Path-integral quantization and cosmological particle production: An example*

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The Feynman path-integral method is applied to the quantization of a scalar field moving in a cosmological background spacetime. The method is illustrated by computing particle production from the vacuum in a spatially flat, Robertson-Walker spacetime with scale factor $R(t) = t$. The result is a distribution of produced particle pairs which at large energies becomes a thermal distribution with a temperature $T = \hbar c/\pi k_B R(t)$. The relation to other methods of quantization is discussed.

I. INTRODUCTION

The process of pair creation may play an important role in the dynamics of the early universe.[1] As a consequence the problem of the production of particles by the dynamical background geometry of cosmological models has been much studied.[2] To determine the amount of particle production in such models one must calculate the overlap amplitude between the initial state of the universe where there may or may not have been particles present and all possible states at late times containing different numbers of particles. The difficulty with doing this lies not so much with solving the quantum-mechanical equations of motion as it does in determining what these states are.

Determining the particle states defined by unaccelerated observers at *late* times is a comparatively simple matter if the universe becomes nearly homogeneous and isotropic and the curvature becomes arbitrarily slowly varying on the scales on which measurements are made. We shall call such a time domain an adiabatic region. The properties of an adiabatic region are close to those of flat space. In particular, one can find WKB solutions of the particle wave equations which have either purely positive or purely negative frequencies throughout the adiabatic region, and which therefore give rise to a natural definition of particle states.[3]

Much more problematical is the definition of the initial particle state of the universe. If the universe had an initial adiabatic region, then one could define initial particle states in a way similar to that discussed above for late times. One could then calculate, for example, the transition amplitude between the initial vacuum and states at late times containing definite numbers of particles. Unfortunately, in the cases of greatest interest the universe does not have an initial adiabatic region but rather an initial singularity. What is the analog of the initial vacuum in these cases? Some approaches to this question have involved procedures which are equivalent to abandoning Einstein's equations and their consequent singularity before some early time t_0. Instead, the universe is joined either smoothly or suddenly, onto an adiabatic regime before t_0. Initial particle states can then be defined, and the transition amplitudes to states with definite numbers of particles at late times can be computed. Results which follow from procedures of this type, however, will depend on the choice of t_0 even if they are insensitive to the *ad hoc* choice of the geometry prior to this time. Another physical principle is therefore needed to fix this transition time.

In this paper we shall present another approach to the problem of defining the initial particle states in universes with singularities which does not involve changing the dynamics of the universe at early times. The approach is a natural extension to cosmological background geometries with initial singularities of the complexified spacetime Feynman path-integral method already applied to particle production in black-hole geometries,[4] de Sitter space,[5] and the Taub-NUT (Newman-Unti-Tamborino) geometry.[6]

To illustrate the application of the Feynman path-integral method to cosmological particle production, let us consider a homogeneous isotropic universe with an initial singularity and calculate the amplitude that a pair of particles are produced in the time since the initial singularity and detected at spacetime points x and x' at very late times. Call the amplitude for this process relative to the amplitude that no particles are produced at all the propagator $K(x,x')$. In Feynman's prescription this is given by

$$K(x,x') \sim \sum_{\text{paths}} e^{iS(\text{path})/\hbar}, \tag{1.1}$$

where S is the classical action for the path. The class of paths which contribute to this sum are illustrated in Fig. 1. They are all paths which connect x and x' and which lie to the future of the initial singularity. Each path is specified by giving the spacetime coordinates as a function of a parameter time. The sum which occurs in Eq. (1.1) is first over all paths which connect x and x' in a definite total parameter time and then over all positive values of this total parameter time.

To give a quantitative meaning to an expression like Eq. (1.1), the sum over paths must be defined. In this paper we will not attempt a totally mathematically rigorous discussion of this question. Rather, we will give a series of steps which will lead us from a formal expression for Eq. (1.1) in terms of an iterated integral to a well-defined procedure for calculating $K(x,x')$ by solving a differential equation with prescribed boundary conditions. We hope that these steps can be given a firm justification in the future. The details of this discussion will be given in Secs. II and III, but the essential points can be summarized here. Expressed as an iterated integral the integrals involved in defining Eq. (1.1) are not well defined because of the oscillatory character of the integrand. We therefore deform the contours of the integration to a domain of complex coordinates where the imaginary part of the action is positive. The integrals are then exponentially damped and the path integral in the complex section can be defined. Its value can be obtained by evaluating the iterated integral or equivalently by solving a differential equation with associated boundary conditions which the iterated integral implies. The physical propagator $K(x,x')$ is the propagator in this complex section analytically continued back to real values of the spacetime coordinates.

The great advantage of this procedure is that the singularity is handled in a natural way. *Ad hoc* specifications of the initial vacuum are replaced by a simply interpretable restriction of the paths summed to those to the future of the initial singularity. Our justification for this path-integral formulation is twofold: First, a similar procedure applied in flat spacetime (Euclidean quantum field theory plus the Euclidean postulate) yields the correct flat spacetime quantum field theory. Second, when this procedure is applied to black-hole geometries with singularities, one obtains[4] in a natural way the thermal radiation already calculated by more standard techniques.

The creation of particles in the time interval since the singularity can also be discussed in the language of fields and states. The sum-over-paths prescription described above defines in a natural way initial and final particle vacuum states for the universe. With this definition states with various numbers of particles already present at the initial time can be defined and the particle creation from these states calculated. No principle to single out any one of these states or any mixture of them as the physical state of the universe is advocated here. We will, however, concentrate on the calculation of the particle production from the vacuum as it is the basic calculation from which the others can be derived.

In the following sections, in addition to putting forward these ideas in greater detail, we will also illustrate them by calculating the production of scalar particles in a particularly simple cosmological model, chosen not so much for its relevance to our own universe, but because the procedure outlined above can be carried out simply and directly. In this example we will not evaluate $K(x,x')$ by carrying out the sum over paths directly. Rather we shall solve the differential equation with associated boundary conditions which are implied by the path-integral expression in the complex section. It would be possible to regard this differential equation and associated boundary conditions as the *starting point* for a definition of the propagator rather than as the consequence of the path-integral prescription as we have done here. Similarly, but alternatively, one could begin with the wave equation for the propagator, continue to a suitable complex section, impose appropriate boundary conditions, and continue back to the real section to define $K(x,x')$. Thus an analytic-continuation algorithm for defining the propagtor could be given without resorting to a formulation of quantum mechanics in terms of path integrals. As yet missing from this type of approach is any physical justification for identifying a particular complex section and for singling out the boundary conditions to be imposed there on those differential equations which determine $K(x,x')$. It is just this physical justification which is supplied in the path-integral formulation by the analysis of which paths contribute to the sum.

In the next section we shall review the details of

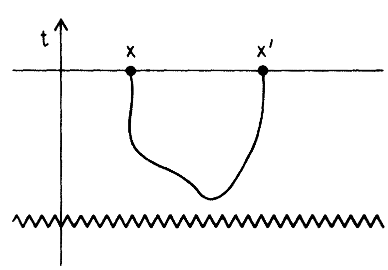

FIG. 1. A typical path corresponding to the production of a pair of particles in the early universe and their detection at x and x' at a late time t.

the path-integral quantization method. In Sec. III we shall calculate the propagator in the simple cosmological model. In Sec. IV the amplitude to produce a particle pair in this model will be deduced. In Sec. V we shall explore the connection with the usual quantum-field-theory methods, and in particular identify the initial particle state discussed above. Section VI contains some brief conclusions.

II. PATH-INTEGRAL QUANTIZATION IN COSMOLOGIES

In the quantum mechanics of a free particle moving in a curved background spacetime a central quantity is the propagator $K(x, x')$. A knowledge of the propagator allows all other physical questions to be answered. In this section we will briefly review, following the discussion of Ref. 4, the procedure for implementing quantitatively the path-integral definition of $K(x, x')$ contained qualitatively in Eq. (1.1). For simplicity we will restrict our attention to scalar particles of rest mass m.

The path-integral prescription for the propagator begins by specifying the amplitude $F(W, x, x')$ for the particle to propagate between two points x' and x both to the future of the initial singularity in a total parameter time W. Formally this is

$$F(W, x, x') = \int \delta x(w) \exp\left(\frac{i}{4}\int_0^W g(\dot{x}, \dot{x}) dw\right). \tag{2.1}$$

The integral is over all paths $x(w)$ which connect x' at $w=0$ to x at $w=W$ and which lie to the future of the initial singularity. The quantity g is the metric on the spacetime and $\dot{x} = dx/dw$. Since the parameter time W is not an observable, the propagator $K(x, x')$ to go from x' to x is a sum over all possible parameter times W with a weight determined so that $K(x, x')$ becomes the usual field theory propagator in flat spacetime,

$$K(x, x') = \lim_{\epsilon \to 0} i \int_0^\infty dW \exp\left(-\frac{\epsilon}{W} - im^2 W\right) \times F(W, x, x'). \tag{2.2}$$

The convergence factor has been inserted so that the integral converges at $W=0$. This is essentially requiring that the particle propagate forward in parameter time. The choice of weights in Eqs. (2.1) and (2.2) is the natural generalization of those which give the usual theory of relativistic scalar particles in flat spacetime. They have no other *a priori* justification.

More concretely, although still formally, Eq. (2.1) may be interpreted as an iterated integral of the propagator $F(e, x, x')$ to go between x' and x in a small parameter time e:

$$F(W, x, x') = \lim_{N\to\infty} \int d^4x_N[-g(x_N)]^{1/2} \cdots \int d^4x_1[-g(x_1)]^{1/2} F(e, x, x_N) F(e, x_N, x_{N-1}) \cdots F(e, x_1, x'). \tag{2.3}$$

Here $(N+1)e = x - x'$ and the integrals extend over the domain to the future of the initial singularity. For very small e, and at least for x and x' sufficiently close together, $F(e, x, x')$ is prescribed to be

$$F(e, x, x') \sim Y(e, x, x') \exp\left(\frac{i}{4}\int_0^e g(\dot{x}, \dot{x}) dw\right), \tag{2.4}$$

where $Y(e, x, x')$ is an appropriate real weight.

As it stands the iterated integral in Eq. (2.3) is not convergent if only because $F(e, x, x')$ does not fall off fast enough at large separations between x and x'. The expression can be given meaning by analytically continuing W, x, x' and distorting the contours of integration in Eq. (2.3) and Eq. (2.4) to a complex domain where the integrals do converge. The value of F for physical values of W, x, x' is then ***defined*** to be the function F in this complex section analytically continued back to real values of its arguments.

In the next section we will illustrate this procedure with a model cosmology for which the analytic continuation can be accomplished by a simple rotation of the x coordinates to values χ for which the analytically continued metric $\gamma(\chi, \chi')$ is real and positive definite. For such cases it is appropriate to rotate W, w, and e by an angle $-\pi/2$. Writing Ω, ω, and ϵ for the rotated values, Eq. (2.3) becomes

$$F(\Omega, \chi, \chi') = \lim_{N\to\infty} \eta \int d^4\chi_N[\gamma(\chi_N)]^{1/2} \cdots \int d^4\chi_1[\gamma(\chi_1)]^{1/2} F(\epsilon, \chi, \chi_N) F(\epsilon, \chi_N, \chi_{N-1}) \cdots F(\epsilon, \chi_1, \chi'), \tag{2.5}$$

where η is a possible phase arising from the rotations. The integrals over χ range over the domain arrived at by rotating into the complex section the real domain lying in the future of the initial singularity. Equation (2.5) is the functional integral

$$F(\Omega, \chi, \chi') = \eta \int \delta\chi(\omega) \exp\left(-\tfrac{1}{4}\int_0^\Omega \gamma(\dot{\chi}, \dot{\chi}) d\omega\right), \tag{2.6}$$

where χ means $d\chi/d\omega$, and the integration is over all paths connecting χ' and χ in the domain just described.

Equations (2.6) and (2.5) define $F(\Omega, \chi, \chi')$. For small Ω the exponent in Eq. (2.6) will be large. If χ and χ' are located so that there is a unique stationary path which connects them, then this stationary (geodesic) path will give the overwhelming contribution to the path integral. For χ and χ' located in this way we can write for small ϵ

$$F(\epsilon, \chi, \chi') \sim \eta Y(\epsilon, \chi, \chi') \exp[-s(\chi, \chi')/4\epsilon] , \quad (2.7)$$

where $s(\chi, \chi')$ is the square of the geodesic distance between χ and χ' and $Y(\epsilon, \chi, \chi')$ is an appropriate weight. The various choices of weight have been discussed by the DeWitts[7] among others and lead to theories of scalar particles in curved spacetime with different amounts of coupling to the scalar curvature. No principle is advocated here for singling out any one of these theories over any other. The path-integral method does not by itself single out a particular theory since the different possible theories can be represented by different choices of the real weight Y. For illustrative purposes we choose the weight which leads to a conformally invariant theory of massless particles.

$$Y(\epsilon, \chi, \chi') = (2\pi)^{-2}[\gamma(\chi)\gamma(\chi')]^{-1/4} D^{1/2}(\chi, \chi') , \quad (2.8)$$

where D is the Van Vleck–Morette determinant

$$D(\chi, \chi') = (4\epsilon)^{-4} \det\left[\frac{\partial^2 s(\chi, \chi')}{\partial\chi^\alpha \partial\chi'^\beta}\right]. \quad (2.9)$$

This choice considerably simplifies the example which will be presented in Sec. III.

Equations (2.5) through (2.9) are sufficient to show (see e.g., Ref. 4) that the function F satisfies a parabolic differential equation

$$\frac{\partial F}{\partial \Omega} = [\gamma(\tilde{\nabla}, \tilde{\nabla}) - \tfrac{1}{6}{}^4R] F , \quad (2.10)$$

where $\tilde{\nabla}$ is covariant differentiation with respect to the metric γ and ${}^4R$ is the scalar curvature. Solving Eq. (2.10) will often prove a more convenient way of evaluating $F(\Omega, \chi, \chi')$ than calculating the iterated integral. The boundary conditions on Eq. (2.10) which single out the solution $F(\Omega, \chi, \chi')$ defined by the path integral are first that, for small values of Ω and χ and χ' located so that they are connected by a unique geodesic, the function F should approach the value given by Eqs. (2.7)–(2.9). In particular,

$$F(0, \chi, \chi') = \eta \delta^{(4)}(\chi, \chi') . \quad (2.11)$$

where $\delta^{(4)}(\chi, \chi')$ is the four-dimensional δ function on the complex section. Second, as the separation between χ and χ' approaches infinity,

$$F(\Omega, \chi, \chi') \to 0 . \quad (2.12)$$

This is implicit in the definition of the path integral as an iterated integral.

The procedure outlined above for defining the propagator in a curved spacetime is the natural generalization of similar quantization procedures in flat spacetime.[8] In the path-integral method one is simply specifying the amplitude for a particle to travel a particular path. Other amplitudes follow from this through the quantum-mechanical law of addition of amplitudes. Specifically, in a cosmology, the amplitude to produce a pair from the analog of the vacuum is given by summing over paths which lie to the future of the initial singularity. The restriction of the sum to this class of paths is equivalent to specifying the initial vacuum state.

In order for such a procedure to have meaning it is necessary that the resulting path integrals be given at least a precise-enough definition that one can compute them. Definition through analytic continuation as employed here is the same procedure which is used either implicitly or explicitly in flat spacetime theories to accomplish this objective.

In Sec. III we shall apply this procedure to calculate the propagator $K(x, x')$ in a simple cosmological model, and in Sec. IV we shall use this propagator to calculate the particle production in this model. The path-integral formulation of these questions has an obvious advantage over the usual field-theoretical methods—it is not necessary to prescribe initial conditions at the singularity. The initial conditions in fact emerge naturally in the calculation from the restriction of the class of paths summed over and are discussed in Sec. V. The one disadvantage of the method is that to actually perform the analytic continuation necessary to calculate F, it appears at present necessary to have an explicit solution to Eq. (2.10). It is for this reason that we have restricted our detailed attention to the simple and unrealistic model presented in Sec. III. Mathematical ingenuity may yet overcome this difficulty.

III. $R(t) = t$ COSMOLOGY

We shall now apply the Feynman path-integral method described in the last section to the propagation of a scalar particle moving in the Robertson-Walker background geometry with scale factor $R(t) = t$. The metric is

$$ds^2 = -dt^2 + t^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] . \quad (3.1)$$

This metric has a curvature singularity at $t = 0$ with the scalar curvature ${}^4R$ given by $6/t^2$. We choose this example not because it is especially relevant to any physical problem, but rather because it gives us a model cosmology with an initial singularity in which we can carry out our procedure exactly.

To begin, we rotate all four coordinates by an angle $\pi/2$ in the complex plane and write $t=i\lambda$, $x^i=i\chi^i$. Real values of λ and $\vec{\chi}$ then define the complex section on which the real positive metric γ is

$$d\sigma^2 = d\lambda^2 + \lambda^2[(d\chi^1)^2 + (d\chi^2)^2 + (d\chi^3)^2]\,. \tag{3.2}$$

The region $t>0$ lying to the future of the initial singularity is thus rotated into the region $\lambda>0$ in the complex section. The range of all integrations in the iterated integral [Eq. (2.5)] is then $0\leq\lambda\leq\infty$, $-\infty\leq\chi^i\leq+\infty$.

The square of the geodesic distance between two points $(\lambda,\vec{\chi})$ and $(\lambda',\vec{\chi}')$, which is the important quantity for determining the small-Ω behavior of F, is easily evaluated in the present example. Because the subspaces $\lambda=$ const are homogeneous and isotropic, $\vec{\chi}$ and $\vec{\chi}'$ can both be taken to lie on one of the spatial coordinate axes, say χ^1, and the geodesic connecting them must also run along this axis. The metric restricted to the two-dimensional subspace spanned by χ^1 and λ is locally the metric of a plane written in polar coordinates with λ the polar radius and χ^1 the polar angle. The geodesics are thus just the straight lines of a flat two-dimensional plane written in polar coordinates, and the square of the geodesic distance is

$$s(\lambda,\lambda',\rho) = \lambda^2 + \lambda'^2 - 2\lambda\lambda'\cos\rho\,. \tag{3.3}$$

where

$$\rho = |\vec{\chi}-\vec{\chi}'| = \left(\sum_i (\chi^i-\chi'^i)^2\right)^{1/2}. \tag{3.4}$$

This expression is valid only for $\rho<\pi$. For $\rho\geq\pi$ there is no geodesic lying in the space $\lambda>0$ connecting the two points. The path of shortest distance for $\rho\geq\pi$ is a curve of constant $\vec{\chi}$ from $(\lambda',\vec{\chi}')$ to the singularity, a curve of zero length along the singularity, and a curve of constant $\vec{\chi}$ from the singularity to $(\lambda,\vec{\chi})$. The length of this curve is always $\lambda+\lambda'$ independent of ρ. Thus, only for $\rho<\pi$ is there a unique geodesic connecting $(\lambda,\vec{\chi})$ and $(\lambda,\vec{\chi}')$ and only for this range of ρ do we require the small-Ω behavior determined by Eqs. (2.7)–(2.9). Explicitly, for $\rho<\pi$ and small ϵ

$$F(\epsilon,\lambda,\lambda',\rho)\sim\eta\frac{1}{(4\pi\epsilon)^2}\frac{\sin\rho}{\rho}\times\exp\left(\frac{1}{4\epsilon}(\lambda^2+\lambda'^2-2\lambda\lambda'\cos\rho)\right). \tag{3.5}$$

It is important to note that this behavior is nonsingular both at $\rho=0$ and also at the singularity at λ or $\lambda'=0$.

Equation (3.5) together with the boundary conditions of Eqs. (2.11) and (2.12) determine $F(\Omega,\chi,\chi')$ as a solution of the differential equation Eq. (2.10). To find this solution it is first convenient to decompose the spatial dependence of $F(\Omega,\chi,\chi')$ into plane waves. Making use of the spatial homegenity and isotropy we write

$$F(\Omega,\chi,\chi') = (2\pi)^{-3}\int d^3\kappa\, e^{i\vec{\kappa}\cdot(\vec{\chi}-\vec{\chi}')}F_\kappa(\Omega,\lambda,\lambda')\,. \tag{3.6}$$

where $\kappa=|\vec{\kappa}|$. Further, it is convenient to make a Laplace transformation of the Ω dependence and write

$$F_\kappa(\Omega,\lambda,\lambda') = \int_0^\infty dp^2 e^{-p^2\Omega}F_{p\kappa}(\lambda,\lambda')\,. \tag{3.7}$$

Equation (2.10) then implies the following differential equation for $F_{p\kappa}(\lambda,\lambda')$:

$$\frac{1}{\lambda^3}\frac{d}{d\lambda}\left(\lambda^3\frac{dF_{p\kappa}}{d\lambda}\right)+\left(p^2+\frac{1-\kappa^2}{\lambda^2}\right)F_{p\kappa}=0\,. \tag{3.8}$$

Two linearly independent solutions of this equation are $\lambda^{-1}J_\kappa(p\lambda)$ and $\lambda^{-1}J_{-\kappa}(p\lambda)$, where J_ν is the Bessel function of order ν. Respectively, these behave at $\lambda=0$ like $\lambda^{-1+\kappa}$ and $\lambda^{-1-\kappa}$. In order to have the regular behavior at $\lambda=0$ for small Ω and $\rho<\pi$ that is dictated by Eq. (3.5), only the solutions which behave like $\lambda^{-1+\kappa}$ near $\lambda=0$ can be incorporated in Eq. (3.7). The symmetry of $F(\Omega,\chi,\chi')$ in χ and χ' (see Ref. 4 for a demonstration) then implies that $F_{p\kappa}(\lambda,\lambda')$ is proportional to

$$(\lambda\lambda')^{-1}J_\kappa(p\lambda)J_\kappa(p\lambda')\,.$$

The proportionality factor can be evaluated from Eq. (2.11) either using the identity (Ref. 9, p. 456)

$$\int_0^\infty p\,dp\,J_\nu(p\lambda)J_\nu(p\lambda') = \delta(\lambda-\lambda')/\lambda\,, \tag{3.9}$$

or directly using Eq. (3.11) below and taking the limit $\Omega\to0$. Since the phase $\eta=i$ for the present rotations the result for $F_{p\kappa}$ is

$$F_{p\kappa}(\lambda,\lambda') = i(\lambda\lambda')^{-1}J_\kappa(p\lambda)J_\kappa(p\lambda')\,. \tag{3.10}$$

Inserting this in Eq. (3.7), doing the integral using the identity on p. 395 of Ref. 9 and inserting the result in Eq. (3.6), one finds

$$F(\Omega,\chi,\chi') = \frac{i}{(2\pi)^3}\frac{1}{2\Omega\lambda\lambda'}e^{-[(\lambda^2+\lambda'^2)/4\Omega]}\times\int d^3\kappa\, e^{i\vec{\kappa}\cdot(\vec{\chi}-\vec{\chi}')}I_\kappa\left(\frac{\lambda\lambda'}{2\Omega}\right). \tag{3.11}$$

Equation (3.11) can be simplified if use is made of the integral representation (Ref. 9, p. 181)

$$I_\nu(z)=\frac{1}{2\pi i}\int_C d\alpha e^{z\cosh\alpha-\nu\alpha} \tag{3.12}$$

where the contour C starts at $\alpha=\infty-i\pi$, ends at $\alpha=\infty+i\pi$ and may be chosen to lie in the right half plane. Inserting this in Eq. (3.11), interchanging the orders of integration, and doing the κ intergration, the result for F can be put in the form

$$F(\Omega,\chi,\chi')=-i\frac{e^{-(\lambda^2+\lambda'^2)/(4\Omega)}}{(2\pi)^3(2\Omega\lambda\lambda'\rho)}\int_C d\alpha e^{\lambda\lambda'\cosh\alpha/(2\Omega)}\left[\frac{1}{(\alpha-i\rho)^2}-\frac{1}{(\alpha+i\rho)^2}\right], \tag{3.13}$$

where $\rho=|\vec{\chi}-\vec{\chi}'|$. Distorting the contour C through the pole at $\alpha=i\rho$ and making use of the symmety of the integrand, this can be written

$$F(\Omega,\chi,\chi')=\frac{i}{(4\pi\Omega)^2}\frac{\sin\rho}{\rho}\exp[-(\lambda^2+\lambda'^2-2\lambda\lambda'\cos\rho)/(4\Omega)]$$
$$+\frac{ie^{-(\lambda^2+\lambda'^2)/(4\Omega)}}{(2\pi)^3(2\Omega\lambda\lambda')\rho}\int_{-\infty}^{+\infty}d\psi\, e^{-\lambda\lambda'\cosh\psi/(2\Omega)}\left(\frac{1}{[\psi+i(\pi-\rho)]^2}-\frac{1}{[\psi+i(\pi+\rho)]^2}\right). \tag{3.14}$$

The first term of this explicit expression dominates the behavior of F as $\Omega\to 0$ and gives exactly the result required in Eq. (3.5). Both terms fall off smoothly as $\rho\to\infty$, $\lambda\to\infty$, or as $\Omega\to\infty$ and are well behaved at $\lambda=0$. To rotate back to physical values of the coordinates one puts $\Omega=iW$, $\lambda=-it$, and $\rho=-ir=-i|\vec{x}-\vec{x}'|$ in either Eqs. (3.13) or (3.14). The resulting expression remains well behaved at $t=0$ and ∞, $r=0$ and ∞, and at $W=\infty$, as in most easily seen from Eq. (3.13).

The propagator $K(x,x')$ is computed from $F(W,x,x')$ by doing the integration over parameter time in Eq. (2.2). An expression for the result is most easily obtained by using Eq. (3.11) for $F(\Omega,\chi,\chi')$. First find $F(W,x,x')$ by rotating $\Omega,\lambda,\lambda',\chi$, and χ' as above and simultaneously rotating κ by an angle $\pi/2$ so that the resulting expression for F is a decomposition in plane waves $\exp(i\vec{k}\cdot\vec{x})$, where $\vec{\kappa}=i\vec{k}$. The result is

$$K(x,x')=\frac{-1}{(2\pi)^3}\int d^3k\,\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{2tt'}\int_0^\infty\frac{dW}{W}e^{-im^2W}e^{(t^2+t'^2)/4iW}I_{ik}\left(\frac{itt'}{2W}\right) \tag{3.15}$$

$$=\frac{1}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\frac{\pi i}{2tt'}H_{ik}^{(2)}(mt_>)J_{ik}(mt_<), \tag{3.16}$$

$t_<$ and $t_>$ are the lesser and the greater of t and t', respectively (see Ref. 9, p. 439 for the W integral). Thus we have determined the propagator for massive particles. For massless particles the propagator can be obtained directly from Eq. (3.15) by putting $m=0$. The result is

$$K(x,x')=\frac{1}{(2\pi)^3}\int d^3k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\frac{1}{2itt'k}\left(\frac{t_<}{t_>}\right)^{ik} \tag{3.17}$$

$$=\frac{-i}{(2\pi)^2}\frac{1}{tt'}\frac{1}{[(\ln t-\ln t')^2+(\vec{x}-\vec{x}')^2+i\epsilon]}. \tag{3.18}$$

The last expression shows that the massless propagator is a conformal factor times the usual Feynman propagator in the conformally related flat spacetime.

IV. AMPLITUDE FOR THE PRODUCTION OF A PAIR

The amplitude A_{ij} that a pair of particles is created by the universe, through the process discussed in the Introduction, and detected at a late time t in states characterized by solutions of the wave equation $f_i(x)$ and $f_j(x)$ is

$$A_{ij}=-A_0\int d\sigma^\mu(x)\int d\sigma^\nu(x')\bar{f}_i(x)\times\overleftrightarrow{\partial}_\mu\bar{f}_j(x')\overleftrightarrow{\partial}'_\nu K(x,x'). \tag{4.1}$$

Here $d\sigma^\mu$ is an element of the constant-t hypersurface over which the integrations are being done and $a\overleftrightarrow{\partial}_\mu b$ has its usual meaning: $a\partial_\mu b-b\partial_\mu a$. The number A_0 is the amplitude that no particles are created at all. This factor occurs because $K(x,x')$ gives the amplitude for pair production only relative to the amplitude that no particles are created. Expression (4.1) can either be regarded as a fundamental interpretative relation of the path-integral method as in Feynman's original papers, or it can be derived by field-theory methods as we shall do in Sec. V.

The amplitude that the universe produces more than one pair of particles can also be calculated. For example, the amplitude to produce two pairs involves a sum over all paths connecting four spacetime points on a late constant-t hypersurface (see Fig. 2). Each of these sums can be broken up into products of those which occur in deriving the propagator. If we write A^c_{ij} for the amplitude A_{ij}/A_0 of producing a pair relative to the amplitude for producing none, then the amplitude for producing $2n$ particles in states $i_1, \ldots, i_{2n}$ is

$$A_{i_1 \cdots i_{2n}} = A_0 \sum_{P} A^c_{i_1 i_2} \cdots A^c_{i_{2n-1} i_{2n}} . \tag{4.2}$$

The sum is over all possible ways of producing n pairs such as the three ways of producing two pairs shown in Fig. 2.

The probability of creating any number of pairs when added to the probability of creating none at all must equal unity. Thus,

$$|A_0|^2 + \sum_{ij} |A_{ij}|^2 + \sum_{ijkl} |A_{ijkl}|^2 + \cdots = 1 , \tag{4.3}$$

where the sum in every case is a sum over a complete set of particle states at late times. This relation together with Eq. (4.2) determines $|A_0|$. Thus the amplitude for creating any number of pairs can be expressed in terms of the propagator. We shall now concentrate on evaluating A^c_{ij} for our model universe.

To evaluate A^c_{ij}, we shall need a complete set of solutions to the wave equation which correspond to the observation of positive-energy particles when

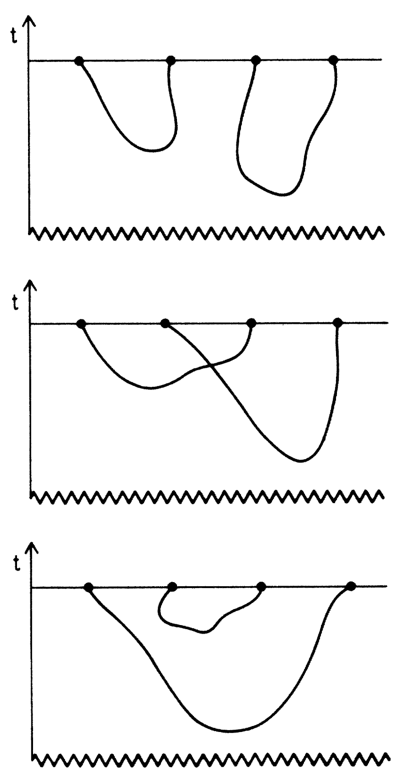

FIG. 2. The three possible classes of paths which contribute to the production of two pairs of particles.

measured by local detectors. It is natural to classify these solutions by a dimensionless wave number $\vec{k}$ corresponding to a space variation $\exp(i\vec{k}\cdot\vec{x})$. For a Robertson-Walker universe with scale factor $R(t)$, the momentum $\vec{p}$ will be $\vec{k}/R(t)$. Localized detectors will be sensitive chiefly to momenta which satisfy $p \gg [R(t)]^{-1}$ or in our case $k \gg 1$. For these values of k the dominant behavior of positive-energy solutions at a late time t should be

$$f_{\vec{k}}(x) = (2\pi t)^{-3/2} (2\omega_k)^{-1/2} \zeta e^{i\vec{k}\cdot\vec{x}} , \tag{4.4a}$$

$$\partial_t f_{\vec{k}}(x) = -i (2\pi t)^{-3/2} (\omega_k/2)^{+1/2} \zeta e^{i\vec{k}\cdot\vec{x}} , \tag{4.4b}$$

with $\omega_k = (p^2+m^2)^{1/2} = (k^2/t^2+m^2)^{1/2}$, and ζ is an irrelevant phase. The constants in front of the exponential have been chosen so that $f_{\vec{k}}$ is normalized to a δ function in $\vec{k}$. At the same time it seems physically reasonable that the solutions of the wave equation which correspond to particle states should be such that there be no particle production in an adiabatic region where the scale factor is varying slowly.

Parker[3] has shown how to construct solutions to the wave equation satisfying these criteria by solving the wave equation [essentially Eq. (3.8) with p^2 replaced by m^2 and $\lambda = -it$] in the late-time adiabatic region using the WKB method. The WKB method gives an accurate solution because at late times the t^{-2} term in Eq. (3.8) is varying slowly on the scale of m^{-1} uniformly in k. Our present considerations are thus limited to massive particles. Defining $W_k(t) = (k^2/t^2+m^2)^{1/2}$ the WKB solutions, valid at late times uniformly in k and which match Eq. (4.4) for large k, are

$$f_{\vec{k}}(x) = (2\pi t)^{-3/2} (2W_k)^{-1/2} \times \exp\left[i\left(\vec{k}\cdot\vec{x} - \int^t W_k(t')dt' \right)\right] . \tag{4.5}$$

It is clear from the form of Eq. (4.5) that for large k it has the form of Eq. (4.4) for all late times. Thus with this definition of particle at late times none will be produced in the adiabatic region. We emphasize that we are not satisfying Eq. (4.4) exactly. This would correspond to choosing particle states which diagonalize the Hamiltonian at every late time and this leads to an infinite number of produced particles at late times, as discussed in Ref. 3. Rather we choose the solutions of Eq. (4.5) for which no particles are produced in the adiabatic region, but which have the behavior of Eq. (4.4) at large k and therefore give correct results for localized measurements.

Using Eq. (4.5) for the states, Eq. (3.16) for the propagator, and standard asymptotic formulas for the Bessel functions it contains, we can evaluate

Eq. (4.1) for the amplitude to produce a pair of particles. One obtains for the amplitude $A_{\vec{k}\vec{k}'}$, at late times a constant term and an oscillating term. For a detector with any localization at all one is interested in the amplitude to produce pairs in positive-energy states which are square integrable superpositions of the states $f_{\vec{k}}$. For such states the oscillating term in $A_{\vec{k}\vec{k}'}$, will give a vanishing contribution at late times. If we write $f_i(\vec{k})$ for the component of state f_i with wave number $\vec{k}$ normalized so that $\int d^3k|f_i(\vec{k})|^2=1$, then

$$A_{ij}=-A_0\int d^3k\bar{f}_i(\vec{k})\bar{f}_j(-\vec{k})e^{-\pi k}\,. \qquad (4.6)$$

This remarkable result shows two things. First, particles are created in pairs with equal and opposite momenta. This is a consequence of the translation invariance of our model geometry. Second, the amplitude of creating a pair of particles, one with wave number $\vec{k}$ and the other with $-\vec{k}$, is $-A_0\exp(-\pi k)$. The probability $P_1(k)$ that a pair is created with wave number $\vec{k}$ is

$$P_1(\vec{k})=|A_0|^2e^{-2\pi k}\,. \qquad (4.7)$$

From Eq. (4.2) one can then obtain the probability that $n(\vec{k})$ pairs are created in this mode

$$P_n(\vec{k})=|A_0|^2e^{-2\pi n(\vec{k})k}\,. \qquad (4.8)$$

Finally from Eq. (4.3) the probability $P_0=|A_0|^2$ that no particles are created can be evaluated

$$P_0=1-e^{-2\pi k}\,. \qquad (4.9)$$

Thus,

$$P_n(\vec{k})=e^{-2\pi n(\vec{k})k}(1-e^{-2\pi k})\,, \qquad (4.10)$$

and the average number in the mode characterized by $\vec{k}$ is then

$$N(\vec{k})=\sum_{n=0}^{\infty}nP_n(\vec{k})=\frac{1}{e^{2\pi k}-1}\,. \qquad (4.11)$$

In terms of the momentum at time t, $p=k/R(t)$, Eq. (4.11) may be written

$$N(\vec{k})=\frac{1}{e^{2\pi pR(t)}-1}\,.$$

The total number of particle pairs produced will be finite and the total energy produced will also be finite. For values of p much larger than the rest mass m, Eqs. (4.10) show that the particle pairs are produced with a thermal probability distribution characterized (since $2p$ is then the energy of a particle pair) by a temperature

$$k_BT=1/\pi R(t)\,, \qquad (4.12)$$

where k_B is Boltzmann's constant. Of course, the final distribution of particles is not actually thermal since only for large values is p the energy and in any event there remain the correlations between the momenta of the particles. Still, Eq. (4.12) is a useful number to characterize the distribution. In more usual units, $k_BT=\hbar c/\pi R(t)$. If our universe had the dynamics of this model, the temperature characterizing this produced distribution today would be $7.8\times10^{-29}(H_0/50)$ °K, where H_0 is present value of the Hubble constant in (km/sec)/Mpc. Thus in this model a very tiny number of particles would be produced.

All of the immediately preceding discussion holds for the production of particles with masses such that $m\gg t^{-1}$. This restriction was necessary for the validity of the WKB approximation in Eq. (4.5) and for the validity of the asymptotic form for the propagator. It is also of interest to examine the case of massless scalar particles. Then since we have adopted a conformally invariant wave and our model geometry is conformally flat, we expect to have no particle production, in accord with the general result of Parker.[2]

In the case of $m=0$ the solutions of the conformally invariant wave equation which correspond to the definition of particle in Eq. (4.5) can be found exactly. They are

$$f_{\vec{k}}(x)=\frac{1}{(2\pi)^{3/2}(2k)^{1/2}t}e^{-ik\ln t}e^{i\vec{k}\cdot\vec{x}}\,. \qquad (4.13)$$

These are positive-frequency solutions measured in a time $\tau=\ln t$. The massless propagator, Eq. (3.17), propagates such positive-frequency solutions in this sense forward in time and negative-frequency solutions backward in time. There will therefore be no massless particle production. An explicit calculation bears this argument out.

V. CONNECTION WITH QUANTUM FIELD THEORY

In the quantum field theory of a scalar particle in a curved background spacetime the probability amplitude to observe two particles at late times in states characterized by solutions of the wave equation $f_i(x)$ and $f_j(x)$ would be written

$$A_{ij}=-\int d\sigma^\mu(x)\int d\sigma^\nu(x')\bar{f}_i(x)\overleftrightarrow{\partial}_\mu\bar{f}_j(x')\overleftrightarrow{\partial}'_\nu \times\langle 0_+|T\varphi(x)\varphi(x')|0_-\rangle\,. \qquad (5.1)$$

Here $\varphi(x)$ is the field operator, $|0_-\rangle$ is the initial vacuum state of the universe, $|0_+\rangle$ is the vacuum state at late times, and T signifies a time-ordered product. In writing this expression we are explicitly assuming the existence of an adiabatic regime at late times, so that there is one vacuum state $|0_+\rangle$ with respect to which particles can be defined for all late times.

In order to compare the path-integral approach to cosmological particle production with others which have been advanced, it will be convenient to cast our results in these quantum-field-theory terms. In particular, we will want to identify the initial and final vacuum states which are implicit in our construction of the propagator $K(x,x')$. To do this for the state $|0_-\rangle$ we first decompose the field $\varphi(x)$ into annihilation and creation operators for this state.

$$\varphi(x)=\sum_i[g_i(x)a_i+\bar{g}_i(x)a_i^\dagger], \tag{5.2}$$

where

$$a_i|0_-\rangle=0\,. \tag{5.3}$$

Specifying the state $|0_-\rangle$ is the same as specifying those solutions of the wave equation $g_i(x)$ which project out from $\varphi(x)$ the operators which annihilate $|0_-\rangle$.

Comparing Eq. (4.1) with Eq. (5.1) the familiar field-theory expression for the propagator can be deduced

$$K(x,x')=i\langle 0_+|T\varphi(x)\varphi(x')|0_-\rangle/\langle 0_+|0_-\rangle\,. \tag{5.4}$$

The factor $\langle 0_+|0_-\rangle$ is just the vacuum persistence amplitude A_0. Inserting Eq. (5.2) in Eq. (5.4) and using Eq. (5.3) we have, taking $t>t'$,

$$K(x,x')-\sum_i[i\langle 0_+|\varphi(x)a_i^\dagger|0_-\rangle/\langle 0_+|0_-\rangle]\bar{g}_i(x')\,. \tag{5.5}$$

If the spatial dependence of the $g_i(x)$ is specified, then Eq. (5.5) together with a knowledge of $K(x,x')$ gives the time dependence of $g_i(x)$ necessary so that they project out of $\varphi(x)$ the annihilation operators of $|0_-\rangle$.

To make this concrete let us consider our model cosmology. The translation invariance of the spatial sections makes it natural to consider modes of the form

$$g_{\vec{k}}(x)=(2\pi)^{-3/2}G_k(t)e^{i\vec{k}\cdot\vec{x}}\,. \tag{5.6}$$

Comparison of Eq. (5.5) with Eq. (3.15) then shows that

$$G_k(t)=\left(\frac{\sinh\pi k}{2\pi k^2}\right)^{1/2}\frac{1}{t}J_{-ik}(mt)\,, \tag{5.7}$$

where the constant has been chosen so that the $g_{\vec{k}}$ are normalized to a δ function in $\vec{k}$. For small t this becomes

$$G_k(t)\sim\text{const}\times t^{-1}e^{-ik\ln t}. \tag{5.8}$$

A similar argument in the case of massless particles leads to exactly the same behavior at small t The content of this result can be simply stated as follows: The spacetime of Eq. (3.1) is conformally related to a flat spacetime in which $\tau=\ln t$ is a Minkowski time coordinate. As measured by the time τ the singularity is located at $\tau=-\infty$. The boundary condition of Eq. (5.8) which defines the propagator is just the statement that in terms of the time τ negative frequencies are propagated into the past.

The functions $h_i(x)$ which project out the part of the field which annihilated $|0_+\rangle$ can be found in a similar way. They are

$$h_{\vec{k}}(x)=\frac{1}{(2\pi)^{3/2}}\frac{\sqrt{\pi}}{2t}e^{\pi(k/2-i/4)}H_{ik}^{(2)}(mt)e^{i\vec{k}\cdot\vec{x}}\,, \tag{5.9}$$

where the constant has been adjusted so that the functions $h_{\vec{k}}$ are normalized to a δ function in $\vec{k}$. For large t these behave exactly as the positive-frequency WKB solutions which were argued on physical grounds in Sec. IV to represent particle states in the large-time adiabatic region. The present discussion can be regarded as a check on those physical arguments. The boundary condition on the propagator is thus that for large times it propagates forward positive frequencies in the WKB sense. Exactly the same result is obtained by a similar analysis of the massless case.

Having obtained the states $|0_-\rangle$and $|0_+\rangle$ one could now proceed to calculate the particle production amplitude of finding the Bogoliubov transformation which takes the initial annihilation and creation operators into the final ones. The production amplitudes are related to the matrix elements of this transformation. The results thus obtained are the same as those we have already quoted.

These results show that our propagator cannot be the same as those obtained by methods which involve joining the universe into an adiabatic regime prior to some early time t_0. Neither our results for the boundary conditions which determine the propagator nor our calculated particle production amplitudes depend on any such time. However, those results which are obtained from prescription which do involve a joining time t_0, but which are insensitive to the value of this time, may well be similar to those calculated here since the condition (5.8) is similar to the condition that negative frequencies be propagated to the past in an initial adiabatic region.

VI. CONCLUSIONS

Path-integral quantization as defined through analytically continuing spacetime gives a powerful method for calculating the production of particles in curved background spacetimes. The great advantage of the method is that it gives a natural definition of the initial vacuum state in a universe with singularity without the necessity of altering the dynamics dictated by Einstein's equations to introduce an initial adiabatic region in an *ad hoc* way. Issues of interpretation of what are meant by particles come in only at late times where the physics is comparatively well understood.

We have demonstrated the method by calculating the production of particles in a model universe, which is homogeneous, isotropic spatially flat, and has a scale factor $R(t)=t$. For this model the path-integral prescription led to an initial particle vacuum which was annihilated at $t=0$ by solutions to the wave equation which had purely positive frequencies in the time of the conformally related flat spacetime $\tau=\log t$. Our result for massive particles was a spectrum of produced particle pairs which at high energies becomes a thermal spectrum with a temperature $T=1/\pi k_B R(t)$. The total number of particles produced and the total energy produced is finite.

The outstanding issue is the generality of these results. What is the general nature of the boundary condition on the propagator at the singularity which is dictated by path-integral quantization and what is its physical interpretation? How general is the quasithermal spectrum of produced pairs obtained in this model? How do the results generalize to anisotropic cosmologies and what is the back reaction of the produced particles on the geometry? We hope that future work will help to resolve these questions.

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