\rho\nu} = 0. \tag{16}$$

Using Eqs. (15) and (16) we see that

$$m\int_{\tau_a}^{\tau_b} d\tau\, \ddot{\xi}_\mu = \int_{\tau_a}^{\tau_b} d\tau \left[gI^\alpha F^\alpha_{\mu\rho}(\xi)\dot{\xi}^\rho + \frac{ig}{2m} I^\alpha \psi^\nu D^{\alpha\beta}_\mu F^\beta_{\rho\nu}\psi^\rho \right]. \tag{17}$$

Since Eq. (17) is valid for arbitrary τ_a and τ_b we have proved that

$$m\ddot{\xi}_\mu = gI^\alpha F^\alpha_{\mu\rho}(\xi)\dot{\xi}^\rho + \frac{ig}{2m} I^\alpha \psi^\nu D^{\alpha\beta}_\mu F^\beta_{\rho\nu}\psi^\rho,$$

which is Eq. (8).

The field equation (1) furthermore implies that

$$D^{\alpha\beta}_\nu J^{\beta\nu}(x) = 0, \tag{18}$$

i.e., the current density obeys the following constraint:

$$\partial_\nu J^{\alpha\nu}(x) - gf^{\alpha\gamma\beta} A^\gamma_\nu(x) J^{\beta\nu}(x) = 0.$$

Using the explicit form of the current, Eq. (11), we obtain after a simple partial integration

$$g\int_{-\infty}^{\infty} d\tau \left\{ \dot{I}^\alpha - i\frac{g}{m}\psi^\mu\psi^\nu I^\gamma f^{\alpha\beta\gamma} A^\beta_\mu(\xi)\partial_\nu - gf^{\alpha\beta\gamma}\left[I^\beta\dot{\xi}^\nu A^\gamma_\nu(\xi) + \frac{i}{m}\psi^\mu\psi^\nu I^\beta A^\gamma_\nu(\xi)\partial_\mu - \frac{ig}{m}\psi^\mu\psi^\nu I^\epsilon f^{\beta\delta\epsilon} A^\gamma_\nu(\xi) A^\delta_\mu(\xi)\right]\right\} \delta^4(x - \xi) = 0. \tag{19}$$

Equation (19) can be simplified further by noticing the identities

$$f_{\alpha\gamma\beta}f_{\beta\delta\epsilon}+f_{\gamma\delta\beta}f_{\beta\alpha\epsilon}+f_{\delta\alpha\beta}f_{\beta\gamma\epsilon}=0 \qquad (20)$$

and

$$\delta^4(x-\xi)\partial_\mu f(\xi)=f(\xi)\partial_\mu\delta^4(x-\xi)-f(x)\partial_\mu\delta^4(x-\xi), \qquad (21)$$

where $f(x)$ is some function defined on space-time. Routine manipulations give us the expression

$$g\int_{-\infty}^{\infty}d\tau\left[\dot{I}^\alpha-gf_{\alpha\beta\gamma}\dot{\xi}^\mu A_\mu^\beta(\xi)I^\gamma-\frac{ig}{2m}\psi^\mu F^\beta_{\mu\nu}(\xi)\psi^\nu I^\gamma\right]\delta^4(x-\xi)=0. \qquad (22)$$

We have proved that the equation of motion (9) follows from the field equations in the sense of Dresden and Chen, i.e., the equations (1) *and* conservation of the energy-momentum tensor.

III. CONSERVATION OF ANGULAR MOMENTUM

In Ref. 6 we constructed a Lagrangian formalism for a Yang-Mills particle interacting with a non-Abelian gauge field. The Nöther energy-momentum tensor, $T^{\mu\nu}$, for the combined system, scalar particle and field, can be calculated in a standard manner and one finds the expression

$$T^{\mu\nu}(x)=-F^{\alpha\mu\rho}(x)\partial^\nu A^\alpha_\rho(x)+\tfrac{1}{4}g^{\mu\nu}F^\alpha_{\rho\sigma}(x)F^{\alpha\rho\sigma}(x)+\int_{-\infty}^{\infty}d\tau[g\dot{\xi}^\mu A^{\alpha\nu}(\xi)I^\alpha+m\dot{\xi}^\mu\dot{\xi}^\nu]\delta^4(x-\xi). \qquad (23)$$

By adding the total divergence[8]

$$\partial_\rho(F^{\rho\mu}(x)A^\nu(x))$$

to the energy-momentum tensor $T^{\mu\nu}$, we obtain a symmetrized energy momentum tensor, $T_B^{\mu\nu}(x)$, which is conserved:

$$T_B^{\mu\nu}(x)=-F^{\alpha\mu\rho}(x)F^{\alpha\nu}{}_\rho(x)+\tfrac{1}{4}g^{\mu\nu}F^\alpha{}_{\rho\sigma}(x)F^{\alpha\rho\sigma}(x)+\int_{-\infty}^{\infty}d\tau\, m\dot{\xi}^\mu\dot{\xi}^\nu\delta^4(x-\xi). \qquad (24)$$

This expression has been used above in order to derive the equations of motion for a particle from the field equations, i.e.,

$$T_B^{\mu\nu}=T^{\mu\nu}_{(m)}+T^{\mu\nu}_{(f)}.$$

A conserved angular-momentum density can easily be constructed:

$$M_B^{\mu\nu\lambda}(x)=-T_B^{\mu\nu}(x)x^\lambda+T_B^{\mu\nu}(x)x^\nu=-T^{\mu\nu}_{(f)}(x)x^\lambda+T^{\mu\lambda}_{(f)}(x)x^\nu+m\int_{-\infty}^{\infty}d\tau(-\dot{\xi}^\mu\dot{\xi}^\nu\xi^\lambda+\dot{\xi}^\mu\dot{\xi}^\lambda\xi^\nu)\delta^4(x-\xi), \qquad (25)$$

which describes the angular-momentum density of the combined field-particle system. We notice that the conservation law of the density $M^{\mu\nu\lambda}(x)$ is identically fulfilled when Eqs. (1), (6), and (7) are valid.

In the case of a spinning Yang-Mills particle an equation of motion of the spin degrees of freedom can be derived in a fashion similar to that leading to Eq. (17). We consider the conservation law

$$\partial_\mu M^{\mu\nu\lambda}_{(f)}(x)=-\partial_\mu M^{\mu\nu\lambda}_{(m)}(x)=-\partial_\mu\int_{-\infty}^{\infty}d\tau\, m\left(-\dot{\xi}^\mu\dot{\xi}^\nu\xi^\lambda+\dot{\xi}^\mu\dot{\xi}^\lambda\xi^\nu-\frac{i}{m}\dot{\xi}^\mu\psi^\nu\psi^\lambda\right)\delta^4(x-\xi), \qquad (26)$$

where the last term in (26) comes from the contribution due to the intrinsic spin of the particle.

We now integrate Eq. (26) over the finite space-time volume Ω and obtain

$$\int_\Omega d^4x[-J^\alpha_\rho(x)F^{\alpha\rho\nu}(x)x^\lambda+J^\alpha_\rho(x)F^{\alpha\rho\lambda}(x)x^\nu]=m\int_\Omega d^4x\int_{-\infty}^{\infty}d\tau\left(-\dot{\xi}^\nu\xi^\lambda+\dot{\xi}^\lambda\xi^\nu-\frac{i}{m}\psi^\nu\psi^\lambda\right)\frac{d}{d\tau}\delta^4(x-\xi). \qquad (27)$$

By taking the explicit form of the current $J^\alpha_\zeta(x)$ into account and by using the constraint (10) we obtain from Eq. (27) after a partial integration

$$m\int_{\tau_a}^{\tau_b}d\tau\left[\frac{d}{d\tau}(\psi^\nu\psi^\lambda)+\frac{g}{m}I^\alpha F^{\alpha\rho\lambda}(\xi)\psi^\nu\psi_\rho-\frac{g}{m}I^\alpha F^{\alpha\rho\nu}(\xi)\psi^\lambda\psi_\rho\right]=0. \qquad (28)$$

We have therefore derived the following equation of motion of the spin variables:

$$\frac{d}{d\tau}(\psi^\nu\psi^\lambda) = \frac{g}{m} I^\alpha[F^{\alpha\lambda\rho}(\xi)\psi^\nu\psi_\rho - F^{\alpha\nu\rho}(\xi)\psi^\lambda\psi_\rho]. \tag{29}$$

If we introduce the Pauli-Lubanski four-vector W_μ defined by

$$W_\mu = -\tfrac{1}{2} im\epsilon_{\mu\nu\rho\sigma}\dot{\xi}^\nu\psi^\rho\psi^\sigma, \tag{30}$$

where $\epsilon_{0123}=1$, we find that

$$\frac{d}{d\tau}W_\mu = \frac{g}{m}F^\alpha_{\mu\nu}I^\alpha W^\nu, \tag{31}$$

where a term proportional to the current vanishes because of the constraint (10). Equation (31) is a version of the Bargmann-Michel-Telegdi equation.[9] The physical interpretation of equation (31) has been discussed in Ref. 6. where it is shown that this equation can be derived from the supersymmetric Lagrangian mentioned above.

IV. THE ENERGY-MOMENTUM TENSOR OF A SPINNING YANG-MILLS PARTICLE

In the preceding two sections we have shown how one can derive the equations of motion of a spinning Yang-Mills particle from the field equations. The derivation was analogous to that of Dresden and Chen except that we used a more complicated form of the current $J^{\alpha\mu}(x)$. The constraint (10) was also essential in the derivation. A more detailed investigation shows, however, that the energy-momentum tensor of the field-particle system in the case of a spinning particle is more complicated as compared to (24), which we have used in Secs. II and III. Using the Lagrangian formalism in Ref. 6 one can derive the following, symmetric, conserved energy-momentum tensor in the case of a spinning Yang-Mills particle in a non-Abelian gauge field:

$$T^{\mu\nu}_B(x) = -F^{\alpha}{}_{\rho}{}^{\mu}(x)F^{\alpha\rho\nu}(x) - \tfrac{1}{4}g^{\mu\nu}F^\alpha{}_{\rho\sigma}(x)F^{\alpha\rho\sigma}(x)$$
$$+\int_{-\infty}^{\infty} d\tau\left[n\dot{\xi}^\mu\dot{\xi}^\nu + \frac{i}{2}\frac{g}{m}I^\alpha(\psi^\mu F^{\alpha\nu\rho}(\xi) + \psi^\nu F^{\alpha\mu\rho}(\xi))\psi_\rho - \tfrac{1}{2}i(\dot{\xi}^\mu\psi^\nu + \dot{\xi}^\nu\psi^\mu)\psi^\rho\partial_\rho\right]\delta^4(x-\xi). \tag{32}$$

In fact, by taking the divergence of Eq. (32) we obtain, using the equations of motion for the field but not the equations of motion for ξ and ψ,

$$\partial_\mu T^{\mu\nu}_B(x) = \int_{-\infty}^{\infty} d\tau\left[\left(m\ddot{\xi}^\nu - gI^\alpha F^{\alpha\nu}{}_\rho\dot{\xi}^\rho + \frac{ig}{2m}\psi^\rho D^{\alpha\beta\nu}F^\beta{}_{\rho\sigma}\psi^\sigma I^\alpha\right)\right.$$
$$\left. - \frac{i}{2}\psi^\nu\left(\dot{\psi}^\rho - \frac{g}{m}I^\alpha F^{\alpha\rho\sigma}\psi_\sigma\right)\partial_\rho - \frac{i}{2}\left(\dot{\psi}^\nu - \frac{g}{m}I^\alpha F^{\alpha\nu\sigma}\psi_\sigma\right)\psi^\rho\partial_\sigma\right]\delta^4(x-\xi). \tag{33}$$

The vanishing of this divergence is precisely equivalent to the equation of motion for the position ξ and the spin $-i\psi_\mu\psi_\nu$.

At first sight it might now seem to be a contradiction that we have used a different form of the energy-momentum tensor in Secs. II and III in order to derive the equations of motion. By noticing that we are actually integrating the four-divergence of the energy-momentum tensor, it is exactly the constraint equation (10) which makes it possible to disregard the spin term. The condition (10), which in the rest frame of the particle reads $\psi^0=0$, makes the spin terms in the energy-momentum tensor vanish identically. If we therefore, correctly, consider the expression (32) in the derivation in the preceding two sections, it will effectively reduce to the energy-momentum tensor (24). The angular-momentum density $M^{\mu\nu\lambda}$ of the combined field-particle system now reads

$$M^{\mu\nu\lambda}_B(x) = -T^{\mu\nu}_B(x)x^\lambda + T^{\mu\lambda}_B(x)x^\nu \tag{34}$$

and will be identically conserved because of (34) and the symmetry of (32).

From (32) the energy-momentum tensor describing the spinning Yang-Mills particle can be extracted:

$$T^{\mu\nu}_{(m)}(x) = \int_{-\infty}^{\infty} d\tau\left[m\dot{\xi}^\mu\dot{\xi}^\nu + \frac{ig}{2m}I^\alpha(F^{\alpha\mu\rho}(\xi)\psi^\nu - F^{\alpha\rho\nu}(\xi)\psi^\mu)\psi_\rho - \tfrac{1}{2}i(\dot{\xi}^\mu\psi^\nu + \dot{\xi}^\nu\psi^\mu)\psi^\rho\partial_\rho\right]\delta^4(x-\xi). \tag{35}$$

In the terminology of Papapetrou[1] (35) describes a pole-dipole particle because of the presence of the derivative of the δ function. In the presence of a gravitational field the equations of motion of a spinning test particle can then be derived. According to Papapetrou the spin of the particle is defined by

$$S^{\nu\lambda} = \int d^3x[-T^{0\nu}_{(m)}(x)(x^\lambda - \xi^\lambda) + T^{0\lambda}_{(m)}(x)(x^\nu - \xi^\nu)],$$

where the integration is over a volume containing the particle which is shrunk to a point after the integration. One finds that

$$S^{\nu\lambda} = -i\psi^{\nu}\psi^{\lambda}, \tag{36}$$

which corresponds to the spin tensor discussed in Sec. III. General covariant equations of motion, in the presence of a gravitational field, have been obtained by Papapetrou[1] and we do not intend to study them further here.

V. SUMMARY AND CONCLUSIONS

In the present paper we have shown that the derivation, due to Dresden and Chen, of the Wong equations of motion can be extended to the case of a spinning Yang-Mills particle interacting with a non-Abelian gauge field. The field equation (1) and the conservation law for energy and momentum implies the equations of motion of the particle. In this sense the Lorentz-Dirac equation in classical electrodynamics can also be derived. The equation of motion of the spin has also been obtained in two different ways. On the one hand, the conservation of the total angular-momentum density implies the Bargmann-Michel-Telegdi equation and on the other hand this equation can be obtained from the energy-momentum tensor (32). A pole-dipole particle has then been constructed and the Papapetrou equations can be written down.

Finally, we would like to stress that the nonlinearity in the field equations (1) is not essential in our derivation of the equations of motion of the spinning Yang-Mills particle.

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