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Particle limit of field theory: A new strong-coupling expansion*

M. B. Halpern and W. Siegel

Department of Physics and Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720

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We study a new semiclassical expansion of field theory. In this expansion, the natural variables are geometrical (particle coordinates), and the expansion is about solutions to the classical mechanics of such variables. It is a *strong*-coupling expansion, and it offers considerable hope for bridging the gap between quantum field theories and theories of quantized geometrical objects.

I. INTRODUCTION

Effort in fundamental physics has turned increasingly toward the problem of quark confinement. There is reason to believe that we know the beginnings of such a program (quantum chromodynamics) and the end (stringlike and/or baglike theories). What is not clear is the path from such a local quantum field theory to such theories of quantized geometrical variables.

We note that *both* types of theories are based on a *classical* formulation: the former in terms of field variables, the latter in terms of particle/string/bag variables. It is of course well known how to regain the semiclassical limit of a quantum field theory in such a manner as to preserve the classical field equations.[1] What is not well known[2] is that there is *another* semiclassical limit of field theory in which classical *mechanics* is (may be) obtained. We shall refer to the former limit as the *field* $\hbar$*-limit*. It is associated with a *weak*-coupling expansion, which we call the *field* $\hbar$*-expansion*. We shall refer to our new limit as the *particle* $\hbar$*-limit*. It is associated with a *strong*-coupling expansion, which we will call the *particle* $\hbar$*-expansion*.

Just as in the field $\hbar$-expansion the natural variables are fields, and the expansion is around solutions to classical field equations, so in the particle $\hbar$-expansion the natural variables are geometrical, and the expansion is around solutions to classical-mechanical equations. It is our feeling then that the particle $\hbar$-expansion provides the *natural language* with which to excavate geometrical objects in field theory. Senjanović and one of the present authors[3,4] have been engaged for some time in a program of rewriting field theories in terms of geometrical variables.[5,6] Indeed, with some approximation, and in two dimensions, a direct bridge was found in this way from gauge theories to strings. One of the purposes of this paper is to put those approximations on firmer footing by embedding them in an organized context—the particle $\hbar$-expansion. By no means, however, is the particle $\hbar$-expansion limited to two dimensions. Indeed, in this paper we shall work primarily in four dimensions. Results for arbitrary dimensions will also be given.

Because they have no charged loops, nonrelativistic field theories offer the simplest applications of the particle $\hbar$-expansion. They also illustrate almost all the general principles necessary to study the relativistic case, and so will be discussed first, in Sec. II. The general program, discussed in detail there, involves three steps: (1) Reexpress the field theory in terms of particle (geometrical) variables; (2) in this language, find the "particle action"; (3) use the particle action to define the particle $\hbar$-limit, in which classical mechanics is obtained. For the nonrelativistic models, because they have no charged loops, we will find ordinary nonrelativistic classical mechanics dominating this strong-coupling limit of the field theory.

The particle $\hbar$-expansion for the relativistic case, discussed in Sec. III, follows the same lines, but is one level more complicated. It is not difficult to isolate a "naive" particle action, *neglecting charged loops*, which has the form of classical relativistic particle dynamics. We will *define* our particle $\hbar$-expansion in such a way (in complete analogy with the nonrelativistic case) that this classical mechanics dominates all structures with no charged loops. (It dominates the strong-coupling limit of processes with no charged loops, e.g., the cracked-eggshell diagrams important in finite QED,[7] whose one loop can be obtained by sewing together the ends of an electron propagator.)

In general, however, we must proceed to study the behavior of the charged loops themselves under this particle $\hbar$-expansion. The question is: Does the classical relativistic dynamics dominate the charged loops in the limit $\hbar\to 0$, or do the charged loops dominate, defining a more sophisticated particle dynamics? A general expansion of the loop contribution in powers of $\hbar$ is developed in Sec. IV, with calculations detailed in the Appendix. The expansion is essentially an expansion in powers of momenta, and it is not difficult to compute exactly the first few terms. The answer to our question is model dependent: In the cases of scalar and spinor quantum electrodynamics, the classical dynamics does in fact dominate the loops; for theories of the type $\mathcal{L}_I = -g\psi^*\psi\phi$, the loops dominate. In the latter case, the particle $\hbar$-limit is a much more complicated dynamics, much more what is expected of strong coupling.

Section V generalizes the discussion of Sec. IV to an arbitrary number of dimensions D. For $D<5$ scalar and spinor QED are still dominated by classical mechanics; for $D\geq 5$, the loops dominate or (for $D=5$) are of the same importance. For $\mathcal{L}_I = -g\psi^*\psi\phi$, the loops dominate in any number of dimensions. A few brief remarks about quantum chromodynamics are also included. This section also deals briefly with some possible directions and some new features of the expansion, including the need for a "classical renormalization" in the relativistic particle $\hbar$-limit.

II. A NONRELATIVISTIC EXAMPLE

The simplest illustration of our new particle $\hbar$-expansion will be in terms of nonrelativistic field theory. We will examine the nonrelativistic analog of scalar electrodynamics, described by the field action

$$S_F = \int d^4x \left(\psi^* \left\{(i\partial_0 - eA_0) - \frac{[-i\vec{\nabla} - (e/c)\vec{A}]^2}{2m} + i\epsilon\right\}\psi - \tfrac{1}{2}A_\mu \overleftrightarrow{\nabla}^2 A^\mu\right). \tag{2.1}$$

The interaction due to the photon is instantaneous. It will be instructive to compare particle and field $\hbar$-limits, so we will first review the usual field $\hbar$-expansion for this model.

A. The field $\hbar$-expansion

To discuss the usual field $\hbar$-limit, it is useful to have the action in three forms:

$$\begin{aligned} S_F &= \int d^4x \left(\psi^* \left\{(i\partial_0 - eA_0) - \frac{[-i\vec{\nabla} - (e/c)\vec{A}]^2}{2m} + i\epsilon\right\}\psi - \tfrac{1}{2}A_\mu \overleftrightarrow{\nabla}^2 A^\mu\right) \\ &\equiv \frac{1}{\hbar}\int d^4x \left(\psi_F^* \left\{(i\partial_0 - e_F A_{F0}) - \frac{[-i\vec{\nabla} - (e_F/c)\vec{A}_F]^2}{2m_F} + i\epsilon\right\}\psi_F - \tfrac{1}{2}A_{F\mu} \overleftrightarrow{\nabla}^2 A_F^\mu\right) \\ &= \int d^4x \left(\frac{\psi_F^*}{\sqrt{\hbar}} \left\{\left(i\partial_0 - \sqrt{\hbar}e_F \frac{A_{F0}}{\sqrt{\hbar}}\right) - \frac{[-i\vec{\nabla} - (\sqrt{\hbar}e_F/c)\vec{A}_F/\sqrt{\hbar}]^2}{2m_F} + i\epsilon\right\}\frac{\psi_F}{\sqrt{\hbar}} - \frac{1}{2}\frac{A_{F\mu}}{\sqrt{\hbar}} \overleftrightarrow{\nabla}^2 \frac{A_F^\mu}{\sqrt{\hbar}}\right), \end{aligned} \tag{2.2}$$

or, more concisely,

$$\begin{aligned} S_F(e, m; \psi, A) &\equiv S_F(e_F, m_F; \psi_F, A_F)/\hbar \\ &= S_F(\sqrt{\hbar}e_F, m_F; \psi_F/\sqrt{\hbar}, A_F/\sqrt{\hbar}). \end{aligned} \tag{2.3}$$

Therefore, we have

$$e = \sqrt{\hbar}e_F, \quad m = m_F, \quad \psi = \psi_F/\sqrt{\hbar}, \quad A = A_F/\sqrt{\hbar}. \tag{2.4}$$

The first form is the $\hbar$-free form which we started with in (2.1). The second form defines the field $\hbar$-limit by the prescription of multiplying by an overall factor of $1/\hbar$. It shows that in this limit ($\hbar \to 0$, e_F and m_F fixed) the theory is dominated by solutions to the classical field equations.[8] We can also see from this form that the field $\hbar$-expansion is a loop expansion: The propagators and vertices (as functions of e_F and m_F) have factors of $\hbar$ and $1/\hbar$, respectively. Finally, in the third form, we see that the field $\hbar$-limit is a weak-coupling limit: It is equivalent to the replacements $e = \sqrt{\hbar}e_F$ and $m = m_F$ in the $\hbar$-free Feynman diagrams.

After finding a "particle action" (Sec. II C), we will treat it just as one treats S_F: We will define the particle $\hbar$-limit as that which is dominated by classical mechanics. The particle $\hbar$-limit will turn out to be totally different, and is in fact a type of strong-coupling limit. Toward finding the particle action, we must first reexpress the field theory in terms of particle mechanics.

B. From field variables to particle variables

We start with the Green's-function generating functional in terms of the $\hbar$-free form of the field action

$$Z(\eta, \eta^*, J_\mu) = \int \mathfrak{D}\psi\,\mathfrak{D}\psi^*\,\mathfrak{D}A^\mu \exp\left[i\left(S_F(\psi, \psi^*, A^\mu) + \int d^4x(\eta^*\psi + \psi^*\eta - J_\mu A^\mu)\right)\right]. \tag{2.5}$$

Our first step is to integrate out the charged field:

$$Z(\eta,\eta^*,J_\mu)=\int \mathfrak{D}A^\mu(\det G)\exp\left[i\int d^4x(-\eta^*G\eta-\tfrac{1}{2}A_\mu\vec{\nabla}^2A^\mu-J_\mu A^\mu)\right],$$
$$G\equiv\left\{(i\partial_0-eA_0)-\frac{[-i\vec{\nabla}-(e/c)\vec{A}]^2}{2m}+i\epsilon\right\}^{-1}. \tag{2.6}$$

G is the propagator for ψ in an external A^μ field. The determinant of G gives the charged-loop contribution to Z. In the relativistic theory this determinant will give corrections to Z, but in nonrelativistic theory $\det G=1$. This is because the nonrelativistic theory has no antiparticles, and therefore no charged loops. More formally, although the $i\epsilon$ prescription above is the usual Euclidean (and therefore time-ordered) prescription, the fact that ψ contains only positive frequencies means that time-ordered products immediately degenerate to retarded products:

$$\langle 0|T(\psi(x)\psi^*(y))|0\rangle=\theta(x^0-y^0)\langle 0|\psi(x)\psi^*(y)|0\rangle. \tag{2.7}$$

Having dealt with $\det G$, we then have the following expression for the Green's functional with $2N$ external ψ lines and the A source:

$$K_N(\vec{z}_k,t'';\vec{y}_k,t';J_\mu)\equiv\prod_{k=1}^{N}\left[\frac{\delta}{i\delta\eta^*(\vec{z}_k,t'')}\frac{\delta}{i\delta\eta(\vec{y}_k,t')}\right]Z(\eta,\eta,J_\mu)\bigg|_{\eta=\eta^*=0}$$
$$=\int\mathfrak{D}A^\mu\sum_{\substack{\text{perm}\\ y_k,z_k}}\left[\prod_{k=1}^{N}iG(\vec{z}_k,t'';\vec{y}_k,t')\right]\exp\left[i\int d^4x(-\tfrac{1}{2}A_\mu\vec{\nabla}^2A^\mu-J_\mu A^\mu)\right]. \tag{2.8}$$

Notice that if we had chosen ψ to be a spinless fermion, the only change in the above expression would have been the inclusion of minus signs for the odd permutations, due to the anticommutativity of fermion sources.

The next step is to introduce particle variables by reexpressing the propagators G in terms of Feynman path integrals[5]:

$$G(\vec{z},t'';\vec{y},t')=-i\theta(t''-t')\int_y^z\mathfrak{D}x\exp\left[i\int_{t'}^{t''}dt\left(\tfrac{1}{2}m\dot{\vec{x}}-eA_0+\frac{e}{c}\dot{\vec{x}}\cdot\vec{A}\right)\right]. \tag{2.9}$$

The A^μ integration in K_N is now Gaussian and can be done exactly:

$$K_N=\theta(t''-t')\int\mathfrak{D}A\sum\left(\prod_k\int_{y_k}^{z_k}\mathfrak{D}x_k\right)$$
$$\times\exp\left\{i\left[\int_{t'}^{t''}dt\sum_k\left(\tfrac{1}{2}m\dot{\vec{x}}_k^{\,2}-eA_0(\vec{x}_k,t)+e\frac{\dot{\vec{x}}_k}{c}\cdot\vec{A}(\vec{x}_k,t)\right)+\int d^4x(-\tfrac{1}{2}A_\mu\vec{\nabla}^2A^\mu-J_\mu A^\mu)\right]\right\}$$
$$=\exp\left(i\int d^4x\tfrac{1}{2}A_{\mu,\text{cl}}\vec{\nabla}^2A^\mu_{\text{cl}}\right)\theta(t''-t')\sum\left(\prod\int\mathfrak{D}x\right)\exp(iS_P), \tag{2.10}$$
$$S_P=\int_{t'}^{t''}dt\left[\sum_k\left(\tfrac{1}{2}m\dot{\vec{x}}_k^{\,2}-eA_{0,\text{cl}}(\vec{x}_k,t)+e\frac{\dot{\vec{x}}_k}{c}\cdot\vec{A}_{\text{cl}}(\vec{x}_k,t)\right)-\tfrac{1}{2}e^2\sum_{j,k}\frac{1-\dot{\vec{x}}_j\cdot\dot{\vec{x}}_k/c^2}{4\pi|\vec{x}_j-\vec{x}_k|}\right],$$
$$A_{\mu,\text{cl}}(x;J)=\int d^3x'\Delta(\vec{x}-\vec{x}')J_\mu(\vec{x}',x_0),\quad \Delta(\vec{x})=-\frac{1}{\vec{\nabla}^2}\delta^3(\vec{x})=\frac{1}{4\pi|\vec{x}|}.$$

In the above, we have used

$$A_\mu(\vec{x}_k,t)=\int d^4xA_\mu(\vec{x},x_0)\delta^3(\vec{x}-\vec{x}_k)\delta(x_0-t). \tag{2.11}$$

The factor with exponent $A_{\text{cl}}\vec{\nabla}^2A_{\text{cl}}$ is the Green's-function generating functional for a free A field: It gives free A propagators disconnected from the rest of the graph. The rest of K_N is the Green's functional of particles interacting through instantaneous Coulomb and Biot-Savart forces, in the presence of an external field A_{cl}. A_{cl} is just the classical A field due to the source J. The divergent interaction terms in S_P for

$j=k$ correspond to renormalization of mass $\delta m=(e^2/c^2)\Delta(0)$ and ground-state energy $\delta E=\frac{1}{2}e^2\Delta(0)$.

We have now completed our transition to particle variables (S_P is our particle action), and are in a position to define the particle $\hbar$-expansion.

C. The particle $\hbar$-expansion

In analogy with our treatment of the field $\hbar$-expansion (in terms of S_F), we write out three forms for S_P:

$$S_P=\int dt\left[\sum_k\left(\tfrac{1}{2}m\dot{\vec{x}}_k^2-eA_{0,\mathrm{cl}}+e\frac{\dot{\vec{x}}_k}{c}\cdot\vec{A}_{\mathrm{cl}}\right)-\tfrac{1}{2}e^2\sum_{j,k}\frac{1-\dot{\vec{x}}_j\cdot\dot{\vec{x}}_k/c^2}{4\pi|\vec{x}_j-\vec{x}_k|}\right]$$

$$\equiv\frac{1}{\hbar}\int dt\left[\sum_k\left(\tfrac{1}{2}m_P\dot{\vec{x}}_k^2-e_PA_{P0,\mathrm{cl}}+e_P\frac{\dot{\vec{x}}_k}{c}\cdot\vec{A}_{P,\mathrm{cl}}\right)-\tfrac{1}{2}e_P^2\sum_{j,k}\frac{1-\dot{\vec{x}}_j\cdot\dot{\vec{x}}_k/c^2}{4\pi|\vec{x}_j-\vec{x}_k|}\right]$$

$$=\int dt\left[\sum_k\left(\frac{1}{2}\frac{m_P}{\hbar}\dot{\vec{x}}_k^2-\frac{e_P}{\sqrt{\hbar}}\frac{A_{P0,\mathrm{cl}}}{\sqrt{\hbar}}+\frac{e_P}{\sqrt{\hbar}}\frac{\dot{\vec{x}}_k}{c}\cdot\frac{\vec{A}_{P,\mathrm{cl}}}{\sqrt{\hbar}}\right)-\frac{1}{2}\left(\frac{e_P}{\sqrt{\hbar}}\right)^2\sum_{j,k}\frac{1-\dot{\vec{x}}_j\cdot\dot{\vec{x}}_k/c^2}{4\pi|\vec{x}_j-\vec{x}_k|}\right], \tag{2.12}$$

or, more concisely,

$$S_P(e,m;A_{\mathrm{cl}})=S_P(e_P,m_P;A_{P,\mathrm{cl}})/\hbar=S_P(e_P/\sqrt{\hbar},m_P/\hbar;A_{P,\mathrm{cl}}/\sqrt{\hbar}). \tag{2.13}$$

We therefore have

$$e=e_P/\sqrt{\hbar}\,,\quad m=m_P/\hbar\,,\quad A_{\mathrm{cl}}=A_{P,\mathrm{cl}}/\sqrt{\hbar}\,. \tag{2.14}$$

We will *define* the particle $\hbar$-limit as the limit $\hbar\to0$ at fixed e_P and m_P. Thus, in this limit [as seen in the second form of (2.12)], the theory is dominated by solutions to classical mechanics. If we allowed scaling of $\vec{x}_k$, we would have a more general (but equivalent) prescription for the particle limit. However, this would be equivalent to scaling $\vec{y}_k$ and $\vec{z}_k$ in K_N, while leaving $\vec{x}$ in $J_\mu(\vec{x})$ unscaled, causing unnecessary complications in notation (similarly, complications arise from scaling t).

We can now rewrite the *field* action in terms of e_P and m_P by substituting (2.14) into the $\hbar$-free form of S_F in (2.1):

$$S_F=\int d^4x\left(\psi^*\left\{\left(i\partial_0-\frac{e_P}{\sqrt{\hbar}}A_0\right)-\frac{[-i\vec{\nabla}-(e_P/\sqrt{\hbar})(1/c)\vec{A}]^2}{2m_P/\hbar}+i\epsilon\right\}\psi-\tfrac{1}{2}A_\mu\vec{\nabla}^2A^\mu\right). \tag{2.15}$$

Thus, we see that our particle $\hbar$-limit is a kind of strong coupling; however, it is *not* the *usual* strong coupling because it also has $m=m_P/\hbar\to\infty$. For comparison with the field $\hbar$/expansion, S_F may also be written as

$$S_F=\int d^4x\left(\frac{\psi_F^*}{\sqrt{\hbar}}\left\{\left(i\partial_0-\frac{e_P}{\sqrt{\hbar}}\frac{A_{F0}}{\sqrt{\hbar}}\right)-\frac{[-i\vec{\nabla}-(e_P/\sqrt{\hbar})(1/c)\vec{A}_F/\sqrt{\hbar}]^2}{2m_P/\hbar}+i\epsilon\right\}\frac{\psi_F}{\sqrt{\hbar}}-\frac{1}{2}\frac{A_{F\mu}}{\sqrt{\hbar}}\vec{\nabla}^2\frac{A_F^\mu}{\sqrt{\hbar}}\right). \tag{2.16}$$

Notice that if the photon had a mass μ [replacing $\vec{\nabla}^2$ with $\vec{\nabla}^2-\mu^2$ in (2.1)], then, by the arguments of this section, we would find $\mu=\mu_P=\mu_F$: The mass μ acts as a field mass, not a particle mass. Therefore, in the particle $\hbar$-limit ψ becomes a classical particle, but A becomes a classical field.

III. EXPANSION FOR RELATIVISTIC FIELD THEORY

Following the nonrelativistic example as closely as possible, we examine scalar electrodynamics, with the field action and Green's-function generating functional

$$\begin{aligned}S_F&=\int d^4x\,\{\psi^*[(i\partial_\mu-eA_\mu)^2-m^2+i\epsilon]\psi+\tfrac{1}{2}A_\mu\Box A^\mu\}\,,\\ Z(\eta;\eta^*,J_\mu)&=\int\mathfrak{D}\psi\mathfrak{D}\psi^*\mathfrak{D}A^\mu\exp\left\{i\left[S_F(\psi,\psi^*,A^\mu)+\int d^4x(\eta^*\psi+\psi^*\eta-J_\mu A^\mu\right]\right\}.\end{aligned} \tag{3.1}$$

We have chosen the Lorentz gauge. In fact, all of our results are gauge independent, and the reader is invited to work things through in an arbitrary gauge.

As in the nonrelativistic case, the field $\hbar$-expansion is described by

$$S_F=\int d^4x\,\{\psi^*[(i\partial-eA)^2-m^2+i\epsilon]\psi+\tfrac{1}{2}A\,\Box A\}$$
$$\equiv\frac{1}{\hbar}\int d^4x\,\{\psi_F^*[(i\partial-e_FA_F)^2-m_F{}^2+i\epsilon]\psi_F+\tfrac{1}{2}A_F\Box A_F\}$$
$$\equiv\int d^4x\left\{\frac{\psi_F^*}{\sqrt{\hbar}}\left[\left(i\partial-\sqrt{\hbar}\,e_F\frac{A_F}{\sqrt{\hbar}}\right)^2-m_F{}^2+i\epsilon\right]\frac{\psi_F}{\sqrt{\hbar}}+\tfrac{1}{2}\frac{A_F}{\sqrt{\hbar}}\Box\frac{A_F}{\sqrt{\hbar}}\right\}, \tag{3.2}$$
$$e=\sqrt{\hbar}\,e_F,\quad m=m_F,\quad \psi=\psi_F/\sqrt{\hbar},\quad A=A_F/\sqrt{\hbar},$$

or, more concisely,

$$S_F(e,m;\psi,A)\equiv S_F(e_F,m_F;\psi_F,A_F)/\hbar=S_F(\sqrt{\hbar}\,e_F,m_F;\psi_F/\sqrt{\hbar},A_F/\sqrt{\hbar}). \tag{3.3}$$

Again the field $\hbar$-limit ($\hbar\to0$ with e_F and m_F fixed) is a weak-coupling limit, dominated by classical field theory.

A. Particle representation of propagator G

To find the *particle* $\hbar$-expansion, we again start by rewriting the Green's functions in terms of the particle variables. The first step is the same, evaluating the ψ and ψ^* integrals in Z:

$$Z(\eta,\eta^*,J_\mu)=\int\mathfrak{D}A^\mu(\det G(A))\exp\left[i\int d^4x(-\eta^*G\eta+\tfrac{1}{2}A_\mu\Box A^\mu-J_\mu A^\mu)\right],$$
$$G=[(i\partial_\mu-eA_\mu)^2-m^2+i\epsilon]^{-1}. \tag{3.4}$$

Now the determinant is not unity. (Its logarithm is the effective action for A due to all one-ψ-loop diagrams, as will be discussed in Sec. IV.) We also have

$$K_N(z_k^\mu,y_k^\mu;J^\mu)=\int\mathfrak{D}A\sum\left[\prod iG(z_k^\mu,y_k^\mu)\right]\exp\{i[d^4x(\tfrac{1}{2}A\,\Box A-JA)]\}(\det G). \tag{3.5}$$

As above, we next need to express G as a path integral over particle variables. The technique is well known. We introduce a proper-time parameter,[5,6,9] canonically conjugate to $\frac{1}{2}m^2$:

$$f(m^2)=\int_{-\infty}^{\infty}dT\,\exp(-iT\tfrac{1}{2}m^2)\tilde{f}(T). \tag{3.6}$$

Then we write a path integral for $\tilde{G}$ instead of G (see Refs. 5 and 6):

$$\tilde{G}=\tfrac{1}{2}[i\partial_T+\tfrac{1}{2}(i\partial_\mu-eA_\mu)^2+i\epsilon]^{-1},$$
$$\tilde{G}(z^\mu,T;y^\mu,0)=-\tfrac{1}{2}i\theta(T)\int_y^z\mathfrak{D}x\exp\left\{i\int_0^T d\tau\,[-\tfrac{1}{2}\dot{x}^2(\tau)-e\dot{x}_\mu(\tau)A^\mu(x(\tau))]\right\}. \tag{3.7}$$

Finally we have the desired path-integral expression for G:

$$G(z,y)=\int_{-\infty}^{\infty}dT\,\exp(-iT\tfrac{1}{2}m^2)\tilde{G}(z,T;y,0)$$
$$=-\tfrac{1}{2}i\int_0^\infty dT\int_y^z\mathfrak{D}x\exp\left[i\int_0^T d\tau(-\tfrac{1}{2}m^2-\tfrac{1}{2}\dot{x}^2-e\dot{x}\cdot A)\right] \tag{3.8}$$

B. Properties of G

It will be useful in what follows to know two properties of the propagator G. The first is the form of the equations of motion at the saddle point:

$$\ddot{x}_\nu(\tau)=e\dot{x}^\mu(\tau)F_{\mu\nu}(x(\tau)),\quad \tfrac{1}{2}\dot{x}^2(T)-\tfrac{1}{2}m^2=0, \tag{3.9}$$

where $F_{\mu\nu}(x)\equiv\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)$. The former comes from varying the action with respect to x, the latter from varying with respect to T. To do this latter variation, the T dependence of $\int_0^T$ *and* of the implicit $\delta(z-x(T))$ (from the end-point constraint) must be considered. The result in total is as expected: A derivative with respect to T brings down the Hamiltonian at the end point, $\frac{1}{2}\dot{x}^2(T)-\frac{1}{2}m^2$, which must then vanish. Taken together, these equations of motion imply $\dot{x}^2(\tau)=m^2$ (for all τ). Thus, at least near the saddle point, τ is the ordinary proper time.[10,11]

The second useful property is that

$$G(z,y)=-\tfrac{1}{2}i\lambda\int_0^\infty dT\int_y^z \mathfrak{D}x\exp\left[i\int_0^T d\tau(-\tfrac{1}{2}\lambda m^2-\tfrac{1}{2}\dot{x}^2(\tau)/\lambda-e\dot{x}\cdot A)\right] \tag{3.10}$$

for arbitrary λ. The proof of this identity follows immediately from the change of variables $T\to\lambda T$, $\tau\to\lambda\tau$, $x(\lambda\tau)\to x(\tau)$.

C. The naive particle action

Inserting (3.8) into (3.5), the expression for K_N is now

$$K_N=\int\mathfrak{D}A\sum\left(\prod\int_0^\infty\frac{dT_k}{2}\int_{y_k}^{z_k}\mathfrak{D}x_k\right)\exp\left\{i\left[-\sum\int_0^{T_k}d\tau(\tfrac{1}{2}m^2+\tfrac{1}{2}\dot{x}_k^2+e\mathring{x}_k\cdot A(x_k))+\int d^4x(\tfrac{1}{2}A\Box A-JA)\right]\right\}(\det G). \tag{3.11}$$

Owing to the determinant, the A integration cannot be done in closed form. Our procedure is as follows: We *define* the *naive* particle action as that which we will obtain (in the manner of Sec. II B) by *neglecting* det$G$. Similarly, we will *define* the particle $\hbar$-limit as that for which K_N, neglecting det$G$, is dominated by solutions of the classical mechanics of that action. Whether the naive particle action in fact dominates the *exact* K_N (including det$G$) is a question of the behavior of det$G$ in this $\hbar$-limit. This is properly the subject of the next section, but it will buoy the reader to know that for certain theories (including scalar electrodynamics), the determinant does *not* contribute to leading order in the particle $\hbar$-expansion we are defining.

Setting the determinant equal to one, the A integration gives[6]

$$\begin{aligned}
K_N&=\exp\left(-i\int d^4x\tfrac{1}{2}A_{\rm cl}\Box A_{\rm cl}\right)\sum\left(\prod\int\frac{dT}{2}\int\mathfrak{D}x\right)\exp(iS_P),\\
S_P&=-\sum_k\int_0^{T_k}d\tau(\tfrac{1}{2}m^2+\tfrac{1}{2}\dot{x}_k^2+e\dot{x}_k\cdot A_{\rm cl}(x_k))-\tfrac{1}{2}e^2\sum_{j,k}\int_0^{T_j}d\tau\int_0^{T_k}d\tau'\dot{x}_j(\tau)\cdot\dot{x}_k(\tau')\Delta_F(x_j(\tau)-x_k(\tau')),\\
A_{\mu,\rm cl}(x)&=\int d^4x'\Delta_F(x-x')J_\mu(x'),\quad \Delta_F(x)=\frac{1}{\Box-i\epsilon}\delta^4(x)=-\frac{i}{(2\pi)^2}\frac{1}{x^2-i\epsilon}.
\end{aligned} \tag{3.12}$$

S_P is our naive particle action. It is similar to the Fokker action[12] for classical, relativistic, charged particles, except that the ranges of integration in τ are finite, and $\Delta_F(x)$ is the Feynman propagator instead of $\delta(x^2)/4\pi$ [so S_P is not even real; also, $(\dot{x}^2)^{1/2}$ is replaced by the *classically* equivalent $\frac{1}{2}\dot{x}^2$].

D. The particle $\hbar$-expansion

Using (3.10), K can be put in the form

$$\begin{aligned}
K_N&=\exp\left(-i\int d^4x\tfrac{1}{2}A_{\rm cl}\Box A_{\rm cl}\right)\lambda^N\sum\left(\prod\int\frac{dT}{2}\int\mathfrak{D}x\right)\exp[iS_P(\lambda)],\\
S_P(\lambda)&=-\sum\int d\tau(\tfrac{1}{2}\lambda m^2+\tfrac{1}{2}\dot{x}^2/\lambda+e\dot{x}\cdot A_{\rm cl})-\tfrac{1}{2}e^2\sum\int d\tau\int d\tau'\dot{x}\cdot\dot{x}'\Delta_F(x-x').
\end{aligned} \tag{3.13}$$

Then, we find,[11] choosing $\lambda=\hbar$,

$$\begin{aligned}
S_P(\hbar)&=-\sum\int d\tau(\tfrac{1}{2}\hbar m^2+\tfrac{1}{2}\dot{x}^2/\hbar+e\dot{x}\cdot A_{\rm cl})-\tfrac{1}{2}e^2\sum\int d\tau\, d\tau'\dot{x}\cdot\dot{x}'\Delta_F(x-x')\\
&\equiv\frac{1}{\hbar}\left[-\sum\int d\tau(\tfrac{1}{2}m_p{}^2+\tfrac{1}{2}\dot{x}^2+e_P\dot{x}\cdot A_{P,\rm cl})-\tfrac{1}{2}e_P{}^2\sum\int d\tau\int d\tau'\dot{x}\cdot\dot{x}'\Delta_F(x-x')\right]\\
&=-\sum\int d\tau\left(\tfrac{1}{2}\hbar\left(\frac{m_P}{\hbar}\right)^2+\tfrac{1}{2}\dot{x}^2/h+\frac{e_P}{\sqrt{\hbar}}\dot{x}\cdot\frac{A_{P,\rm cl}}{\sqrt{\hbar}}\right)-\frac{1}{2}\left(\frac{e_P}{\sqrt{\hbar}}\right)^2\sum\int d\tau\int d\tau'\,\dot{x}\cdot\dot{x}'\Delta_F(x-x').
\end{aligned} \tag{3.14}$$

This is the analog of Eq. (2.12). Thus we identify

$$e=e_P/\sqrt{\hbar}\,,\quad m=m_P/\hbar,\quad A_{\rm cl}=A_{P,\rm cl}/\sqrt{\hbar}\,, \tag{3.15}$$

and

$$S_P(\hbar; e, m; A_{\rm cl}) \equiv \frac{1}{\hbar} S_P(1; e_P, m_P, A_{P,{\rm cl}}) = S_P(\hbar; e_P/\sqrt{\hbar}, m_P/\hbar; A_{P,{\rm cl}}/\sqrt{\hbar}), \qquad (3.16)$$

which is the analog of (2.13). Finally, then

$$\begin{aligned} K_N &= \exp\left(-i\int d^4x \tfrac{1}{2} A_{\rm cl} \Box A_{\rm cl}\right) \sum \left(\prod \int \frac{dT}{2}\int \mathfrak{D}x\right) \exp[iS_P(1; e, m, A_{\rm cl})] \\ &= \exp\left(-i\int d^4x \tfrac{1}{2} A_{\rm cl} \Box A_{\rm cl}\right) \hbar^N \sum \left(\prod \int \frac{dT}{2}\int \mathfrak{D}x\right) \exp[iS_P(\hbar; e, m, A_{\rm cl})] \\ &= \exp\left(-\frac{i}{\hbar}\int d^4x \tfrac{1}{2} A_{P,{\rm cl}} \Box A_{P,{\rm cl}}\right) \hbar^N \sum \left(\prod \int \frac{dT}{2}\int \mathfrak{D}x\right) \exp\left[\frac{i}{\hbar} S_P(1; e_P, m_P; A_{P,{\rm cl}})\right]. \end{aligned} \qquad (3.17)$$

In the last step we used Eq. (3.16), and

$$\begin{aligned} S_P(1; e_P, m_P, A_{P,{\rm cl}}) &\equiv \tilde{S}_P \\ &= -\sum \int d\tau \left(\tfrac{1}{2} m_P{}^2 + \tfrac{1}{2}\dot{x}^2 + e_P \dot{x}\cdot A_{P,{\rm cl}}\right) - \tfrac{1}{2} e_P{}^2 \sum \int d\tau \int d\tau' \dot{x}\cdot \dot{x}' \Delta_F(x - x'). \end{aligned} \qquad (3.18)$$

Clearly then, we want to define our particle $\hbar$-limit as $\hbar \to 0$ with e_P and m_P fixed. *If* the naive form dominates (i.e., if $\det G$ can be neglected as $\hbar \to 0$), then the theory will be dominated by the classical mechanics of $\tilde{S}_P$.

In fact, the classical mechanics is what was presaged in (3.9). As $\hbar \to 0$ with e_P and m_P fixed, we find the equations of motion (varying with respect to x_k and T_k)

$$\begin{aligned} &\ddot{x}_{k\nu}(\tau) = e\dot{x}_k^\mu(\tau)\hat{F}_{\mu\nu}(x_k(\tau)), \quad \tfrac{1}{2}\dot{x}_k{}^2(T_k) - \tfrac{1}{2} m^2 = 0, \\ &\hat{A}_\mu(x) = A_{\mu,{\rm cl}}(x) - e\sum_k \int_0^{T_k} d\tau\, \dot{x}_{k\mu}(\tau)\Delta_F(x - x_k(\tau)). \end{aligned} \qquad (3.19)$$

Therefore, τ is again the proper time.

The prescription $e = e_P/\sqrt{\hbar}$ again identifies the particle $\hbar$-expansion as a strong-coupling expansion, with the modification $m = m_P/\hbar$: We can rewrite the field action as

$$S_F = \int d^4x \left\{ \psi^* \left[\left(i\partial - \frac{e_P}{\sqrt{\hbar}} A\right)^2 - \left(\frac{m_P}{\hbar}\right)^2 + i\epsilon\right]\psi_F + \tfrac{1}{2} A \Box A \right\}. \qquad (3.20)$$

We see that, in the relativistic theory, the particle $\hbar$-expansion for the case $m = 0$ is formally a *pure* strong-coupling expansion. This distinction may disappear after renormalization. (If we renormalize about $p^2 = M^2$ to avoid infrared divergences, should we choose $M = M_P/\hbar$?) We can also rewrite the field action as

$$S_F = \int d^4x \left\{ \frac{\psi_F^*}{\sqrt{\hbar}} \left[\left(i\partial - \frac{e_P}{\sqrt{\hbar}}\frac{A_F}{\sqrt{\hbar}}\right)^2 - \left(\frac{m_P}{\hbar}\right)^2 + i\epsilon\right] \frac{\psi_F}{\sqrt{\hbar}} + \frac{1}{2}\frac{A_F}{\sqrt{\hbar}} \Box \frac{A_F}{\sqrt{\hbar}} \right\}. \qquad (3.21)$$

Again, we see that, if the photon had a mass μ, we would have $\mu = \mu_P = \mu_F$: In the particle $\hbar$-limit ψ becomes a classical particle but A becomes a classical field.

We will show in the next section that, to leading order in our particle $\hbar$-expansion for scalar QED, the determinant is negligible. However, note that the naive action has some applications in its own right (regardless of the behavior of the determinant): It is not hard to see that the cracked-egg-shell graphs, popular in finite QED,[7] are summed as

$$\begin{aligned} &\left[\frac{\hbar}{i}\frac{\delta}{\delta J_\mu(u)}\right]\left[\frac{\hbar}{i}\frac{\delta}{\delta J_\nu(v)}\right] \int d^4y\, K_1(y, y; J)\bigg|_{J=0} \\ &\quad \cong \hbar \int d^4y \int_0^\infty \frac{dT}{2}\int_y^y \mathfrak{D}x \left[\int_0^T d\tau\, \dot{x}^\mu \Delta_F(x - u)\right]\left[\int_0^T d\tau\, \dot{x}^\nu \Delta_F(x - v)\right] \\ &\quad\quad \times \exp\left\{-\frac{i}{\hbar}\left[\int_0^T d\tau\left(\tfrac{1}{2} m_P{}^2 + \tfrac{1}{2}\dot{x}^2\right) + \tfrac{1}{2} e_P{}^2 \int_0^T d\tau \int_0^T d\tau'\, \dot{x}\cdot\dot{x}' \Delta_F(x - x')\right]\right\}. \end{aligned} \qquad (3.22)$$

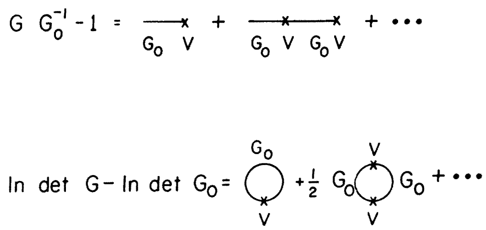

FIG. 1. Expansion of determinant.

Our method then points the way to a strong-coupling approximation for these graphs. (One needs to search for periodic classical solutions, quite possibly in Euclidean space. Could these be the old runaways?)

IV. PARTICLE $\hbar$-EXPANSION OF DETERMINANT

In defining our (strong-coupling) particle $\hbar$-expansion in the relativistic case, we temporarily suspended charged loops. In this way, we defined a "naive" particle action: Essentially-ordinary (or "naive") relativistic classical mechanics dominates the no-loop dynamics as $\hbar\to 0$ at fixed e_P and m_P. An attractive feature of our naive contribution to the action was that the photon field was quadratic and could be integrated out.

Our task now is to compute and compare the strength of the loop contribution in various theories. For some theories, such as scalar electrodynamics, we will find that our naive action dominates the determinant. Thus, in the particle $\hbar$-limit scalar electrodynamics is dominated by ordinary relativistic classical mechanics. In other theories, things are not as simple. For example, in the case of $\mathcal{L}_I=-g\psi^*\psi\phi$ the determinant itself provides the leading order in our expansion, undercutting the "naive" contribution. We still have a particle-variable description, but things are much more complicated: The appearance of the leading terms is much more what is usually expected in a strong-coupling expansion, and we cannot explicitly integrate out the ϕ field (even for the leading term).

We begin by reviewing the fact that $\ln\det G$ is (up to a constant) just the sum of one-ψ-loop diagrams in an external A^μ field. This can easily be seen by comparing its perturbation expansion with that of G itself. We can look in general at the Green's function $G=(G_0^{-1}-V)^{-1}$ of a particle in an external field, with free Green's function G_0 and interaction V (in the case of scalar electrodynamics, V represents both one-photon and two-photon vertices). The expansions are

$$G=(G_0^{-1}-V)^{-1}=G_0+G_0VG_0+G_0VG_0VG_0+\cdots, \tag{4.1}$$

$$\begin{aligned}\ln\det G-\ln\det G_0&=-\ln\det(1-G_0V)\\&=-\mathrm{tr}\ln(1-G_0V)\\&=\mathrm{tr}(G_0V+\tfrac{1}{2}G_0VG_0V\\&\quad+\tfrac{1}{3}G_0VG_0VG_0V+\cdots).\end{aligned}$$

From this we see that $\ln\det G-\ln\det G_0$ is just $GG_0^{-1}-1$ with its ends sown together, along with a combinatoric factor $1/n$ for n interactions (due to the symmetry of the one-loop diagram under rotation). These expansions are shown diagrammatically in Fig. 1.

In order to define the particle $\hbar$-expansion, we made the replacement $S_P\to S_P/\hbar$, and reexpressed this as the identifications $e=e_P/\sqrt{\hbar}$, $m=m_P/\hbar$. This in turn determines the $\hbar$-dependence of $\det G$, through its dependence on e and m. We therefore expect an expression for K_N of the form

$$\begin{aligned}K_N&=\int\mathfrak{D}A_F\Big(\prod\int\frac{\hbar\,dT}{2}\int\mathfrak{D}x\Big)\\&\quad\times\exp\Big[i\sum\hbar^nS_n(y,z;T;A_F)\Big],\\S_{-1}&=-\sum\int d\tau(\tfrac{1}{2}m_P{}^2+\tfrac{1}{2}\dot{x}^2+e_P\dot{x}\cdot A_F)\\&\quad+\int d^4x(\tfrac{1}{2}A_F\Box A_F-JA_F)+\cdots.\end{aligned} \tag{4.2}$$

The limits of summation for $\sum\hbar^nS_n$ will be determined by analyzing the $\hbar$ dependence of $\ln\det G$. We have chosen to use $A_F=\sqrt{\hbar}A$. Since A is only an integration variable, this choice is arbitrary, but it is the choice which makes *explicit*, even *before* gluon integration, that the naive contribution is order $1/\hbar$:

$$\int\mathfrak{D}A_F\sum\Big(\prod\int\frac{\hbar dT}{2}\int\mathfrak{D}x\Big)\exp\Big(\frac{i}{\hbar}S_{-1}\Big)=\sum\Big(\prod\int\frac{\hbar dT}{2}\int\mathfrak{D}x\Big)\exp\Big[\frac{i}{\hbar}\Big(\tilde{S}_P-\tfrac{1}{2}\int d^4xA_{P,\mathrm{cl}}\Box A_{P,\mathrm{cl}}\Big)\Big]. \tag{4.3}$$

Here we have included only the first two terms of S_{-1} from (4.2) (i.e., the naive contribution). Any other choice of $\hbar$ dependence for A would hide this fact.

We therefore want to evaluate

$$\det G=\det\{[(i\partial-eA)^2-m^2+i\epsilon]^{-1}=\det\{[(i\partial-e_PA_F/\hbar)^2-m_P{}^2/\hbar^2+i\epsilon]^{-1}\}\sim\det\{[(i\hbar\partial-e_PA_F)^2-m_P{}^2+i\epsilon]^{-1}\}. \tag{4.4}$$

(The proportionality constant is unimportant, since we are only concerned with $\det G/\det G_0$.) If we define $A_F(x)=\bar{A}_F(x/\hbar)$, then since x is only a dummy variable (the determinant is over x), we can replace x with $\bar{x}=x/\hbar$. We then have

$$\det\{[(i\hbar\partial - e_p A_F(x))^2 - m^2 + i\epsilon]^{-1}\} = \det\{[(i\bar{\partial} - e_p\bar{A}_F(\bar{x}))^2 - m^2 + i\epsilon]^{-1}\}. \tag{4.5}$$

All explicit $\hbar$ dependence has been eliminated (it is hidden in $\bar{A}$).

The next step is to expand $\ln\det G - \ln\det G_0$ in momenta about $p=0$:

$$\ln\det G - \ln\det G_0 = \int d^4\bar{x}\{f_0[\bar{A}(\bar{x})] + \tfrac{1}{2}[\bar{\partial}\bar{A}(\bar{x})]^2 f_1[\bar{A}(\bar{x})] + \cdots\} \tag{4.6}$$

(Lorentz indices are suppressed; f_i are ordinary functions, not functionals). According to the Landau-Cutkosky rules, such an expansion exists when the charged particle is massive (i.e., there are no singularities at $p=0$). Finally, we restore the $\hbar$ dependence by returning from $\bar{A}$ and $\bar{x}$ to A and x:

$$\ln\det G - \ln\det G_0 = \hbar^{-4}\int d^4x\{f_0[A(x)] + \hbar^2\tfrac{1}{2}[\partial A(x)]^2 f_1[A(x)] + \cdots\}. \tag{4.7}$$

It is now clear that *the particle $\hbar$-expansion for the determinant is just the expansion in derivatives*: There is an $\hbar$ for each derivative and an overall $\hbar^{-4}$ (in 4 dimensions) for $\ln\det G$. Compare this with the field $\hbar$-expansion of the proper-vertex functional, used in studying spontaneous breakdown. There the expansion is in *two* variables: field-$\hbar$ (the number of loops) and the number of derivatives. The no-derivative term (effective potential) contains all orders of field-$\hbar$ (all numbers of loops). The *particle* $\hbar$-expansion of the determinant is much simpler: $\ln\det G$ can be computed entirely from *one*-loop diagrams, and there is only one variable, particle-$\hbar$, whose power *is* the power of momentum.

We now see that, in four dimensions, the exponent of K_N in (4.2) is $i\sum_{n=-4}^{\infty}\hbar^n S_n$. In general (*but see below*), the lowest-order term is S_{-4}, which dominates the naive contribution to S_{-1} in the limit $\hbar\to 0$. As an explicit example, we consider the theory where the photon interaction is replaced by the scalar interaction $\mathcal{L}_I=-g\psi^*\psi\phi$. In that case we find (see the Appendix)

$$S_{-4}(\phi) = -\frac{1}{32\pi^2}\int d^4x[(m^2+g\phi)^2\ln(1+g\phi/m^2) - (\text{finite counterterms})]. \tag{4.8}$$

Therefore, the particle $\hbar$-limit of this theory is not classical mechanics. This theory illustrates a *typical strong-coupling form*: Kinetic-energy terms are not regained until a higher order $(1/\hbar)$. One could say that this theory has no classical-mechanical limit, though it does have a *quantum* particle mechanics.

On the other hand, in scalar electrodynamics *gauge invariance* restricts the form of the expansion. The determinant is itself gauge invariant:

$$\begin{aligned}\det G' &= \det\{[(i\partial - eA')^2 - m^2 + i\epsilon]^{-1}\}\\ &= \det\{[(e^{i\lambda}(i\partial - eA)e^{-i\lambda})^2 - m^2 + i\epsilon]^{-1}\}\\ &= \det\{e^{i\lambda}[(i\partial - eA)^2 - m^2 + i\epsilon]^{-1}e^{-i\lambda}\}\\ &= \det G.\end{aligned} \tag{4.9}$$

Therefore, $\det G$ depends on A only through $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The expansion for $\ln\det G$ thus becomes

$$\ln\det G - \ln\det G_0 = i\hbar^{-4}\int d^4x\left[f_0 + \hbar^2\tfrac{1}{2}F^2 f_1 + \hbar^4\left(\frac{1}{4!}aF^4 + \tfrac{1}{2}bF\Box F\right)f_2 + \cdots\right], \tag{4.10}$$

where the f_i are *constants* [Lorentz indices are again suppressed; the f_2 term actually consists of many terms, one for $(F_{\mu\nu}F^{\mu\nu})^2$, one for $(\epsilon_{\mu\nu\sigma\tau}F^{\mu\nu}F^{\sigma\tau})^2$, etc.]. In fact, $f_0=0$, since $\ln\det G(A=0) - \ln\det G_0 = 0$. Also, the f_1 term is only a renormalization of the free A action, and so can be absorbed into the similar term of S_{-1}. Explicitly, it is (see the Appendix), in $4-2\epsilon$ dimensions,

$$\tfrac{1}{2}F^2 f_1 = -\frac{1}{192\pi^2}\left(\frac{1}{\epsilon} - \gamma\right)e^2 F_{\mu\nu}F^{\mu\nu}. \tag{4.11}$$

Hereafter, we shall ignore such renormalizations. Therefore, the lowest-order remaining contribution of $\ln\det G$ is S_0. S_{-1} consists only of the naive contribution $\tilde{S}_P$, which dominates $\ln\det G$ in the particle $\hbar$-limit: *The (strong-coupling) particle $\hbar$-limit of scalar electrodynamics is the (naive, rela-*

tivistic) classical mechanics of charged particles.

In the following section we will extend our results to arbitrary dimension, and to other theories.

V. GENERALIZATIONS AND DIRECTIONS

The results of Sec. IV can easily be generalized to arbitrary dimension D: The only change in the derivation of the form of the particle $\hbar$-expansion of the determinant is the generalization of the overall factor $\hbar^{-4}$ to $\hbar^{-D}$ [in (4.7)]. This means that the dominant term in the exponent of K_N is now S_{-D} for the $\psi^*\psi\phi$ theory, and S_{4-D} for scalar QED. Therefore, the naive action never dominates for $\psi^*\psi\phi$ theory. However, for scalar QED the situation is dimension dependent: For $D<5$, the naive action dominates. For $D=5$, the naive action and lowest-order contribution of the determinant are of the same order, and a *non-naive* classical mechanics results in the particle $\hbar$-limit. For $D>5$, the determinant dominates.

The generalization to include fermions is also simple. The path integral for the fermion propagator in an external field can be written with the aid of anticommuting particle variables, in addition to the commuting particle coordinates.[4] The $\hbar$ counting is the same: In particular, for spinor QED we have the same expansion (4.10) as for scalar QED, with different values for the constants f_i (see the Appendix).

As a special case, we see that in the massive Schwinger model the naive action S_{-1} dominates the lowest-order determinant contribution S_2. This justifies the neglect of the determinant in a paper by Senjanović and one of the authors.[3] It was shown in that reference that the resulting (dominant) "naive relativistic classical mechanics" is the classical mechanics of the two-dimensional string of Bardeen, Bars, Hanson, and Peccei (BBHP).[13]

In the massless Schwinger model, the masslessness of the fermion causes a singularity at $p=0$, and the expansion itself needs modification. For that model we have

$$\ln\det G-\ln\det G_0=\frac{i}{\hbar^2}\int d^2x\left(\frac{e^2}{\pi}\right)A^\mu\left(g_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)A^\nu$$
$$=\frac{i}{\hbar^2}\int d^2x\left(\frac{e^2}{2\pi}\right)F_{\mu\nu}\Box^{-1}F^{\mu\nu}. \qquad (5.1)$$

The $\hbar$-counting and gauge-invariance arguments are still correct, but the expansion begins (and ends) with a negative power.

The particle $\hbar$-expansion can also be extended to two-dimensional quantum chromodynamics with massive fermions, since, in linear gauges, the field Lagrangian simplifies to a form similar to that of the massive Schwinger model. Again the naive action dominates the determinant, justifying the neglect of quark loops in Ref. 4. The naive classical mechanics is again essentially the classical BBHP string.

If we extend the particle $\hbar$-expansion without modification to QCD (with $D>2$), we find the expansion has characteristics similar to the $\psi^*\psi\phi$ theory. Gauge invariance again restricts the particle $\hbar$-expansion of the determinant to the form (4.10), where now

$$F_{\mu\nu}=(\partial_\mu A_\nu-\partial_\nu A_\mu)+\frac{g}{\hbar}[A_\mu,A_\nu].$$

Thus, we have all powers of A contributing to order $\hbar^{-D}$. However, a modification of the particle $\hbar$-expansion which brings it into accord with non-Abelian gauge invariance may improve this situation. It is of interest to note that a *phenomenological* model of Cornwall and Tiktopoulos[14] uses a first-quantized path-integral formalism for QCD which neglects the determinant and still describes the leading-logarithm infrared behavior of the theory. This lends support to the possibility of a modified particle $\hbar$-expansion in which the naive action dominates. Furthermore, the particle $\hbar$-expansion may have particular relevance to QCD, since the particle $\hbar$-limit probes the infrared behavior: It is not only a strong-coupling limit $g=g_P/\sqrt{\hbar}\to\infty$, but also a large-distance limit, since the length scale $1/m=\hbar/m_P\to 0$ (i.e., dimensionless lengths $mx\to\infty$).

The particle $\hbar$-expansion may also be the natural expansion for the Abelian field theory of charges and monopoles.[15] The relevance of the particle $\hbar$-expansion to magnetic flux quantization already shows at the classical-field level: In the Nielsen-Olesen model of vortices,[16] magnetic charge is quantized even in the classical field solutions—no $\hbar$'s appear. However, when the theory is written in terms of e_P instead of e_F, the magnetic flux quantization takes the form of a Bohr-Sommerfeld quantization, familiar from the first-quantized (i.e., particle) monopole theory of Dirac. Explicitly (since $e=\sqrt{\hbar}e_F=e_P/\sqrt{\hbar}$, $\vec{A}=\vec{A}_F/\sqrt{\hbar}$),

$$\oint d\vec{S}\cdot(e\vec{B})=\oint d\vec{S}\cdot(e_F\vec{B}_F)=2\pi n,$$
$$\oint d\vec{S}\cdot(e_P\vec{B}_F)=2\pi n\hbar. \qquad (5.2)$$

Thus, what may be viewed on the one hand as a "purely classical quantization" may *also* be seen as a (particle-$\hbar$) semiclassical quantization.

Another indication of the relevance of the particle $\hbar$-expansion to monopoles is the fact that there is no classical, relativistically invariant field theory of electrically and magnetically charged fields. This can easily be seen by comparing the form of the charge-quantization condition for charge-monopole field theories in terms of particle and field couplings. By the symmetry of the Lagrangian with respect to charges and monopoles, we see that besides $e=\sqrt{\hbar}e_F=e_P/\sqrt{\hbar}$ we also have $g=\sqrt{\hbar}g_F=g_P/\sqrt{\hbar}$. The charge-quantization condition is thus

$$eg=\sqrt{\hbar}\,e_F g_F=e_P g_P/\hbar=2\pi n\,. \tag{5.3}$$

Therefore, the field $\hbar$-limit is $n\to 0$, which does not satisfy the charge-quantization condition (and therefore violates relativistic invariance), since n is not an integer ($0<n\ll 1$). On the other hand, the particle $\hbar$-limit is $n\to\infty$, familiar from quantum mechanics, where the classical limit is always the limit of quantum numbers becoming large.

We can also see the advantages of the particle $\hbar$-expansion for Abelian charge-monopole field theory by applying the methods of Sec. IV. In monopole theory, there are two determinants, one for charged loops and one for monopole loops. The charged-loop determinant is the same as for QED; the monopole-loop determinant is of the same form, but with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ replaced with $\partial_\mu B_\nu-\partial_\nu B_\mu$, where B_μ is the monopole vector potential. Therefore, *the naive action again dominates*. Also, the determinants are manifestly covariant: *All string dependence is isolated in the naive action*; higher-order corrections are independent of the string direction. Explicitly, the particle $\hbar$-expansion for that theory is

$$e=e_P/\sqrt{\hbar}\,,\quad g=g_P/\sqrt{\hbar}\,,\quad m_e=m_{e,P}/\hbar\,,\quad m_g=m_{g,P}/\hbar\,.$$

The utility of the particle $\hbar$-expansion may be enhanced for theories of the $\psi^*\psi\phi$ type (where the determinant dominates) by the simultaneous use of another expansion, the $1/N$ expansion.[17] By giving the Lagrangian a (global) U(N) symmetry, and by choosing N to tend to infinity as an appropriate inverse power of $\hbar$, the naive action can be made to dominate the one-charged-loop graphs of the determinant.

We conclude with a discussion of the need for a "classical renormalization" in the particle $\hbar$-limit. When calculating classical solutions in order to find the leading behavior of a theory in its particle $\hbar$-limit, divergences are found in the classical action. The source of trouble is the $\Delta_F(x_j(\tau)-x_j(\tau'))$. The divergences are of the form of self-interaction divergences found in, e.g., classical relativistic electrodynamics. Such troubles could have been anticipated from the point of view that our expansion is a strong-coupling expansion, and, as such, includes much loop structure in the leading approximation. By using a regularization (such as a cutoff for small proper times), we have shown that the divergences can be absorbed by renormalizations of the mass and action (addition of a constant term to the action is equivalent to wave-function renormalization, since $K_N\sim e^{iS}$). The resulting renormalized (naive) action involves a principal-value prescription for the singularity. This will be reported more fully elsewhere.

Note added in proof. After acceptance of this paper for publication, we received a report by R. Brandt, F. Neri, and D. Zwanziger. These authors have independently recognized the advantages of particle variables in the monopole problem (our Sec. V).

APPENDIX: CALCULATION OF DETERMINANT

1. General method

We will now describe the general method for calculating the determinant to finite order in the particle $\hbar$-expansion, with explicit examples. The first step, as in Sec. III, is to reexpress the determinant in terms of the proper-time Green's function $\tilde{G}$. We use the identity[9]

$$\ln\det G-\ln\det G_0=\ln\det\frac{1}{H-i\epsilon}-\ln\det\frac{1}{H_0-i\epsilon}$$

$$=\int d^Dx\int_0^\infty\frac{dT}{T}\,[\langle x|\exp(-iHT/\hbar)|x\rangle-\langle x|\exp(-iH_0T/\hbar)|x\rangle]\,. \tag{A1}$$

We therefore need to evaluate

$$\langle x|\exp(-iHT/\hbar)|x\rangle\equiv\tilde{G}(x,x;T,0)\,. \tag{A2}$$

Of course, we cannot evaluate $\tilde{G}$ exactly, since H involves an arbitrary external field [$\phi(x)$ or $A^\mu(x)$]. However, we can evaluate an arbitrary, finite number of terms in the semiclassical expansion of $\tilde{G}$; as shown in Sec. III, this is the particle $\hbar$-expansion. Since we also know (from Sec. IV) that this expansion is an expansion in the number of derivatives of the external field, it will clearly be helpful to employ the expansion[18]

$$\phi(X)=\phi(x)+[(X-x)\cdot\partial]\phi(x)+\cdots\,. \tag{A3}$$

Here X is the position operator in the Hamiltonian $H(P,X)$, and x is the c number in

$$\langle x|\exp[-iH(P,X)T/\hbar]|x\rangle\,.$$

Inserting this expansion into H, we have H as an explicit function only of P, $X-x$, and $\phi(x)$ (and derivatives). Exhibiting this dependence explicitly as $H(P, X-x, \phi(x))$, we have the further simplification

$$\langle x|\exp[-iH(P,X-x,\phi(x))T/\hbar]|x\rangle = \langle 0|\exp[-iH(P,X,\phi(x))T/\hbar]|0\rangle . \quad \text{(A4)}$$

This result follows immediately from translation invariance.

From either the path-integral formalism[19] or the usual operator formalism[20] we know that $\tilde{G}$ can be evaluated exactly when H is quadratic in P and X. In that case, the result is

$$\langle x|\exp[-iH(P,X)T/\hbar]|y\rangle = \left(\det \frac{\partial^2}{\partial x \partial y}\frac{iS}{2\pi\hbar}\right)^{1/2} e^{iS/\hbar} , \quad \text{(A5)}$$

where S is the classical action. Therefore, we can easily evaluate the determinant for the first few orders in $\hbar$ by keeping only as many terms in the expansion (A3) as will keep H quadratic. For the cases of $\mathcal{L}_I = -g\psi^*\psi\phi$ and scalar electrodynamics, respectively, we then have

$$H(P,X,\phi(x)) \cong -\tfrac{1}{2}[P^2 - m^2 - g\phi(x) - X\cdot\partial\phi(x) - \tfrac{1}{2}X^\mu X^\nu \partial_\mu\partial_\nu\phi(x)] , \quad \text{(A6)}$$

$$H(P,X,A_\mu(x)) \cong -\tfrac{1}{2}[(P_\mu - e\tfrac{1}{2}X^\nu F_{\nu\mu}(x))^2 - m^2] .$$

Here we have used gauge invariance to drop some terms in the expansion

$$\begin{aligned} A_\mu(X) &= A_\mu(x) + (X-x)^\nu\partial_\nu A_\mu(x) \\ &= \tfrac{1}{2}(X-x)^\nu(\partial_\nu A_\mu - \partial_\mu A_\nu)(x) \\ &\quad + \frac{\partial}{\partial X^\mu}[(X-x)^\nu A_\nu(x) + \tfrac{1}{4}(X-x)^\nu(X-x)^\sigma \\ &\quad \times(\partial_\nu A_\sigma + \partial_\sigma A_\nu)(x)] . \end{aligned} \quad \text{(A7)}$$

By the arguments of Sec. IV, we see that this approximation will give us all of S_{-4} and S_{-2} (in arbitrary D, S_{-D}, and S_{2-D}), plus parts of higher orders [the rougher approximation $\phi(X)\cong\phi(x)$ would give us all of S_{-4} plus parts of higher orders]. Therefore (for $D=4$), along with the naive action S_{-1} with which we started, we can easily calculate all contributions to the particle action of order $\hbar^n$ with $n\le -1$. To calculate higher orders, we can consider the nonquadratic part of H as a *perturbation* to the quadratic part, and use either old-fashioned perturbation theory in the operator formalism or Feynman-diagramlike perturbation theory in the path-integral formalism. In the cases of scalar and spinor QED, owing to gauge invariance, there is a simpler method: Since the determinant depends on A_μ only through $F_{\mu\nu}$ (which is itself a first derivative of A_μ), to finite order in $\hbar$ it consists of only a finite number of Feynman diagrams. Specifically, S_{2n-D} depends only on one-loop diagrams with at most $2n$ external lines.

2. Explicit calculations

In this section we will use the method described above—the quadratic approximation to the Hamiltonian—to calculate the lowest orders in $\hbar$. Combining Eqs. (A1), (A4), and (A5), we have

$$\ln\det G - \ln\det G_0 = \int d^Dx \int_0^\infty \frac{dT}{T}\left[\left(\det\frac{\partial^2}{\partial y_\mu \partial z_\nu}\frac{iS}{2\pi\hbar}\right)^{1/2} \times e^{iS/\hbar} - (S\to S_0)\right]\Bigg|_{y=z=0} . \quad \text{(A8)}$$

Here $S(y,z,T;\phi(x))$ is determined from $H(P,X,\phi(x))$ by using

$$L(p,\dot q) = p\dot q - H(p,q) , \quad \dot q = \partial H/\partial p \quad \text{(A9)}$$

solving for $q(T)$ in terms of $q(0)$ and $\dot q(0)$, integrating

$$S(q(0),\dot q(0),T) = \int_0^T d\tau\, L(q(0),\dot q(0),\tau) , \quad \text{(A10)}$$

and reinserting $q(T)$ to find $S(q(0),q(T),T) = S(y,x,T)$. Here we use p and q as the c numbers corresponding to P and X; $\phi(x)$ and its derivatives are considered as constants until the final $\int d^Dx$.

As explicit examples of these methods, we consider the calculation of the lowest-order contribution for $\psi^*\psi\phi$ theory (S_{-D}), and the lowest nonzero contributions for scalar and spinor QED (S_{2-D}, which is merely a wave-function renormalization, but illustrates the method). For the former case we need only the approximation $\phi(X)\cong\phi(x)$ (lowest order means no derivatives), so we have simply the free Hamiltonian $H = -\frac{1}{2}(P^2 - m^2 - g\phi(x))$ [remember that $\phi(x)$ is a constant as far as P and X are concerned; $m^2 + g\phi(x)$ is a fixed-(mass)^2 term]. After a trivial calculation (since the classical equation of motion is $\ddot q = 0$), we find

$$S(y,z,T;\phi(x)) = -\tfrac{1}{2}[(m^2+g\phi)T + (z-y)^2/T] ,$$

so (A8) becomes

$$\begin{aligned} \ln\det G - \ln\det G_0 &= i\int d^Dx \int_0^\infty \frac{dT}{T}(2\pi\hbar i T)^{-D/2}\exp(-\tfrac{1}{2}im^2T/\hbar)[\exp(-\tfrac{1}{2}ig\phi T/\hbar) - 1] \\ &= i(4\pi)^{-D/2}\hbar^{-D}\Gamma\left(-\frac{D}{2}\right)\int d^Dx[(m^2+g\phi(x))^{D/2} - m^D] . \end{aligned} \quad \text{(A11)}$$

We have analytically continued in D in order to apply dimensional regularization. As usual, we take $D \to D - 2\epsilon$, $\epsilon \to 0$, and use

$$\Gamma(\epsilon - D/2) = \begin{cases} (-1)^{D/2} \dfrac{1}{\Gamma(D/2+1)} (1/\epsilon + \psi(D/2+1)) & \text{for } D \text{ even}, \\ (-1)^{(D+1)/2} \dfrac{1}{\Gamma(D/2+1)} \pi & \text{for } D \text{ odd}, \end{cases} \quad \text{(A12)}$$

which follows from $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$ ($\psi(x) = (d/dz)\ln\Gamma(z)$). Note that the result (A11) is already finite for D odd. The result is therefore

$$\ln\det G - \ln\det G_0 = i(4\pi)^{-D/2}\hbar^{-D} \frac{1}{\Gamma(D/2+1)}$$
$$\times \begin{cases} (-1)^{D/2} \displaystyle\int d^D x \left\{ \left(\frac{1}{\epsilon} + \psi(D/2+1)\right) [(m^2 + g\phi(x))^{D/2} - m^D] \right. \\ \qquad \left. + \left[(m^2 + g\phi(x))^{D/2} \ln\left(\frac{m^2 + g\phi(x)}{4\pi\hbar^2}\right) - m^D \ln(m^2/4\pi\hbar^2) \right] \right\} & \text{for } D \text{ even}, \\ (-1)^{(D+1)/2} \pi \displaystyle\int d^D x [(m^2 + g\phi(x))^{D/2} - m^D] & \text{for } D \text{ odd}. \end{cases} \quad \text{(A13)}$$

Assuming the $1/\epsilon$ term can be canceled by a renormalization counterterm (true for $D \leq 6$; for $D > 6$, $\mathcal{L}_I = -g\psi^*\psi\phi$ is nonrenormalizable), the final result becomes

$$\ln\det G - \ln\det G_0 = i(4\pi)^{-D/2}\hbar^{-D} \frac{1}{\Gamma(D/2+1)}$$
$$\times \begin{cases} (-1)^{D/2} \displaystyle\int d^D x [(m^2 + g\phi)^{D/2} \ln(1 + g\phi/m^2) - (\text{finite counterterms})] & \text{for } D \text{ even}, \\ (-1)^{(D+1)/2} \pi \displaystyle\int d^D x [(m^2 + g\phi)^{D/2} - (\text{finite counterterms})] & \text{for } D \text{ odd}. \end{cases} \quad \text{(A14)}$$

For the case of scalar QED, we use $A_\mu(X) \cong \frac{1}{2} X^\nu F_{\nu\mu}(x)$ [see (A6)],

$$L = -\tfrac{1}{2}(\dot{X}^2 + m^2) - e\dot{X}^\mu A_\mu(X) = -\tfrac{1}{2}[\dot{X}^2 + m^2 + eX^\mu \dot{X}^\nu F_{\mu\nu}(x)].$$

After a simple calculation, we find

$$S(y, z, T; F(x)) = -\tfrac{1}{2}[m^2 T - (z-y)eF(1 - e^{eFT})^{-1}(z-y)],$$

using matrix notation for Lorentz indices. The solution for all orders in F (which includes parts of all orders in $\hbar$, since F is a first derivative of A) is then

$$\ln\det G - \ln\det G_0 = i\int d^D x \int_0^\infty \frac{dT}{T} (2\pi i\hbar T)^{-D/2} \exp(-\tfrac{1}{2} i m^2 T/\hbar) \left[\left(\det \frac{e^{eFT} - 1}{eFT}\right)^{-/2} - 1 \right]. \quad \text{(A15)}$$

Since we are only interested in the lowest order in $\hbar$ (i.e., the lowest order in F), we expand in F and get [using $\det M = \exp(\text{tr} \ln M)$]

$$\ln\det G - \ln\det G_0 = i\int d^D x \int_0^\infty \frac{dT}{T} (2\pi i\hbar T)^{-D/2} \exp(-\tfrac{1}{2} i m^2 T/\hbar) [(1 - \tfrac{1}{48} e^2 T^2 \text{tr} F^2) - 1]$$
$$= -\tfrac{1}{12} i (4\pi)^{-D/2} e^2 \hbar^{2-D} \Gamma(2 - \tfrac{1}{2}D) m^{D-4} \int d^D x F_{\mu\nu} F^{\mu\nu}. \quad \text{(A16)}$$

As stated above, the whole term is a wave-function renormalization, and can be absorbed into the similar term in S_{-1}.

The generalization to spinor QED is simple because[9]

$$\text{tr} \ln \frac{1}{\not{P} - e\not{A} - m + i\epsilon} = \tfrac{1}{2} \text{tr} \int_0^\infty \frac{dT}{T} \exp[\tfrac{1}{2} i((\not{P} - e\not{A})^2 - m^2)T/\hbar] \quad \text{(A17)}$$

and

$$(\not{P} - e\not{A})^2 = (P_\mu - eA_\mu)^2 - \tfrac{1}{2} ie\gamma^\mu \gamma^\nu F_{\mu\nu}. \quad \text{(A18)}$$

Since in the leading approximation [Eq. (A6)] F is a constant, the matrix $\gamma^\mu\gamma^\nu F_{\mu\nu}$ commutes with everything, and so can be treated as a nonmatrix (for higher orders, we can use path-integral methods for first quantization with spin[4]). Effectively we just change the (mass)^2, $m^2 \to m^2 - \frac{1}{2}ie\gamma^\mu\gamma^\nu F_{\mu\nu}$. We then have, instead of (A15),

$$\ln \det G - \ln \det G_0 = i2^{D/2-1}\int d^Dx \int_0^\infty \frac{dT}{T}(2\pi\hbar i T)^{-D/2}\exp(-\tfrac{1}{2}im^2T/\hbar)\,.$$
$$\times\left[\left(\det\frac{e^{eFT}-1}{eFT}\right)^{-1/2}\left(\frac{1}{2^{D/2}}\mathrm{tr}\,\exp(-\tfrac{1}{4}e\gamma^\mu\gamma^\nu F_{\mu\nu}T/\hbar)\right)-1\right], \tag{A19}$$

where the remaining trace is a matrix trace. The factor of 2 for $D=4$ is from the $\frac{1}{2}$ in (A17) and the 4 from $\mathrm{tr}\,1 = 2^{D/2} = 4$; physically, it arises because spin $\frac{1}{2}$ has twice as many spin components as spin 0. Again expanding in F, we have

$$\ln \det G - \ln \det G_0 = i2^{D/2-1}\int d^Dx \int_0^\infty \frac{dT}{T}(2\pi\hbar i T)^{-D/2}\exp(-\tfrac{1}{2}im^2T/\hbar)[(1-\tfrac{1}{48}e^2T^2\mathrm{tr}F^2+\tfrac{1}{24}e^2T^2\mathrm{tr}F^2)-1]$$
$$=+\tfrac{1}{12}i(2\pi)^{-D/2}e^2\hbar^{2-D}\Gamma(2-\tfrac{1}{2}D)m^{D-4}\int d^Dx\,F_{\mu\nu}F^{\mu\nu}\,. \tag{A20}$$

Also, owing to Fermi statistics, it is actually $-(\ln\det G - \ln\det G_0)$ which contributes to the exponent in the functional integral over A.

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