Pseudoparticles and massless fermions in two dimensions

Curtis G. Callan, Jr.*† and Roger Dashen*
The Institute for Advanced Study, Princeton, New Jersey 08540

David J. Gross‡
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
(Received 26 May 1977)

The role of pseudoparticles in the breakdown of chiral U(N) symmetry is studied in a two-dimensional model. Chiral U(1) is always destroyed by the axial-vector anomaly. For $N = 2$ chiral SU(N) is also spontaneously broken yielding massive fermions and three (decoupled) Goldstone bosons. For $N \geq 3$ the fermions remain massless. Realistic four-dimensional theories are believed to behave in a similar way but the critical N above which the fermions cease to be massive is not known in four dimensions.

I. INTRODUCTION AND SUMMARY OF RESULTS

It has recently been discovered[1-4] that gauge theories exhibit the peculiar phenomenon of tunneling between distinct vacuum states via the pseudoparticle mechanism. The existence of this effect means that the standard perturbation theory vacuum is not really a vacuum state (cluster will be lost). The correct vacuum (θ vacuum) is constructed by an appropriate superposition of naive perturbation theory vacuums.[3,4] Although previous arguments have been more kinematic than dynamic in nature, we have been able to identify[3] a number of interesting ways in which the θ vacuums will have qualitatively new properties.

By far the most interesting of these qualitative features arises when massless fermions are present. Then a vacuum-tunneling event automatically produces fermion pairs of nonzero chirality [hence the breaking of chiral U(1) invariance by the pseudoparticle] and would appear to suppress pure vacuum tunneling. However, a tunneling followed by an antitunneling (to absorb the pair) is not forbidden but does have an amplitude which falls off (owing to the massless fermion propagators) as a power of the separation between the two events. In short, the massless fermions produce a strong long-range correlation, or effective potential, between pseudoparticles. It is therefore necessary to ask whether the qualitative properties of the vacuum are not quite different from what we found in the more general case where the pseudoparticles do not interact significantly.

The answer to this question is both yes and no, and to obtain a clear idea of what is going on it is necessary to go somewhat beyond the rather kinematical weak-coupling arguments so far developed to study this subject. Of course, sufficiently powerful general methods do not at the moment exist and we are obliged to turn for guidance to semisoluble special cases.

The special case which is the subject of this paper and which will turn out to be very instructive is charged scalar electrodynamics in two space-time dimensions. It possesses pseudoparticles (Euclidean Nielsen-Olesen vortices[5]) and, while not soluble, turns out to be quite manageable. In the absence of fermions, and in the weak-coupling limit, the vacuum of the no-fermion model is accurately described as a noninteracting gas of low density (particles are pseudoparticles and the chemical potential is the classical action of a single pseudoparticle). One of our main points is that when N species of massless fermion are added, an effective Coulomb interaction between pseudoparticles appears and the vacuum functional becomes essentially the partition function of a Coulomb gas at temperature $\beta \propto N$. We then make use of existing statistical mechanics arguments[6,7] to show that the system has two quite different phases: a dielectric phase for large N in which the fermion remains massless and vacuum-tunneling effects are strongly suppressed and a conducting, or plasma, phase for small N in which the fermions acquire a spontaneously generated mass and vacuum-tunneling effects are *not* suppressed. The phase transition is such that it must have a four-dimensional analog which could provide a four-dimensional mechanism for breaking not just chiral U(1) but chiral SU(N).

Actually, the phase transition uncovered in this way is more general than the particular problem (massless fermions in two-dimensional models) which called it to our attention. (It also occurs in a number of two dimensional problems[6] in statistical mechanics and in the one-dimensional Ising model with a $|j-j'|^{-2}$ interaction.) Being logarithmic in nature this kind of phase transition is natural in a scale-invariant theory such as four-dimensional quantum chromodynamics. We have, in fact, already suggested[8] that a phase transition of this kind could be responsible for quark confinement. This will be briefly discussed at the end of the paper.

II. TWO-DIMENSIONAL MODEL WITHOUT FERMIONS

The model we will use to study the phenomena mentioned in the introduction is the familiar one of charged scalar electrodynamics in two space-time dimensions. The Lagrangian is

$$\mathcal{L} = \tfrac{1}{4}(\partial_0 A_1 - \partial_1 A_0)^2 + (D_\mu\phi)^*(D_\mu\phi) - V(\phi^*\phi)\,, \tag{2.1}$$

where $D_\mu\phi = (\partial_\mu - ieA_\mu)\phi$ and $V(\phi^*\phi) = -\mu^2\phi^*\phi + \frac{1}{2}\lambda(\phi^*\phi)^2$ with both μ^2 and λ taken positive. The minimum of the potential V is not unique as long as $\mu^2 > 0$ [it occurs for any ϕ such that $|\phi| = \phi_0 = (\mu^2/\lambda)^{1/2}$] and one has the usual degenerate-vacuum–spontaneous-symmetry-breaking problem. The standard treatment assumes that by a choice of gauge one can bring any relevant field configuration to a form which is a small perturbation on $\phi = \phi_0$ everywhere. Then an examination of the small perturbations about this vacuum shows that of the two degrees of freedom of ϕ one is a bona fide massive scalar while the other is a would-be Goldstone boson which, by the Higgs mechanism, combines with the electric field degree of freedom to create a massive "photon" of mass $\mu_w = e\phi_0$. These two particles are manifestly neutral and one usually concludes that there is no way of introducing charged sources into the system because the long-range Coulomb interaction is screened (since the "photon" is massive).

However, as we have recently learned from the study of the topology of gauge theory vacuums, one must consider field configurations which cannot be brought to the form $\phi = \phi_0$ by a nonsingular gauge transformation. This is best understood by considering the vacuum functional integral of the theory in two-dimensional Euclidean space-time. The requirement of finite action imposes only the constraint that on the circle at infinity $|\phi| = \phi_0$ and $\oint dx_\mu A_\mu = 2\pi n/e$ (in a gauge where $A_r = 0$, for instance, this implies that $A_\theta \sim n/er$, $\phi \sim \phi_0 e^{in\theta}$ for large r). The finite-action field configurations thus fall into topologically distinct classes indexed by n and it turns out that to construct a true vacuum functional it is necessary to add together the contributions of all possible classes with phases $e^{in\alpha}$ (α fixed but arbitrary).

The basic configuration out of which we construct everything else is the minimum-energy configuration in the $n = \pm 1$ sector. It is not too hard to see that this is just the Nielsen-Olesen vortex,[5] thought of as a pseudoparticle in 2-dimensional Euclidean space rather than as a soliton in (2+1)-dimensional Minkowski space-time. The vortex solution has a region, which may be centered anywhere, of radius $\sim \mu_w^{-1}$ in which $\partial_0 A_1 - \partial_1 A_0 = E \neq 0$ and $|\phi| \neq \phi_0$. Outside this core region, ϕ and A_μ approach vacuum values exponentially rapidly [in the Landau gauge, if the vortex is centered at $x = 0$, $\phi \sim \phi_0 e^{i\theta}, A_\mu \sim (1/e)(\epsilon_{\mu\nu}x_\nu/x^2)$] and the total action of the configuration is $S_0 \sim \mu^2/\lambda$. There is also an antipseudoparticle with opposite sign for A. Since the region of nonvacuum field is well localized it is easy to construct approximate solutions of any desired n by superposing n_+ pseudoparticles at locations x_i^+ and $n_- = n_+ - n$ antipseudoparticles at locations x_i^- so long as all separations are large compared to μ_w^{-1}. The method of superposition is just to add the pseudoparticle vector potentials $[A_\mu = \sum^{n_+} A_\mu^+(x - x_i^+) + \sum^{n_-} A_\mu^-(x - x_i^-)]$ and to construct ϕ outside the vortex cores according to $|\phi| = \phi_0$, $D_\mu\phi = 0$. This is equivalent to $\phi(x) = \phi_0 \exp(i\int_{x_0}^{x} dx^\mu A_\mu)$, the integral being taken along any path avoiding vortex cores. Since $\oint dx^\mu A_\mu$ is a multiple of 2π this leads to no ambiguity.

It is then easy to construct a reasonable approximation to the true vacuum functional by summing over all such configurations and doing Gaussian functional integration over small perturbations about them. Up to exponentially small corrections the action of n widely separated pseudoparticles is just nS_0 and the θ-vacuum functional is well approximated by

$$\langle\theta|e^{-HT}|\theta\rangle = \sum_{n_+ n_-}^{\infty} \int \frac{\prod^{n_+} d^2x_i^+}{(n_+)!}\,\frac{\prod^{n_-} d^2x_i^-}{(n_-)!}\left(\frac{e^{-S_0}}{V_0}\right)^{n_+ + n_-} e^{i\theta(n_+ - n_-)}\,. \tag{2.2}$$

The $x_i^\pm$ integrations are taken over a finite but large volume LT and V_0 is a normalization factor of order μ_w^{-2} summarizing the result of doing the small fluctuation integral about the multipseudoparticle configuration. Apart from the phase factor $e^{i\theta(n_+ - n_-)}$ this is nothing other than the grand canonical ensemble expression for the partition function of *noninteracting* particles of chemical potential S_0 per particle. The partition function of a noninteracting gas is trivial to evaluate and one finds

$$\langle\theta|e^{-HT}|\theta\rangle \simeq \exp\left(\frac{V}{V_0}\,2\cos\theta\, e^{-S_0}\right)\,, \tag{2.3}$$

the most probable configurations being those where the mean pseudoparticle density is approximately $V_0^{-1}e^{-S_0}$.

Now that vacuum topologies have been properly accounted for, there are some important qualitative features of the physics of the model which differ from expectations based on naive perturbation theory. First of all, in the standard perturbation treatment, there is a sense in which the scalar field has a nonzero vacuum expectation value $\simeq \phi_0$: Although ϕ is gauge variant, one can find a gauge in which, up to small quantum fluctuations, $\phi = \phi_0$. Once one has properly taken account of transitions between different vacuum topologies by summing over the pseudoparticles, it becomes clear that the only possible vacuum expectation value for ϕ is zero. The magnitude of ϕ is ϕ_0 essentially everywhere, but the phase varies randomly in a way which cannot be undone by a nonsingular gauge transformation when one averages over pseudoparticle positions. In the present context, however, nothing of physical interest depends on the vacuum expectation value of ϕ (the heavy photon mass is proportional to $\langle\phi^*\phi\rangle$, which quantity is still approximately equal to ϕ_0^2). On the other hand, the assertion, based on naive perturbation theory, that the system cannot support a long-range Coulomb field and that any charge introduced into the system must be completely screened appears to be falsified by the proper inclusion of pseudoparticles. Indeed, the distinguishing feature of a θ vacuum is that it is a state in which the expectation of $E = \partial_0 A_1 - \partial_1 A_0$ is nonzero. In the noninteracting plasma approximation of the previous paragraph, one finds explicitly $\langle E\rangle = -2\sin\theta\, e^{-S_0} V_0^{-1}$. Since such an electric field must have a nonzero source, it is manifestly possible to introduce incompletely screened charges into the system.

A particularly revealing way of discussing this question is to calculate the energy of two widely separated charges $\pm Q$ in a θ vacuum. By the well-known argument of Wilson this amounts to calculating the θ-vacuum expectation value of $\exp(iQ\oint dx^\mu A_\mu)$, the closed loop being of dimensions large compared to the characteristic lengths of the system. But this just amounts to the construction of a system in a θ vacuum outside the loop and in a $\theta + 2\pi(Q/e)$ vacuum (recall that e is the charge carried by the Higgs field) inside the loop. Since the θ vacuum is characterized by an energy per unit volume, the energy of the loop is proportional to the area of loop, which corresponds by Wilson's argument to a linear or confining potential between external charges Q which are nonintegral multiples of e. Note that if Q is an integral multiple of e, the energy is zero because the vacuum energy is periodic in θ with period 2π. Indeed, any external source whose charge is a multiple of the charge carried by ϕ can obviously be screened by the formation of neutral bound states. Note also that the field set up by noninteger Q (which is a sign of incomplete screening) is a purely quantum effect since E is proportional to $e^{-S_0} \sim e^{-1/\hbar}$. Therefore, even though classical arguments would indicate that any Q is completely screened, the qualitative nature of the screening process is changed by the quantum-mechanical vacuum-tunneling process.

III. MASSLESS FERMIONS

For the reasons outlined in the Introduction we would now like to add to the system just described N identical species of massless fermion, coupled only to the gauge field

$$\Delta\mathcal{L} = \sum_{i=1}^{N} \bar\psi_i \gamma^\mu(\partial_\mu - ieA_\mu)\psi_i \,. \tag{3.1}$$

The vacuum functional integral is now

$$\langle 0|e^{-HT}|0\rangle = \int DA_\mu e^{-S(A)}[\mathrm{Det}(A_\mu)]^N \,, \tag{3.2}$$

where $\mathrm{Det}(A_\mu)$ is the functional determinant of the operator $\gamma^\mu(\partial_\mu - ieA_\mu)$ and $\mathrm{Det}(A_\mu)$ appears raised to the Nth power because we have N independent species of fermion. We still expect the important A_μ configurations to be (at least for weak coupling) superpositions of multiple pseudoparticles and antipseudoparticles, but we expect the properties of the vacuum functional itself to be quite different because of nontrivial dependence of the determinant of $\gamma\cdot(\partial - ieA)$ on pseudoparticle locations. Thus our major problem is to evaluate $\mathrm{Det}(A_\mu)$.

Before turning to that problem we should comment briefly on the symmetry properties of the system. Formally, the N identical massless fermions support a global $\mathrm{U}(N)\otimes\mathrm{U}(N)$ symmetry. We will see that in certain cases there is spontaneous generation of fermion mass which would normally, because of the spontaneous breaking of the chiral symmetry, lead to Goldstone bosons. Since massless bosons are forbidden in two dimensions such mass generation would seem to be forbidden by general principles. The resolution of this paradox is provided by the peculiarly two-dimensional possibility of boson representation of Fermi fields. Each ψ_i may be written as the exponential of a massless boson ϕ_i and the total current $\sum_i \bar\psi_i\gamma_\mu\psi_i$ may be shown to be a function only of the normalized field

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\phi_i = \Phi \,.$$

The other $N-1$ fermion degrees of freedom, independent of $\sum_{i=1}^{N}\phi_i$, remain free and massless even in the presence of A_μ and provide a basis for the symmetry $\mathrm{SU}(N)\times\mathrm{SU}(N)$. The remaining chiral symmetry is just $\mathrm{U}(1)$ and corresponds to the freedom to translate Φ: $\Phi \to \Phi + \alpha$. The chiral

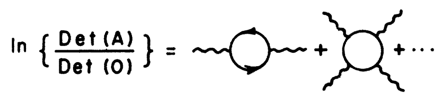

FIG. 1. Graphical expansion of the fermion determinant for an external vector potential. The wavy line stands for the specified external field and the loop is constructed out of free massless fermion propagators.

anomaly automatically reduces this continuous symmetry to a *discrete* translation group which for $N>1$ is enough to forbid a fermion mass term. The point is that the effective determinant interaction generated by the anomaly can easily be seen to depend only on Φ and to have the explicit form $\cos(\pi N)^{1/2}\Phi$. This now supports only the *discrete* translation symmetry $\Phi \to \Phi + 2\sqrt{\pi}\, n/\sqrt{N}$ for integer n. Since a mass term would have the form $\cos\sqrt{\pi}\,\Phi$, this discrete symmetry suffices, for $N>1$, to forbid a mass. The spontaneous symmetry breaking we shall find is a breakdown only of a *discrete* chiral symmetry and generates no Goldstone bosons.

Let us turn now to the evaluation of $\mathrm{Det}(A_\mu)$. For a sufficiently well-behaved A_μ, $\mathrm{Det}(A_\mu)$ has an obvious graphical expansion, illustrated in Fig. 1. For massless fermions in two dimensions the four- (and higher-) point current correlation functions vanish identically. (This is why the Schwinger model is soluble.) Therefore

$$\ln\left(\frac{\mathrm{Det}(A)}{\mathrm{Det}(0)}\right) \equiv -\tfrac{1}{2}e^2 \int dx\, dy A_\mu(x) G_{\mu\nu}(x-y) A_\nu(y)\,. \tag{3.3}$$

The current two-point function, if we include the anomaly term needed to guarantee current conservation, has the value

$$G_{\mu\nu} = \frac{1}{\pi}\, g_{\mu\nu}\, \delta(x) - \frac{1}{4\pi^2}\, \partial_\mu \partial_\nu \ln x^2\,. \tag{3.4}$$

If we adopt the gauge $\partial \cdot A = 0$, only the δ-function term contributes, and we have the simple explicit result

$$\ln\left(\frac{\mathrm{Det}(A)}{\mathrm{Det}(0)}\right) = -\frac{e^2}{2\pi} \int d^2x\, A_\mu{}^2(x)\,. \tag{3.5}$$

An immediate consequence of this result concerns the relative contribution of the different topological classes of A_μ mentioned in Sec. II. Recall that except in the $n=0$ class, A_μ must fall off as r^{-1} for large r $(2\pi n = \oint dx \cdot A)$. Therefore for $n\neq 0$, $\int d^2x A^2$ must diverge logarithmically and $\mathrm{Det}(A)$ must vanish. This is the two-dimensional analog of 't Hooft's discovery[2] that massless fermions have a zero eigenvalue in a topologically nontrivial gauge field configuration and therefore vanishing determinant. It signifies the suppression of vacuum tunneling as an asymptotic process (though not as an intermediate process with finite lifetime) and tells us that in computing the vacuum-to-vacuum amplitude we need only include $n=0$ configurations (equal number of pseudoparticles and antipseudoparticles).

For such configurations we shall adopt the vector field trial function

$$A_\mu(x) = \sum_{i=1}^{n} [A_\mu^{(0)}(x - x_i^+) - A_\mu^{(0)}(x - x_i^-)]\,, \tag{3.6}$$

where $A_\mu^{(0)}$ is the Nielsen-Olesen vortex solution in the $\partial \cdot A = 0$ gauge. In this gauge we may, of course, write $A_\mu^0 = \epsilon_{\mu\nu}\partial_\nu \phi$ with $\phi = (1/e)\ln r\mu_w$ outside the vortex core, $\nabla^2(e\phi) = \rho$, with $\rho = 0$ outside the vortex core, and $\int d^2x\rho = 1$. Then as long as no vortex cores overlap, the quantity $\frac{1}{2}e^2\int d^2x A^2$ is identical to the two-dimensional Coulomb energy of n charges $+1$ at locations x_i^+ and n charges -1 at locations x_i^-. Explicitly,

$$\begin{aligned}\frac{e^2}{2\pi} \int d^2x A_\mu{}^2 &\equiv \sum_{i,j} \ln \mu_w{}^2 (x_i^+ - x_i^-)^2 - \sum_{i>j} \ln\mu_w{}^2 (x_i^+ - x_j^+)^2 \\ &\quad - \sum_{i>j} \ln \mu_w{}^2 (x_i^- - x_j^-)^2 \\ &= U_c(\{x_i^+\}, \{x_j^-\})\end{aligned} \tag{3.7}$$

and

$$\frac{\mathrm{Det}(A)}{\mathrm{Det}(0)} = \exp[-U_c(\{x_i^+\}, \{x_i^-\})]\,. \tag{3.8}$$

Having evaluated the fermion determinant we may now integrate over the relevant gauge field configurations to construct the vacuum functional itself. As in the no-fermion case, we expect that we need only integrate over the locations of pseudoparticles and antipseudoparticles as well as summing over their number (keeping net pseudoparticle number equal to zero). The result for the θ-vacuum energy is, for N massless fermions,

$$\langle \theta | e^{-HT} | \theta \rangle \cong \sum_n \int \prod^n \frac{d^2x_i^+}{(n!)} \prod^n \frac{d^2 x_i^-}{(n!)} \left(\frac{e^{-S_0}}{V_0}\right)^{2n} \exp[-NU_c(\{x_i^+\}, \{x_i^-\})]\,. \tag{3.9}$$

Of course since $n_+ = n_-$ there is no longer any explicit dependence on θ. Also, we never allow vortex cores to overlap in integrating over $x_i^\pm$. More importantly, we see that if in the no-fermion case, the vacuum is essentially a grand canonical ensemble of *noninteracting* particles with chemical potential $V_0{}^{-1}e^{-S_0}$, N species of massless fermion convert the system into a two-dimensional Coulomb gas with the same chemical potential, but a temperature $\beta q^2 = N$. We will see that the qualitative features of the system can be very different from the $N=0$ case and that phase transitions in this classical statistical mechanical system lead to drastic changes in the qualitative physics.

Consider for instance a nearby pseudoparticle-antipseudoparticle pair at locations x and y. As long as they are far from all other pseudoparticles we will have

$$e^{-NU_c} \propto [\mu_w^2(x-y)^2]^{-N}. \tag{3.10}$$

In other words, the larger N, the more strongly the pseudoparticles and antipseudoparticles will clump together in pairs whose mean separation is the order of the vortex core size. Outside each pair, since its net topological charge is zero, the vacuum is indistinguishable from the standard perturbation-theory vacuum and the previously discussed qualitative effects of vacuum tunneling are suppressed. Another way of saying this is that because of the interactions between pseudoparticle and antipseudoparticle, each vacuum-tunneling event is immediately followed by an antitunneling which cancels out any physical effects of either event. In order for the effects of tunneling to be felt the correlation between tunneling and antitunneling should not be too close. We reduce the correlation by decreasing N (increasing T) and the key question is whether a phase transition to a phase in which pseudoparticles and antipseudoparticles are essentially decorrelated occurs for a physically useful value of N ($N>1$).

IV. EQUIVALENT GRAPHICAL METHODS

The classical two-dimensional Coulomb gas may, of course, be studied directly in order to determine the location and nature of its phase transitions. We shall find it convenient to first pass by another route which makes use of the fact that for us, only integer values of N are relevant. In that case we can find an equivalent graphical representation for the partition function which makes explicit the notion that each pseudoparticle is, because of the chiral anomaly, the source of massless fermions in a state of nonzero chirality. The resulting graphical rules will give us a useful intuitive preview of the subject of phase transitions and will allow us to see in advance what new physics arises at a phase transition.

Recall that we found that the fermion determinant in the vector potential corresponding to n pseudoparticles and n antipseudoparticles at locations $\{x_i^+\}$ and $\{x_i^-\}$ was just $\exp[-U_c(\{x_i^+\},\{x_i^-\})]$. Given the explicit form of U_c as a sum of logarithms we may easily show that for n pseudoparticles and n antipseudoparticles

$$(\mu_w^2)^n e^{-U_c} = \frac{[\prod_{i>j}^n (x_i^+ - x_j^+)^2][\prod_{i>j}^n (x_i^- - x_j^-)^2]}{[\prod_{i,j}^n (x_i^+ - x_j^-)^2]}. \tag{4.1}$$

It is well-known from studies of the Thirring model and other two-dimensional models with massless fermions that this rational function of coordinate differences can be reexpressed in a form that has a simple interpretation in terms of fermion Feynman graphs. To construct this expansion, it is best to introduce the notion of a "cycle," an ordered set of $2m$ points $(x_{i_1}^+ x_{i_1}^- x_{i_2}^+ x_{i_2}^- \cdots x_{i_m}^+ x_{i_m}^-)$ which is thought of as invariant under cyclic permutation. The entire set of $2n$ points can be decomposed in a number of ways into a product of cycles. To each cycle we assign a value by the rule

$$\begin{aligned}(x_{i_1}^+ x_{i_1}^- x_{i_2}^+ \cdots x_{i_m}^+ x_{i_m}^-) &= (x_{i_1}^+ x_{i_1}^-)(x_{i_1}^- x_{i_2}^+) \cdots (x_{i_m}^+ x_{i_m}^-)(x_{i_m}^- x_{i_1}^+),\\ (x^+x^-) &= \frac{1}{z^+ - z^-},\\ (x^-x^+) &= \frac{1}{(z^+ - z^-)^*},\end{aligned} \tag{4.2}$$

where $z(x)$ is a complex number constructed out of the two-vector x according to $z(x) = x_1 + ix_0$. To a given decomposition of the $2n$ points into a product of cycles we assign a value equal to the product of the values of each cycle. Then the theorem is that the rational function $(\mu^2)^n e^{-U_c}$ is equal to the sum over all possible decompositions into cycles of the $2n$ points $\{x_i^+\}, \{x_i^-\}$ (values being assigned to each cycle decomposition as above).

It is perhaps helpful to consult Fig. 2 where this theorem is stated pictorially for the case $n=2$. It is not too hard to recognize the vacuum diagrams for multiple insertions of $\bar\psi \frac{1}{2}(1+\gamma_5)\psi$ (corresponding to the x^+) and $\bar\psi \frac{1}{2}(1-\gamma_5)\psi$ (corresponding to the x^-). The oriented propagators (x^+x^-) and (x^-x^+) are nothing more than the Euclidean-rotated and γ_5-projected massless fermion propagators. We will use this graphical interpretation of the partition function to construct a particularly useful interpretation of the Coulomb gas at the special temperatures corresponding to integer N.

Consider now the partition function for $N=1$. The decomposition theorem just described may be used directly: We have to integrate over the $2n$ coordinates $\{x_i^+, x_i^-\}$ and then sum over n. At the same time we have in principle to remember the cutoff instruction that forbids any coordinate difference to be less than twice the vortex radius. The chemical potential and associated normalization factors in Eq. (3.9) give a weight $(\mu_w V_0 e^{S_0})^{-1} \sim (\mu_w^{-1} e^{S_0})^{-1}$

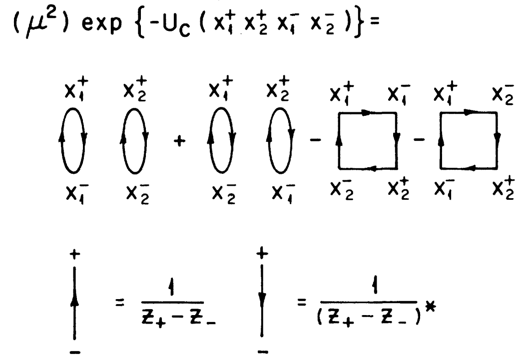

FIG. 2. Graphical interpretation of the contribution of two pseudoparticles and two antipseudoparticles to the partition function in the case of one flavor.

to each of the vertices labeled by $x_i^\pm$ and the various terms in the partition function expansion have an obvious interpretation as multiple insertions of the vertices $(\mu_w V_0 e^{S_0})^{-1}\bar{\psi}\,\frac{1}{2}\,(1 \pm \gamma_5)\psi$ on fermion loops. These operator insertions summarize the effect of the pseudoparticle on the system and we see explicitly in the case $N=1$ that the pseudoparticle (antipseudoparticle) is replaced by an effective interaction $(\mu_w V_0 e^{S_0})^{-1}\bar{\psi}[(1\pm\gamma_5)/2]\psi(\mu_w V_0 e^{S_0})^{-1} \times \bar{\psi}[(1\mp\gamma_5)/2]\psi$ which is precisely the sort of chiral-invariance-breaking term found by 't Hooft in four dimensions.[2] It should be noted that the technique for constructing this effective interaction in four dimensions which relies on finding *normalizable* zero-energy solutions of the Dirac equation would not work here: Although zero-energy solutions can be found, they are not normalizable and there is no clear way of deciding on the correct normalization.

With this simple graphical interpretation of the $N=1$ partition function it is easy to recognize that it is just the exponential of the connected loop graphs constructed according to the same rules (Fig. 3). But the sum of the connected graphs is obviously just the expansion in powers of mass of the vacuum loop graph for a free fermion of mass operator $F(q^2)(\mu_w V_0 e^{S_0})^{-1}$ [where $F(q^2)$ is a momentum-space transform of the position-space cutoff instruction used in integrating over $x_i^\pm$: $F(0)=1$, $F(\infty)=0$, and F passes from 1 to 0 roughly at $q^2=\mu_w{}^2$]. Owing to the cutoff, the loop integral, otherwise logarithmically divergent, is convergent, and there are no divergence difficulties in defining the partition function. More importantly,

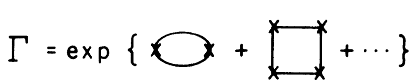

FIG. 3. Resummation of the partition function for one flavor.

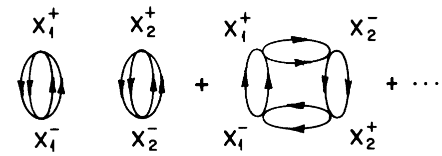

FIG. 4. Graphical interpretation of the two-pseudoparticle–two-antipseudoparticle contribution to the partition function for two flavors.

the fermion clearly behaves as if it has acquired a mass $(\mu_w V_0 e^{S_0})^{-1}$, and all correlation lengths must be finite and roughly of the order $\mu_w V_0 e^{S_0}$, large insofar as S_0 is large, but finite. For a temperature corresponding to $N=1$, the Coulomb gas must therefore be in the plasma phase with a finite screening length of the order of $\mu_w V_0 e^{S_0}$. On the other hand, for large N, as argued before, the system must be in a dielectric phase, with infinite screening length, and there must be some intermediate value of N at which a phase transition occurs.

Insofar as the plasma phase is associated with a nonzero, spontaneously generated fermion mass m_F, we should find that as we increase N from $N=1$, m_F should decrease and at some critical value, N_c, should vanish. For integer N greater than 1 the graphical treatment of the partition function is significantly different from the discussion we have just given. Consider first the case $N=2$. For a given set of pseudoparticles and antipseudoparticle we need the graphical expansion of $[\mu^{2n}\exp\{-U_c(\{x^+\},\{x^-\})]^2$ (i.e., its expansion as a sum of products of fermion propagators). The general rule is easily extracted from the particular example of Fig. 2. The square of that sum of graphs is displayed schematically in Fig. 4 and is clearly the sum of vacuum graphs constructed out of quadrillinear vertices $\bar{\psi}_1(1+\gamma_5)\psi_1\bar{\psi}_2(1+\gamma_5)\psi_2$ for the pseudoparticles and $\bar{\psi}_1(1-\gamma_5)\psi_1\bar{\psi}_2(1-\gamma_5)\psi_2$ for antipseudoparticles. To construct the full partition function, it is necessary to weight each vertex with $(\mu_w{}^2 V_0 e^{S_0})^{-1}$ and sum over numbers and locations of pseudoparticles as before.

The partition function is again the exponential of the sum of connected graphs constructed out of these new vertices. Some of the possible structures are displayed in Fig. 5. Now, of course, the connected graphs have a rather complicated structure, and in particular cannot be interpreted as generating a fermion mass in any obvious way.

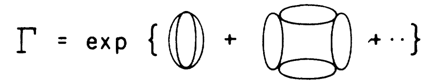

FIG. 5. Resummation of the partition function for two flavors.

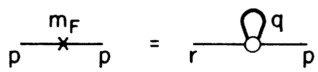

FIG. 6. Graphical expression of the Hartree-Fock equation for two flavors.

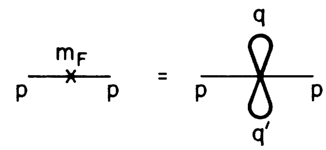

FIG. 7. Hartree-Fock equation for three flavors.

This is in line with our discussion of the chiral-symmetry properties of the effective fermion interaction generated by the pseudoparticles: Although it "breaks" the continuous chiral symmetry, it leaves unbroken a discrete symmetry which suffices to exclude a fermion mass operator. However we still may ask whether this formal discrete symmetry is not spontaneously broken, and a fermion mass generated in a nonperturbative fashion.

The simplest, and probably reasonably accurate approximation to the integral equation for the fermion mass operator is shown in Fig. 6 where the left-hand side is the mass operator, the internal line on the right-hand side is the full fermion propagator including the mass operator, and the vertex is the pseudoparticle-generated effective fermion interaction. The resulting integral equation is

$$m_F(p)=\xi e^{-S_0}\int \frac{d^2q}{(2\pi)^2}\, m_F(q)\,\frac{F(p)F(q)}{q^2+m_F^2(q^2)}, \qquad (4.3)$$

where ξ is a pure number of order 1 and we have included the cutoff function $F(p)F(q)$ in the definition of the vertex. The structure of the equation is such that $m_F(p)=m_F F(p)$ and m_F is either zero (of no interest) or satisfies

$$1=e^{-S_0}\int \frac{d^2q}{(2\pi)^2}\,\frac{F^2(q)}{q^2+m_F^2F^2(q)}$$
$$\simeq e^{-S_0}\int^{\mu_w} \frac{d^2q}{(2\pi)^2}\,\frac{1}{q^2+m_F^2}\,. \qquad (4.4)$$

Because of the infrared divergence as $m_F\to 0$, it is clear that this equation has a solution with $m_F \sim \mu_w \exp(-\xi e^{S_0})$. In other words, for $N=2$, there is a solution with nonzero (although exceptionally small) spontaneously generated fermion mass and the system is still in the plasma phase with finite correlation lengths. On the other hand, the extreme smallness of m_F suggests that we are near the critical point where m_F vanishes and that for $N=3$ or greater we should not be able to find a solution with $m_F\neq 0$.

Indeed, for $N=3$, the same crude approximation to the integral equation for the mass operator yields the system pictured in Fig. 7. Proceeding in the same way as above, we obtain the equation

$$m_F=\xi\,\frac{m_F^2}{\mu_w}\,e^{-S_0}\left[\int \frac{d^2q}{(2\pi)^2}\,\frac{F^2(q)}{q^2+m_F^2F^2(q)}\right]^2. \qquad (4.5)$$

Under the assumption that m_F/μ_w is small, this becomes, approximately

$$e^{S_0}=\left[\frac{m_F}{\mu_w}\,\ln\!\left(\frac{m_F}{\mu_w}\right)\right]^2, \qquad (4.6)$$

which manifestly does not have a solution when $S_0\gg 1$, which is the basic validity criterion for all the approximations we have been making. So, for $N=3$ (and by extension for any $N>3$) we have $m_F=0$ and the system is no longer in the plasma phase.

V. RENORMALIZATION-GROUP METHODS

Needless to say, the two-dimensional Coulomb gas has been studied intensively on its own merits by the statistical mechanics community and we should be able to verify the picture we have developed in Sec. IV by comparison with known results. In this connection we find the work of Kosterlitz[7] most helpful and in this section we would like to paraphrase his results on the two-dimensional Coulomb gas in a way which will, we hope give weight to the picture we extracted from crude diagrammatic arguments.

The Coulomb gas partition function has the explicit form

$$Z=\sum_n \frac{1}{(n!)^2}K^{2n}$$
$$\times\int_\tau \prod^{2n} d^2x_i \exp\left[\,\beta Q^2 \textstyle\sum \epsilon_i\epsilon_j \ln\left|(x_i-x_j)/\tau\right|^2\right], \qquad (5.1)$$

where K is the chemical potential, $\epsilon_i=\pm$ according to the sign of the charge located at x_i, and the integrations are carried out within a large volume V with the instruction that a circle of radius τ around each charge is excluded. For us $\beta Q^2=N$, $K=V_0^{-1}e^{-S_0}$, and $\tau\sim\mu_w^{-1}$, the pseudoparticle radius. On dimensional grounds alone we may say that $Z=Z(V/\tau^2, \tau^2K, N)=Z(V/\tau^2, \bar{K}, N)$. The quantity $\bar{K}$ is the dimensionless chemical potential, and in our case essentially equal to e^{-S_0}. Thus, varying τ, holding $\bar{K}, N$ fixed, is the same as varying V and should cause no change in intensive quantities such as m_F (assuming here to be a thermodynamic limit). On the other hand, Koslerlitz has established rescaling equations, valid for small chemical potential, which show that small changes in τ may be replaced by small changes in the parameters $\overline{K}$ and N. For quantities like m_F which are independent of τ, the rescaling equations therefore define trajectories in $\overline{K}, N$ space along which these quantities are constant.

Kosterlitz's rescaling equations are easily stated in terms of the parameters $y=4\pi\overline{K}$, $x=N-2$ (they are valid only for $y \ll 1$ and $t=\ln\tau$)

$$\frac{dx}{dt}=\tfrac{1}{4}y^2(x+2)^2, \qquad \frac{dy}{dt}=-yx\,. \tag{5.2}$$

These equations have a fixed point at $x=y=0$ and standard renormalization-group lore suggests that the trajectory passing through the fixed point is associated with a phase transition. We already expect a phase transition associated with the vanishing of the spontaneously generated fermion mass somewhere near $N=2$, and the important question is how $N_{\rm crit}$ depends on density. In the immediate neighborhood of the fixed point we may replace the rescaling equations by

$$\frac{dx}{dt}=-y^2, \quad \frac{dy}{dt}=-yx\,. \tag{5.3}$$

The solution of this system which passes through $x=y=0$ is $x^2-y^2=0$ and we find that

$$N_{\rm crit}=2+4\pi\overline{K}>2\,. \tag{5.4}$$

The critical value of N is precisely 2 for zero density, but for small $\overline{K}(\sim e^{-S_0})$, $N_{\rm crit}$ is slightly larger than 2 and the physically interesting value $N=2$ lies in the plasma regime, which is one of the important things we wanted to show.

We may also get an impression of how the spontaneously generated fermion mass varies with N from the following argument. For $N<2$ $(x<0)$ and $y \ll 1$ (small density) we can easily show that to first order in y_0, the solution of the trajectory equations is

$$x(\tau)=x_0, \quad y(\tau)=y_0(\tau/\tau_0)^{-x_0}\,. \tag{5.5}$$

On dimensional grounds the spontaneously generated fermion mass must have the functional form $m_F=(1/\tau)\Phi(x,y)$. Along a trajectory m_F must be independent of τ. Thus

$$\frac{1}{\tau}\,\Phi\left(x_0, y_0\left(\frac{\tau}{\tau_0}\right)^{-x_0}\right)=\frac{1}{\tau_0}\,\Phi(x_0,y_0) \tag{5.6}$$

or

$$\Phi(x,y)=\phi(x)y^{-1/x}\,. \tag{5.7}$$

The interesting thing about this is that it tells us how the spontaneously generated mass depends on the chemical potential for varying N. For $N=1$ $(x=1)$ we see that $m_F \propto (y)^{+1} \propto e^{-S_0}$ exactly as specified by our diagrammatic argument. For $N=2$ we cannot use the above formulas since we are close to the critical point. The Hartree-Fock result that at $N=2$, $\log m_F \propto e^{S_0}$, is not unreasonable, since, according to (5.7) as N approaches 2, m vanishes as a higher and higher power of e^{-S_0}. On the other hand, rescaling arguments do not seem to be powerful enough to prove the precise form of the Hartree-Fock result.

In sum, the rescaling argument of Kosterlitz shows that there is a phase transition at N slightly greater than 2. (The separation from 2 decreases with decreasing density.) For $N>N_{\rm crit}$ the fermion mass is zero (dielectric phase of the gas), while for $N<N_{\rm crit}$ the spontaneously generated fermion mass is nonzero (plasma phase, finite correlation lengths) and varies with N in a manner consistent with the findings of our diagrammatic calculations.

VI. CONCLUSIONS AND SPECULATIONS

Let us now summarize the results of our discussion before attempting to speculate on their meaning for more physically relevant theories. Our first achievement was to extract explicitly the basic effective interaction between the pseudoparticle and the massless fermions. Just as in four dimensions, the pseudoparticle necessarily creates massless fermion pairs of nonzero chirality (as many pairs as there are flavors of fermion). The technique of evaluating this vertex by finding zero-energy solutions of the Dirac equation does not work in two dimensions (the zero-energy solution exists, but it is not normalizable) and we must resort to special trickery. This then allows us to construct graphical rules for the vacuum functional and establish that it is identical to the partition function of the classical Coulomb gas at special temperatures.

From the structure of the partition function it is clear that adding massless fermions to the original Higgs model changes the physics of the pseudoparticle gas representation of the vacuum quite drastically. Without fermions, the pseudoparticles behave as statistically independent noninteracting particles. With massless fermions there are strong forces between pseudoparticles and antipseudoparticles and in the large-N limit one must expect pseudoparticles to be tightly bound to antipseudoparticles in pairs. Outside such a pair the fields are those of a non-pseudoparticle vacuum, and for all practical purposes, the pseudoparticles have no effect on the system. For physical effects of the pseudoparticle to manifest themselves some finite fraction of the bound pairs must "ionize." Since the binding potential is logarithmic (and therefore rises without limit) ionization per se is not possible, but by increasing the temperature (decreasing N) one causes the mean separation of the pairs to increase and one may expect that at some critical N, the pseudoparticles become sufficiently uncorrelated to reestablish significant physical effects.

As a sign of this phase transition, we focus on the mass of the fermion. The fermions are added to the system with zero mass but the possibility remains that spontaneous breaking of chiral symmetry may cause m_F to be nonzero. If $m_F \neq 0$, then the long-range correlations between pseudoparticles and antipseudoparticles which suppressed their effects on the physics go away. For $m_F \neq 0$ all correlations are short range, and the pseudoparticles reappear as a significant dynamical variable. By a combination of diagrammatic and statistical mechanical arguments we find that indeed, for $N=1$ and $N=2$, m_F is nonzero and the original chiral symmetry of the theory is violated spontaneously. For $N=1$, this symmetry breaking is identical to that arising from the chiral anomaly via the pseudoparticle, while for $N=2$ it is a purely dynamical phenomenon happening on top of the anomaly. It signifies that despite their binding by an infinite-range potential, the pseudoparticle pairs have partially ionized and their constituents behave in an uncorrelated fashion. For $N>2$ this does not happen: The fermion mass is zero.

The existence of two regimes in the number of fermion flavors—one like the no-fermion case because the pseudoparticles are in an uncorrelated plasma phase, the other like old-fashioned perturbation theory because the pseudoparticles are so tightly bound to antipseudoparticles that it is as if there were no pseudoparticles at all is of interest in its own right in order to understand the physics of the two-dimensional gauge theory. More interesting are the suggestions it makes about various aspects of four-dimensional gauge theories. Consider for instance the question of spontaneous breaking of chiral SU(N) flavor in four dimensions. With N massless fermion flavors, the chiral anomaly only breaks the chiral U(1), leaving a chiral SU(N) and leaving open the possibility that dynamical effects further break this down to ordinary SU(N). In the "dilute pseudoparticle gas" approximation, the diagrammatic picture is nearly identical to that presented in Sec. IV. For $N=1$, the fermion directly acquires a mass from the anomaly. For $N \gg 1$, clearly the pseudoparticles and antipseudoparticles will be tightly bound in pairs, there will be no surviving effects of pseudoparticles and m_F will be zero. Somewhere in between there clearly must be a phase transition. If it happens for $N>2$, then there will be at least one case ($N=2$) in which the pseudoparticles not only solve the U(1) problem, but are responsible for the dynamics of ordinary chiral symmetry breaking. In four dimensions we would expect Goldstone bosons which do not decouple. However, the four-dimensional pseudoparticle gas plus fermion system is not equivalent to anything as simple as the Coulomb gas and we do not have a quick way of estimating N_{crit}.

Finally we note that the key feature behind the phase transition discussed above is not the dimensionality of space but that the potentials are logarithmic. In an earlier paper[8] we pointed out that in four dimensions a pseudoparticle can split into two half pseudoparticles (merons) at a cost in action which is proportional to the logarithm of the distance between the merons at small coupling (low temperature). The merons are permanently pairwise bound into pseudoparticles. However, because the potential is only logarithmic it is likely that at moderate or large coupling (high temperature) there will be a phase transition in which the merons become free and the quarks become confined. Whatever its intrinsic merits, the theory studied here is an interesting model for the phase transition which we believe to be responsible for the dynamics of confinement in four dimensions.

One of us (CGC) gratefully acknowledges the hospitality of the Laboratoire de Physique Theorique of the Ecole Normale Superieur, where part of the work reported here as done.

---

*Research sponsored in part by ERDA under Grant No. E(11-1)2220.

†On leave from Princeton University, 1976–77.

‡Research sponsored in part by the National Science Foundation under Grant No. MPS75-22514.

[1]A. Belavin *et al.*, Phys. Lett. 59B, 85 (1975).

[2]G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976).

[3]C. G. Callan, R. Dashen, and D. Gross, Phys. Lett. 63B, 334 (1976).

[4]R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976).

[5]H. Nielsen and P. Olesen, Nucl. Phys. B61, 45 (1972).

[6]J. Kosterlitz and D. Thouless, J. Phys. C 6, 1181 (1973).

[7]J. Kosterlitz, J. Phys. C 7, 1046 (1974).

[8]C. G. Callan, R. Dashen, and D. Gross, Phys. Lett. 66B, 375 (1977).