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Field theories in terms of particle-string variables: Spin, internal symmetries, and arbitrary dimension

M. B. Halpern*
Department of Physics, University of California, Berkeley, California 94720

A. Jevicki†
Institute for Advanced Study, Princeton, New Jersey 08540

P. Senjanović*
Department of Physics, University of California, Berkeley, California 94720
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We provide essential tools for a program of rewriting field theories in terms of particle-string variables. The general methods are illustrated in the case of quantum chromodynamics: (1) We find the particle-trajectory representation for the quark Green's functional. (2) Thus, we derive directly correct end-point terms for quarks at the ends of strings. (1) and (2) are for any number of dimensions. (3) In two dimensions, we find a functional bridge from quantum chromodynamics to the Bardeen-Bars-Hanson-Peccei string.

I. INTRODUCTION

In recent years, the thrust of fundamental theory has turned increasingly toward the problem of quark confinement—the extraction of hadrons from local quantum field theory. Evidence is mounting that we may already know the beginning [quantum chromodynamics (QCD)] and the end (stringlike and baglike theories) of such a program. Yet the path from field variables to particle-string-bag variables has remained elusive.

In 1950, Feynman[1] made the first step in this direction when he showed how to express the Green's functionals of scalar field theories in terms of particle variables. In a previous publication,[2] we pointed out that these particle variables $x_\mu(\tau)$ can be identified as the trajectory of the end points of a string. Indeed, in two dimensions, where the gluon variables can be integrated explicitly, we demonstrated this by providing a direct functional bridge from certain Abelian field theories to the Bardeen-Bars-Hanson-Peccei (BBHP) string.[3]

Our goal in this paper is to provide the tools for a program of rewriting general field theories in terms of particle and particle-string variables. The first step in such a program is to find particle-trajectory representations for Green's functionals of fields carrying spin and internal symmetry in an arbitrary number of dimensions. The methods we use will suffice for any such fields; for simplicity, we choose to illustrate all our work with the case of QCD.

This is the subject of Sec. II. There we find the particle-trajectory functional representation of the quark Green's functional in QCD. We find that each quark is associated with an $x_\mu(\tau)$ (end-point trajectory) and an anticommuting trajectory variable $\psi(\tau)$. The quantity $\bar{\psi}\psi$ is conserved and equal to 1 for a single quark. The derivation thus provides *correct end-point terms for quarks at the ends of strings*.

In Sec. III, we discuss the same problem in light-cone variables. In Sec. IV, we apply the formalism, in the case of two dimensions, to find a functional bridge from QCD to the BBHP string. There is also an Appendix, where we give details of the derivation of the fermionic functional integrals.

II. QUARK GREEN'S FUNCTIONAL AND QUARK-END-POINT TERMS FROM QCD

We consider QCD in $D \geqslant 2$ dimensions:

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - e\tfrac{1}{2}\lambda_\alpha \not{A}_\alpha - M)\psi - \tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha , \tag{1}$$

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - ef^{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu . \tag{2}$$

The color group may be SU(N) or U(N), and the desired number of flavors is assumed implicitly. As discussed in Ref. 2, the Green's functions of the theory can be expressed as functional integrals over quark Green's functionals. As an example, the quark four-point function, shown in Fig. 1, is given by

$$\begin{aligned} G_4^{\alpha_1\alpha_2\alpha_3\alpha_4} &= \langle 0|T(\bar{\psi}_{\alpha_1}(z_1)\psi_{\alpha_2}(z_2)\psi_{\alpha_3}(z_3)\bar{\psi}_{\alpha_4}(z_4))|0\rangle \\ &= -\int \mathfrak{D}A^\alpha_\mu(\Delta\delta)(\det G_F{}^{-1}) \\ &\quad\times \exp\left[i\int d^Dx(-\tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha\right] \\ &\quad\times[G_F^{\alpha_2\alpha_4}(z_2,z_4;A)G_F^{\alpha_3\alpha_1}(z_3,z_1;A) \\ &\qquad - G_F^{\alpha_3\alpha_4}(z_3,z_4;A)G_F^{\alpha_2\alpha_1}(z_2,z_1;A)] . \end{aligned} \tag{3}$$

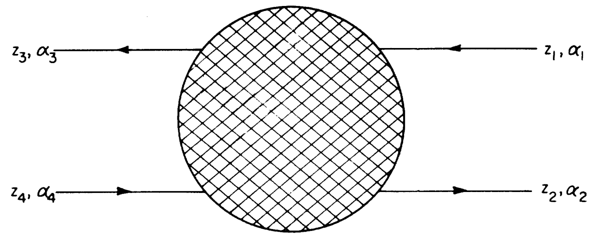

FIG. 1. The quark four-point function.

Here α_i are indices labeling spin, color (and flavor), while $G_F^{\alpha\rho}(x,y;A)$ is the quark Green's functional:

$$(i\not\partial - e\not A^{\alpha}\tfrac{1}{2}\lambda_\alpha - M)^{x}_{\rho\gamma} G_F^{\gamma\beta}(x,y;A) = \delta_{\rho\beta}\,\delta^D(x-y)\,. \qquad (4)$$

$(\delta\Delta)$ is some gauge-fixing and Faddeev-Popov determinant. The correct time-ordering prescription is obtained via $M \to M - i\epsilon$. In finding a particle-trajectory representation for G_F, the quark field variables will be entirely eliminated from the theory in favor of particle variables.

The method for finding this representation follows that of Ref. 2, but there are complications due to spin and internal symmetry. The first step toward the desired representation is to invert Eq. (4).

Toward this end, we introduce a preliminary operator formalism. We define position and momentum operators $P_{\mu,\mathrm{op}}, x_{\mu,\mathrm{op}}$, and coordinate eigenstates:

$$\begin{aligned} &x_{\mu,\mathrm{op}}|x\rangle = x_\mu|x\rangle, \quad \langle x|y\rangle = \delta^D(x-y)\,, \\ &\langle x|P_{\mu,\mathrm{op}}|y\rangle = -i\partial_\mu^x\langle x|y\rangle\,. \end{aligned} \qquad (5)$$

We will also introduce anticommuting quark operators $\psi_{\alpha,\mathrm{op}}, \overline{\psi}_{\beta,\mathrm{op}}$ such that

$$[\psi_{\alpha,\mathrm{op}}, \overline{\psi}_{\beta,\mathrm{op}}]_+ = \delta_{\alpha\beta}\,. \qquad (6)$$

Here $\overline{\psi} = \psi^\dagger\gamma^0$ but, so that ψ may be a spinor under Lorentz transformations, we have taken the $[\psi,\overline{\psi}]_+$ algebra. Such representations were first introduced for dual models, and for the same reason, in Ref. 4.

We construct a $\psi_{\mathrm{op}}, \overline{\psi}_{\mathrm{op}}$ Hilbert space by multiple application of $\overline{\psi}_{\mathrm{op}}$ on a state $|0\rangle$, defined by

$$\psi_{\alpha,\mathrm{op}}|0\rangle = 0, \quad \langle 0|0\rangle = 1\,. \qquad (7)$$

Most useful to us will be the product states

$$\begin{aligned} &|x\alpha\rangle \equiv \overline{\psi}_{\alpha,\mathrm{op}}|x,0\rangle\,, \\ &|x,0\rangle \equiv |x\rangle \times |0\rangle\,, \\ &\langle x\beta| \equiv \langle x,0|\overline{\psi}_{\beta,\mathrm{op}}\,, \\ &\langle x\alpha|y\beta\rangle = \delta^D(x-y)\delta_{\alpha\beta}\,. \end{aligned} \qquad (8)$$

We also define an operator $G_{\mathrm{op}}^{\alpha\beta}$ such that

$$\langle x|G_{\mathrm{op}}^{\alpha\beta}|y\rangle = G^{\alpha\beta}(x,y;A)\,. \qquad (9)$$

We now have the formalism to incorporate the spin and internal-symmetry indices in the operator statement; define further

$$G_{\mathrm{op}} = \overline{\psi}_{\alpha,\mathrm{op}} G_{\mathrm{op}}^{\alpha\beta}\psi_{\beta,\mathrm{op}}\,. \qquad (10)$$

Then it is immediate that

$$\langle x\alpha|G_{\mathrm{op}}|y\beta\rangle = G^{\alpha\beta}(x,y;A)\,. \qquad (11)$$

In this notation it is not hard to see that Eq. (4) is equivalent to

$$-\overline{\psi}_{\mathrm{op}}[\not P_{\mathrm{op}} + e\not A^{\alpha}(x_{\mathrm{op}})\tfrac{1}{2}\lambda_\alpha + M]\psi_{\mathrm{op}} G_{\mathrm{op}} = 1_{\mathrm{op}}\,. \qquad (12)$$

The verification proceeds by sandwiching Eq. (12) between $\langle x\alpha|$ and $|y\beta\rangle$. You must note that

$$H_{\mathrm{op}} \equiv \overline{\psi}_{\mathrm{op}}[\not P_{\mathrm{op}} + e\not A^{\alpha}(x_{\mathrm{op}})\tfrac{1}{2}\lambda_\alpha + M]\psi_{\mathrm{op}} \qquad (13)$$

does not change the particle number

$$N \equiv \overline{\psi}_{\mathrm{op}}\psi_{\mathrm{op}}\,, \qquad (14)$$

so only intermediate states with $N=1$ can contribute.

The desired inversion is then

$$G_{\mathrm{op}} = -\frac{1_{\mathrm{op}}}{\overline{\psi}_{\mathrm{op}}[\not P_{\mathrm{op}} + e\not A^{\alpha}(x_{\mathrm{op}})\frac{1}{2}\lambda_\alpha + M - i\epsilon]\psi_{\mathrm{op}}}\,, \qquad (15)$$

where we have chosen the time-ordered boundary condition. Further, then

$$\begin{aligned} G_{\alpha\beta}(x,y;A) &= \langle x\alpha|G_{\mathrm{op}}|y\beta\rangle \\ &= -\left\langle x\alpha\left|\frac{1_{\mathrm{op}}}{H_{\mathrm{op}} - i\epsilon}\right|y\beta\right\rangle \\ &= -i\int_0^\infty dT\langle x\alpha|e^{-iTH_{\mathrm{op}}}|y\beta\rangle\,. \end{aligned} \qquad (16)$$

To get the $(H_{\mathrm{op}} - i\epsilon)^{-1}$ form, we again used the fact that H_{op} does not change particle number, and that the external states have $N=1$.

Equation (16), together with Eq. (13), is in large part the attainment of our goal. The quark Green's functional is expressed in terms of (operator) particle variables.

For further application, as in Ref. 2, it is valuable to put Eq. (16) in a functional integral form. This is a matter of defining anticommuting c numbers on a suitable grid. The calculation is technically involved, and there are some tricky points, especially in regard to the external wave functions. Details are given in the Appendix; here we state the final result:

$$\begin{aligned} &\langle x\alpha|e^{-iTH_{\mathrm{op}}}|y\beta\rangle \\ &\quad = \int \mathfrak{D}x_\mu\mathfrak{D}P_\nu\mathfrak{D}\overline{\psi}\mathfrak{D}\psi\Phi^*_{x,\alpha}(x(T),\overline{\psi}(T),\psi(T)) \\ &\quad\quad \times\Phi_{y,\beta}(x(0),\overline{\psi}(0),\psi(0))e^{iS}\,, \end{aligned} \qquad (17)$$

$$S=\int_0^T (P\cdot\dot{x}+\tfrac{1}{2}i\bar{\psi}\overleftrightarrow{\partial}_\tau\psi-H)d\tau\,, \tag{18}$$

$$H=\bar{\psi}[\not{P}+e\not{A}^\alpha(x)\tfrac{1}{2}\lambda_\alpha+M]\,\psi\,. \tag{19}$$

Here Φ,Φ^* are the external wave functions

$$\Phi_{y,\beta}(x(0),\bar{\psi}(0),\psi(0))=e^{-\bar{\psi}(0)\psi(0)/2}\,\bar{\psi}_\beta(0)\delta^D(x(0)-y)\,,$$
$$\Phi^*_{x,\alpha}(x(T),\bar{\psi}(T),\psi(T))=e^{-\bar{\psi}(T)\psi(T)/2}\psi_\alpha(T)\,\delta^D(x(T)-x)\,. \tag{20}$$

The functional integral is over the location of the quark trajectory $x_\mu(\tau)$, as a function of some "proper" time τ, and over anticommuting c numbers $\psi,\bar{\psi}$.

Except for the details of the external wave functions, the functional quark dynamics is what one might guess from Eq. (13). In operator language, using $[x_{\mu,\mathrm{op}},P_{\nu,\mathrm{op}}]=ig_{\mu\nu}$ and $[\psi_{\alpha,\mathrm{op}},\bar{\psi}_{\beta,\mathrm{op}}]_+=\delta_{\alpha,\beta}\,,\partial_\tau O=i[H,O]$, the Hamiltonian equations of motion are

$$\begin{aligned}
&i\partial_\tau\psi_{\mathrm{op}}(\tau)=[\not{P}_{\mathrm{op}}+M+e\not{A}^\alpha(x_{\mathrm{op}})\tfrac{1}{2}\lambda_\alpha]\psi_{\mathrm{op}}(\tau)\,,\\
&i\partial_\tau\bar{\psi}_{\mathrm{op}}(\tau)=-\bar{\psi}_{\mathrm{op}}(\tau)[\not{P}_{\mathrm{op}}+M+e\not{A}^\alpha(x_{\mathrm{op}})\tfrac{1}{2}\lambda_\alpha]\,,\\
&\dot{x}_{\mu,\mathrm{op}}(\tau)=\bar{\psi}_{\mathrm{op}}\gamma_\mu\psi_{\mathrm{op}}\,,\\
&\dot{P}_{\mu,\mathrm{op}}(\tau)=-\bar{\psi}_{\mathrm{op}}e\frac{\partial}{\partial x^\mu_{\mathrm{op}}}\not{A}^\alpha(x_{\mathrm{op}})\tfrac{1}{2}\lambda_\alpha\psi_{\mathrm{op}}\,.
\end{aligned} \tag{21}$$

From these, it follows that $N=\bar{\psi}_{\mathrm{op}}\psi_{\mathrm{op}}$ is conserved, as expected

$$\partial_\tau(\bar{\psi}_{\mathrm{op}}\psi_{\mathrm{op}})=0\,. \tag{22}$$

This also follows from an application of Noether's theorem to the invariance $\psi\to e^{i\lambda}\psi$. In the sector we are considering, it is consistent to set $\bar{\psi}\psi=1$ in the Hamiltonian, and take instead

$$H'=\bar{\psi}[\not{P}+e\not{A}^\alpha(x)\tfrac{1}{2}\lambda_\alpha]\psi+M\,. \tag{23}$$

This can be done inside the functional integral, but one must *not* tamper with the external wave functions, as given in Eq. (20).

Another remark worth making is about *Zitterbewegung*. The $\dot{x}_{\mu,\mathrm{op}}$ equation of motion is showing that phenomenon: in $|\psi\rangle,\langle\bar{\psi}|$ states, $\langle\dot{x}_\mu\rangle\sim\gamma_\mu$. This can also be seen directly by doing the $\mathfrak{D}P_\mu$ integration. Thus, we have not only "ordinary" *Zitterbewegung* ($\mu=1,2,\ldots D-1$), but an "x^0 *Zitterbewegung*" as τ goes on. Apparently, the fermion is switching back and forth between particle and antiparticle. (What is constant is $N=1$, but N cannot tell the difference between fermion forward in τ and antifermion backward in τ.)

An important by-product of our result, Eq. (17), is that we have derived *correct end-point terms from QCD for quarks at the ends of strings*. In fact, of course, we do not yet know how to integrate the non-Abelian gluon field (except for $D=2$). Proceeding formally however, by putting Eq. (17) back into Eq. (3), we derive for the string-plus-end-points action

$$S_{\mathrm{total}}=S^{(0)}_{\mathrm{quark}}+S^{(0)}_{\mathrm{antiquark}}+e^2S_{\mathrm{string}}\,. \tag{24}$$

Here

$$S^{(0)}_{\mathrm{quark}}=\int_0^{T_1}d\tau_1[\dot{x}_1\cdot P_1+\tfrac{1}{2}i\bar{\psi}_1\overleftrightarrow{\partial}_{\tau_1}\psi_1-\bar{\psi}_1(\not{P}_1+M)\psi_1] \tag{25}$$

and the same for $S^{(0)}_{\mathrm{antiquark}}$, with $T_2,\tau_2,x_2,P_2,\psi_2,\bar{\psi}_2$. (The difference at this stage is only in the external wave functions. See also Sec. IV.) We do not have an explicit expression for $e^2\,S_{\mathrm{string}}$ (the result of the gluon integration). We do know, however, that it is $O(e^2)$, and it is additive. From a general point of view $e^2\,S_{\mathrm{string}}$ is an extremely complicated functional of $x_1,P_1,x_2,P_2,\psi_1,\bar{\psi}_1,\psi_2,\bar{\psi}_2$. We speculate that it will be convenient *not* to integrate A^α_μ out, but rather to *change variables* $\mathfrak{D}A^\alpha_\mu\to\mathfrak{D}X_\mu(\sigma,\tau)$ to stringlike variables. $e^2\,S_{\mathrm{string}}$ will then also be a functional of these variables.

The reader should recall that $T_{1,2}$ are finally integrated over, as in Eq. (16). The Bars-Hanson[5] end-point terms have no such additional integration. Thus, the connection of our end-point terms with those of Bars and Hanson deserves further investigation. In fact, we can show such a connection in two dimensions (see Ref. 2 and Sec. IV of the present paper). In an arbitrary number of dimensions, a fruitful approach may be to consider the *semiclassical limit*[6] of our end-point terms: if one *also* varies with respect to T, it is easy to show that, for each quark, the additional equation of motion

$$H=\bar{\psi}(\not{P}+M+e\not{A}^\alpha\tfrac{1}{2}\lambda_\alpha)\psi=0 \tag{26}$$

is obtained. The solution of the system is then very close to that of an ordinary Dirac equation. In particular, one obtains a "pseudoclassical" dynamics, similar to that studied by Berezin and Marinov and other workers.[7] We will, at the end of Sec. III, make some further remarks about the difficulties of showing correspondence between our end-point terms and those of other workers.

III. LIGHT-CONE TREATMENT

Again we begin with the action for QCD in $D\geq 2$ dimensions [Eq. (1)]. This time, we introduce light-cone coordinates[8]

$$\begin{aligned}
&A^\pm=\frac{A^0\pm A^{D-1}}{\sqrt{2}}\,,\quad x^\pm=\frac{x^0\pm x^{D-1}}{\sqrt{2}}\,,\\
&\gamma^\pm=\frac{\gamma^0\pm\gamma^{D-1}}{\sqrt{2}}\,,\quad(\gamma^+)^2=(\gamma^-)^2=0,\quad(\gamma^+,\gamma^-)_+=2\,,\\
&R_\pm=\tfrac{1}{2}\gamma^\mp\gamma^\pm,R_++R_-=1,\quad R_+R_-=0\,,\\
&\psi_\pm\equiv R_\pm\psi,\quad A^\alpha_\mu\tfrac{1}{2}\lambda_\alpha\equiv A_\mu\,.
\end{aligned} \tag{27}$$

After a little algebra, we obtain

$$\mathcal{L} = \sqrt{2}\,(\psi_-)^\dagger(i\partial_- - eA^+)\psi_- + \sqrt{2}\,(\psi_+)^\dagger(i\partial_+ - eA^-)\psi_+ - \frac{1}{\sqrt{2}}(\psi_-)^\dagger(i\gamma^i\partial_i - eA^i\gamma_i + M)\gamma^+\psi_+ - \frac{1}{\sqrt{2}}(\psi_+)^\dagger(i\gamma^i\partial_i - eA^i\gamma_i + M)\gamma^-\psi_- . \tag{28}$$

Here $1 \le i \le D-2$ denotes transverse variables. As is well known, ψ_- is a dependent variable

$$\sqrt{2}\,(i\partial_- - eA^+)\psi_- - \frac{1}{\sqrt{2}}(i\gamma^i\partial_i - eA_i\gamma^i + M)\gamma^+\psi_+ = 0\,, \tag{29}$$

and can be eliminated from the dynamics.

We intend to compute Green's functions involving external ψ_+'s only, so we begin with the generating functional

$$Z[\rho,\rho^\dagger] = \mathfrak{N}\int \mathfrak{D}A\,(\delta\Delta)\mathfrak{D}\psi_-^\dagger\mathfrak{D}\psi_-\mathfrak{D}\psi_+^\dagger\mathfrak{D}\psi_+ \times \exp\left[i\int d^Dx(\mathcal{L} + 2^{1/4}\rho^+\psi_+ + 2^{1/4}\psi_+^\dagger\rho)\right]. \tag{30}$$

The factors $2^{1/4}$ have been introduced for convenience, and $\mathfrak{N}$ is the customary normalization. We now rescale

$$\psi_\pm \to 2^{-1/4}\psi_\pm\,, \quad \psi_\pm^\dagger \to 2^{-1/4}\psi_\pm^\dagger \tag{31}$$

and integrate over ψ_-, $\psi_-^\dagger$. The result is

$$Z[\rho,\rho^\dagger] = \mathfrak{N}\int \mathfrak{D}A\,(\delta\Delta)\mathfrak{D}\psi_+^\dagger\mathfrak{D}\psi_+\det(i\partial_- - eA^+) \times \exp\left[i\int d^Dx(\mathcal{L}_+ + \rho^\dagger\psi_+ + \psi_+^\dagger\rho)\right], \tag{32}$$

$$\mathcal{L}_+ = \psi_+^\dagger(i\partial_+ - eA^-)\psi_+ - \tfrac{1}{2}\psi_+^\dagger K_T(i\partial_- - eA^+)^{-1}K_T^\dagger\psi_+\,,$$
$$K_T = i\gamma^i\partial_i - eA^i\gamma_i + M\,, \tag{33}$$
$$K_T^\dagger = -i\gamma^i\partial_i + eA^i\gamma_i + M\,.$$

In the usual way, one then expresses Green's functions in terms of the quark Green's functional. For the light-cone ordered four-point function, we find

$$\begin{aligned}\tilde{G}_4^{\alpha_1\alpha_2\alpha_3\alpha_4} &= 2\langle 0|T[\psi_{+\alpha_1}^\dagger(z_1)\psi_{+\alpha_2}(z_2)\psi_{+\alpha_3}(z_3)\psi_{+\alpha_4}^\dagger(z_4)]|0\rangle \\ &= -\int \mathfrak{D}A_\mu^\alpha(\Delta\delta)\det(i\partial_- - eA^+)(\det\tilde{G}_F^{-1})\exp\left[i\int d^Dx(-\tfrac{1}{4}F_{\mu\nu}^\alpha F_\alpha^{\mu\nu})\right] \\ &\quad\times[\tilde{G}_F^{\alpha_2\alpha_4}(z_2,z_4;A)\tilde{G}_F^{\alpha_3\alpha_1}(z_3,z_1;A) - \tilde{G}_F^{\alpha_3\alpha_4}(z_3,z_4;A)\tilde{G}_F^{\alpha_2\alpha_1}(z_2,z_1;A)]\,.\end{aligned} \tag{34}$$

Here the light-cone ordered quark Green's functional $\tilde{G}_F$ satisfies

$$\left[i\partial_+ - eA^- - \tfrac{1}{2}(i\gamma^i\partial_i - eA_i\gamma^i + M)\frac{1}{i\partial_- - eA^+}(-i\gamma^i\partial_i + eA_i\gamma^i + M)\right]\tilde{G}_F(A) = R_+\delta^D\,. \tag{35}$$

The (light-cone) time-ordering prescription is, as usual, $M \to M - i\epsilon$ (or $K_T \to K_T - i\epsilon$, $K_+^\dagger \to K_+^\dagger - i\epsilon$). Because $(R_+,\gamma^i) = 0$, it is easy to show from Eq. (35) that $R_+\tilde{G}_F R_+ = \tilde{G}_F$, as it should be. It is our job now to invert $\tilde{G}_F$, and express the result in particle variables.

In this form, we are going to have trouble with one of our inversion tricks: If we are to use again the simple identity

$$[i(H_{\rm op} - i\epsilon)]^{-1} = \int_0^\infty dT\,e^{-iH_{\rm op}T}\,,$$

we must have the $i\epsilon$ term of definite sign. In Sec. II, this was true; we found $i\epsilon\bar{\psi}_{\rm op}\psi_{\rm op} \sim i\epsilon$ in the sector of interest. In the present light-cone formulation, the $i\epsilon$ term is loaded with structure of unknown sign: we need only worry about the $i\epsilon$ term at $e=0$, because other ϵ structure is part of the vertices and therefore irrelevant. But even at $e=0$, the $i\epsilon$ term in the square brackets of Eq. (35) has the form $(i\partial_-)^{-1}i\epsilon M$ (and will be worse when we introduce the fermion variables).

To circumvent this, we employ the trick of Ref. 2. Define another, more "bosonic," Green's functional by

$$\tilde{G}_F \equiv 2(i\partial_- - eA^+)\bar{G}\,, \tag{36}$$

$$\left[2(i\partial_+ - eA^-)(i\partial_- - eA^+) - K_T\frac{1}{i\partial_- - eA^+}K_T^\dagger(i\partial_- - eA^+)\right]\bar{G} = R_+\delta^D\,,$$
$$K_T \to K_T - i\epsilon\,, \quad K_T^\dagger \to K_T^\dagger - i\epsilon\,. \tag{37}$$

Now the $i\epsilon$ term at $e=0$ has the form $+i\epsilon 2M \simeq +i\epsilon$, and this will suffice for the inversion. We record

$$\left[-2(i\partial_+ - eA^-)(i\partial_- - eA^+) + K_T\frac{1}{i\partial_- - eA^+}K_T^\dagger(i\partial_- - eA^+) - i\epsilon\right]G = -R_+\delta^D. \tag{38}$$

It will also be useful to have the equation in another form: multiplying the equation by R_+ from the left and from the right, and noticing that $(R_+, \gamma^i)=0$, we can write

$$R_+[\cdots]R_+R_+\overline{G}R_+=-R_+\delta^D\,, \tag{39}$$

where $[\cdots]$ is exactly the quantity in square brackets in Eq. (38). The R_+'s will not prevent the inversion.

Following Sec. II, we next introduce an operator formalism. For $x_\mu, P_\mu, x_{\mu,\text{op}}, P_{\mu,\text{op}}$, we take over the definitions of Sec. II. For the fermionic structure, we introduce

$$\begin{aligned}&[\psi_{+\alpha,\text{op}}\psi^\dagger_{+\beta,\text{op}}]_+=(R_+)_{\alpha\beta}\,,\\ &R_+\psi_{+,\text{op}}=\psi_{+,\text{op}}\,,\\ &\psi^\dagger_{+,\text{op}}R_+=\psi^\dagger_{+,\text{op}}\,.\end{aligned} \tag{40}$$

The relevant states (and operator Green's functional) are

$$\begin{aligned}&|x\alpha\rangle=\psi^\dagger_{+\alpha,\text{op}}|x,0\rangle,\quad |x,0\rangle=|x\rangle\times|0\rangle\,,\\ &\psi_{+\beta,\text{op}}|0\rangle=0,\quad \langle 0|0\rangle=1,\\ &\langle x\alpha|y\beta\rangle=\delta^D(x-y)(R_+)_{\alpha\beta}\,,\\ &\overline{G}_\text{op}\equiv\psi^\dagger_{+\alpha,\text{op}}\overline{G}^{\alpha\beta}_\text{op}\psi_{+\beta,\text{op}}\,,\\ &\langle x\alpha|\overline{G}_\text{op}|y\beta\rangle=(R_+\overline{G}R_+)_{\alpha\beta}=\overline{G}_{\alpha\beta}\,.\end{aligned} \tag{41}$$

The operator statement equivalent to Eq. (39) is now

$$(H_\text{op}-\tfrac{1}{2}i\epsilon\psi^\dagger_{+,\text{op}}\psi_{+,\text{op}})\overline{G}_\text{op}=-\tfrac{1}{2}1_\text{op}\,, \tag{42}$$

$$\begin{aligned}H_\text{op}=&-\psi^\dagger_{+,\text{op}}[P^-_\text{op}+eA^-(x_\text{op})][P^+_\text{op}+eA^+(x_\text{op})]\psi_{+,\text{op}}\\ &+\tfrac{1}{2}\psi^\dagger_{+,\text{op}}[-\gamma^iP_{i,\text{op}}-eA_i(x_\text{op})\gamma^i+M]\frac{1}{P^+_\text{op}+eA^+(x_\text{op})}\\ &\times[\gamma^iP_{i,\text{op}}+eA_i(x_\text{op})\gamma^i+M][P^+_\text{op}+eA^+(x_\text{op})]\psi_{+,\text{op}}\,.\end{aligned} \tag{43}$$

As in Sec. II, the operator $N_+=\psi^\dagger_+\psi_+$ commutes with H_op and is equal to 1 on the states $|x\alpha\rangle$, so we have

$$\begin{aligned}\langle x\alpha|\overline{G}_\text{op}|y\beta\rangle&=[R_+\overline{G}(x,y)R_+]_{\alpha\beta}=\overline{G}_{\alpha\beta}(x,y)\\ &=-\tfrac{1}{2}\left\langle x\alpha\left|\frac{1}{H_\text{op}-\frac{1}{2}i\epsilon\psi^\dagger_{+,\text{op}}\psi_{+,\text{op}}}\right|y\beta\right\rangle\\ &\cong-\tfrac{1}{2}\left\langle x\alpha\left|\frac{1}{H_\text{op}-\frac{1}{2}i\epsilon}\right|y\beta\right\rangle\\ &=-\frac{i}{2}\int_0^\infty dT\langle x\alpha|e^{-iH_\text{op}T}|y\beta\rangle\,.\end{aligned} \tag{44}$$

A calculation almost identical to that of the Appendix yields the functional integral form

$$\langle x\alpha|e^{-iH_\text{op}T}|y\beta\rangle=\int\mathfrak{D}\psi^\dagger_+\mathfrak{D}\psi_+\mathfrak{D}P\mathfrak{D}x\Phi^{T*}_{x\alpha}(\psi_+,\psi^\dagger_+,x)\times\Phi^0_{y\beta}(\psi_+,\psi^\dagger_+,x)e^{iS}\,, \tag{45}$$

$$S=\int_0^T d\tau(P\cdot\dot{x}+\tfrac{1}{2}\psi^\dagger_+\overleftrightarrow{\partial}_\tau\psi_+-H)\,, \tag{46}$$

where H is the same form as Eq. (43) with all $x_{\mu,\text{op}}P_{\mu,\text{op}}$ replaced by c numbers x_μ, P_μ, and fermonic operators $\psi^\dagger_+, \psi_{+,\text{op}}$ replaced by anticommuting c numbers $\psi^\dagger_+, \psi_+$. These are taken to satisfy $R_+\psi_+=\psi_+$, $\psi^\dagger_+R_+=\psi^\dagger_+$. The Φ's are external wave functions:

$$\begin{aligned}&\Phi^0_{y\beta}(\psi_+,\psi^\dagger_+,x)=\delta^D(x(0)-y)\Psi_{0,\beta}(\psi_+,\psi^\dagger_+)\,,\\ &\Psi_{0,\beta}(\psi_+,\psi^\dagger_+)=\exp[-\tfrac{1}{2}\psi^\dagger_+(0)\psi_+(0)]\psi^\dagger_{+\beta}(0)\,,\\ &\Phi^{T*}_{x\alpha}(\psi_+,\psi^\dagger_+,x)=\delta^D(x(T)-x)\Psi^*_{T,\alpha}(\psi_+,\psi^\dagger_+)\,,\\ &\Psi^*_{T,\alpha}(\psi_+,\psi^\dagger_+)=\exp[-\tfrac{1}{2}\psi^\dagger_+(T)\psi_+(T)]\psi_{+\alpha}(T)\,.\end{aligned} \tag{47}$$

As in Sec. II, one easily recovers $\partial_\tau(\psi^\dagger_{+,\text{op}}\psi_{+,\text{op}})=0$ from the equations of motion. Further, if desired, $\psi^\dagger_+\psi_+$ may be set to 1 in H (inside the functional integral). This completes our task. Equations (44) to (47) express the quark Green's functional as a path integral over particle trajectories in light-cone variables.

Light-cone quark end-point terms can again be read off from the $e=0$ form of Eq. (45); for quark or antiquark

$$S^{(0)}=\int_0^T d\tau[\dot{x}\cdot P+\tfrac{1}{2}i\psi^\dagger_+\overleftrightarrow{\partial}_\tau\psi_++\psi^\dagger_+P^-P^+\psi_+-\tfrac{1}{2}\psi^\dagger_+(-\gamma^iP_i+M)(\gamma^iP_i+M)\psi_+]\,. \tag{48}$$

Again, the connection with Bars-Hanson end-point terms is obscure, except for $D=2$. Drawing on our experience in Ref. 2, we can make a few remarks about why this is so.

In the light-cone gauge, $A^+=0$, we can easily do the P^- integration in Eq. (45), obtaining a factor

$$\delta(\dot{x}^++\psi^\dagger_+P^+\psi_+)\approx\delta(\dot{x}^++P^+)\,. \tag{49}$$

Since P^+ has arbitrary sign, so also will $\dot{x}^+$. In Ref. 2, we argued that the end-point terms of Bars and Hanson[5] correspond to a single sign of $\dot{x}^+$ (positive for quarks, negative for antiquarks), and we explored a method (the "chopping" procedure of Ref. 2) of eliminating the sign changes of $\dot{x}^+$. Indeed, if sign changes of P^+, $\dot{x}^+$ are ignored, the δ function of Eq. (49) is enough to do the T integration and get very close to Bars and Hanson's end-point terms. Unfortunately, the "chopping" procedure is Lorentz-invariant only for $D=2$; in other dimensions the connection between our end-point terms and those of Bars and Hanson is not yet clear. As mentioned in Sec. II, it is our feeling that a study of the semiclassical limit of our dynamics may provide the connection with the terms of Bars and Hanson: it is physically reasonable to expect that sign changes of $\dot{x}^+$ (or $\dot{x}^0$ in Sec. II) would be suppressed in that limit.

IV. BRIDGE FROM QCD$_2$ TO STRING

In this section, we shall specialize the results of Sec. III to $D=2$ and proceed to find a functional bridge from two-dimensional quantum chromodynamics to the BBHP string. We will assume some familiarity with the methods of Ref. 2, where we detailed a similar transition for Abelian gauge theories in two dimensions.

In two dimensions, ignoring annihilation graphs and quark loops,[9] we have

$$\begin{aligned}\tilde{G}_4^{\alpha_1\alpha_2\alpha_3\alpha_4}(z_1,z_2,z_3,z_4) &= 2\langle 0|T[\psi^\dagger_{+\alpha_1}(z_1)\psi_{+\alpha_2}(z_2)\psi_{+\alpha_3}(z_3)\psi^\dagger_{+\alpha_4}(z_4)]|0\rangle \\ &= \int DA^+_\alpha DA^-_\alpha(\delta\Delta)\exp\left(\frac{i}{2}\int d^2x\, F^\alpha_{+-}F^\alpha_{+-}\right) \\ &\quad\times \tilde{G}_F^{\alpha_2\alpha_4}(z_2 z_4;A)\tilde{G}_F^{\alpha_3\alpha_1}(z_3,z_1;A)\,. \end{aligned}\tag{50}$$

We know further that

$$\tilde{G}_F = 2(i\partial_- - eA^+)\overline{G}\,,\tag{51}$$

$$\overline{G}_{\alpha\beta}(x,y) = -\frac{i}{2}\int_0^\infty dT\langle x\alpha|e^{-iH_{\rm op}T}|y\beta\rangle\,,\tag{52}$$

$$\langle x\alpha|e^{-iH_{\rm op}T}|y\beta\rangle = \int \mathfrak{D}\psi^\dagger_+\mathfrak{D}\psi_+\mathfrak{D}P\mathfrak{D}x\Phi^*_{x\alpha}{}^T(\psi_+,\psi^\dagger_+,x) \times\Phi^0_{y\beta}(\psi_+,\psi^\dagger_+,x)e^{iS}\,,\tag{53}$$

$$S = \int_0^T d\tau(P\cdot\dot{x} + \tfrac{1}{2}i\psi^\dagger_+\overleftrightarrow{\partial}_\tau\psi - H)\,,\tag{54}$$

$$H = -\psi^\dagger_+(P^- + eA^-)(P^+ + eA^+)\psi_+ + \tfrac{1}{2}M^2\psi^\dagger_+\psi_+\,.\tag{55}$$

As in Ref. 2, we choose to "chop" out the "pure" quark part of $\tilde{G}_F^{\alpha_3\alpha_1}(z_3,z_1;A)$ by the Lorentz-invariant,[10] gauge-invariant insertion $\theta(\dot{x}^+)$ inside the functional integral. This procedure was discussed in detail in Ref. 2. In terms of trajectories, we are requiring the particle *always* to go forward in proper time. In light-cone gauge diagrams, it is not hard to show that the chopping amounts to a change in the propagator,

$$\begin{aligned}S_F(z_3-z_1) &\to \theta(z_3^+ - z_1^+)S_F(z_3-z_1) \\ &= \int\frac{d^2p}{(2\pi)^2}\,\frac{\exp[-ip\cdot(z_3-z_1)]}{\not{p}-M+i\epsilon}\,\theta(p^+)\,,\end{aligned}\tag{56}$$

which suppresses all light-cone Z graphs. Similarly, we will (later) chop out the pure antiquark part of $\tilde{G}_F^{\alpha_2\alpha_4}(z_2,z_4)$ by the insertion $\theta(-\dot{x}^+)$: "Pure" antiquarks moving forward in τ are like quarks moving always backward in proper time; this corresponds to

$$\begin{aligned}S_F(z_2-z_4) &\to \theta(z_4^+ - z_2^+)S_F(z_2-z_4) \\ &= \int\frac{d^2p}{(2\pi)^2}\,\frac{\exp[+ip\cdot(z_2-z_4)]}{-\not{p}-M+i\epsilon}\,\theta(p^+)\,.\end{aligned}\tag{57}$$

Both choppings thus correspond to $\theta(p^+)$ insertions on all Fermi lines. As mentioned in Ref. 2 it is a fact that the 't Hooft integral equation "chops itself" during solution: The same solution is obtained for that equation whether the extra $\theta(p^+)$'s are fed in or not. We further define formally in coordinate space $\theta(0)=0$. This suppresses (light-cone gauge) all mass and vertex renormalizations. The chopping procedure is quite appropriate to get to the BBHP string—which, e.g., *neglects* mass renormalizations. On the other hand, the procedure is presumably only an interim measure for the present non-Abelian case, as we are not taking full advantage of the N^{-1} expansion.[11] (The N^{-1} expansion, by itself, suppresses vertex corrections.)

For the quark Green's functional, then, we wish to study

$$\tilde{G}_{FC}^{\alpha_3\alpha_1}(z_3,z_1;A) = D_-^{z_3}\int_0^\infty dT\int_{\substack{x(T)=z_3\\ x(0)=z_1}}\mathfrak{D}\psi^\dagger_+\mathfrak{D}\psi_+\mathfrak{D}x^+\mathfrak{D}x^-\mathfrak{D}P^-\mathfrak{D}P^+\Psi^*_{T,\alpha_3}(\psi_+,\psi^\dagger_+)\Psi_{0,\alpha_1}(\psi_+,\psi^\dagger_+)\theta(\dot{x}^+)e^{iS},$$

$$D_-^{z_3} \equiv \partial_-^{z_3} + ieA^+(z_3)\,,\tag{58}$$

where S is given in Eq. (54). The subscript C on $\tilde{G}_F$ denotes "chopped." It is our option, if we choose, to set $\psi^\dagger_+\psi_+=1$ in S.

The following manipulation (on the quark Green's functional) follows quite closely the procedure of Ref. 2: Choosing the light-cone gauge ($A^+=0$) and doing the P^- integration, we obtain

$$\begin{aligned}\tilde{G}_{FC}^{\alpha_3\alpha_1}(z_3,z_1;A) = \int_0^\infty dT\int_{\substack{x^+(T)=z_3^+\\ x^+(0)=z_1^+\\ x^-(T)=z_3^-\\ x^-(0)=z_1^-}} &\mathfrak{D}\psi^\dagger_+\mathfrak{D}\psi_+\mathfrak{D}P^+\mathfrak{D}x^+\mathfrak{D}x^-\Psi^*_{T,\alpha_3}(\psi_+,\psi^\dagger_+)\Psi_{0,\alpha_1}(\psi_+,\psi^\dagger_+)iP^+(T)\delta(\epsilon_\tau(\dot{x}^+ + P^+)) \\ &\times\theta(-P^+)\exp\left[i\int_0^T d\tau(P^+\dot{x}^- + \tfrac{1}{2}i\psi^\dagger_+\overleftrightarrow{\partial}_\tau\psi_+ + e\psi^\dagger_+P^+A^-\psi_+ - \tfrac{1}{2}M^2)\right]\,.\end{aligned}\tag{59}$$

In this form, the factor ∂_- [of Eq. (58)] has been brought inside the functional integral by the standard method $[\partial_- \to iP^+(T)]$. ϵ_τ is the size of the τ grid, as in Ref. 2. Because the chopping does not allow $P^+=0$ (no mass or vertex renormalizations) the following change of variable is well-defined[12]:

$$\tau \equiv -\int_0^\lambda \frac{d\bar\lambda}{\bar P^+(\bar\lambda)}, \quad T\equiv -\int_0^\Lambda \frac{d\bar\lambda}{\bar P^+(\bar\lambda)}, \tag{60}$$

$$P^+(\tau)\equiv \bar P^+(\lambda), \quad x^\pm(\tau)\equiv \bar x^\pm(\lambda), \quad \psi_+(\tau)\equiv\bar\psi_+(\lambda), \quad \psi_+^\dagger(\tau)\equiv\bar\psi_+^\dagger(\lambda). \tag{61}$$

The minus signs are necessary to maintain $\lambda\geq 0$. Then, as in Ref. 2,

$$\delta[\epsilon_\tau(\dot x^+ + P^+)] = \delta(z_3^+ - z_1^+ - \Lambda)\prod_{0<\lambda<\Lambda}\delta(\bar x^+(\lambda)-\lambda - z_1^+). \tag{62}$$

These δ functions are just enough to do the $\bar x^+$ and Λ integrations, with the result

$$\tilde G_{FC}^{\alpha_3\alpha_1}(z_3,z_1;A) = -i\theta(z_3^+ - z_1^+)\int_{\substack{\bar x^-(z_3^+-z_1^+)=z_3^- \\ \bar x^-(0)=z_1^-}} \mathfrak{D}\bar P^+ \mathfrak{D}\bar x^- \theta(-\bar P^+) e^{iS}\Psi^*_{z_3^+-z_1^+,\alpha_3}[\bar\psi_+,\bar\psi_+^\dagger]\Psi_{0,\alpha_1}[\bar\psi_+,\bar\psi_+^\dagger], \tag{63}$$

$$S=\int_0^{z_3^+-z_1^+} d\lambda\left\{\frac{M^2}{2\bar P^+} + \frac{i}{2}\bar\psi_+^\dagger \overleftrightarrow{\partial}_\lambda \bar\psi_+ - \bar\psi_+^\dagger eA^-(\lambda+z_1^+,\bar x^-(\lambda))\bar\psi_+ + \bar P^+ \dot{\bar x}^-\right\}. \tag{64}$$

Note that, as promised, the chopped G_F is nonzero only for $z_3^+ > z_1^+$. A last change of variables,

$$\lambda + z_1^+ \equiv \tau_1,\ \bar P^+(\lambda)\equiv -P_1^+(\tau_1), \quad \bar x^-(\lambda)\equiv x_1^-(\tau_1), \quad \bar\psi_+(\lambda)\equiv\psi_{+1}(\tau_1), \quad \bar\psi_+^\dagger(\lambda)\equiv\psi_{+1}^\dagger(\tau_1), \tag{65}$$

brings us to a resting place for the chopped quark Green's functional

$$\tilde G_{FC}^{\alpha_3\alpha_1}(z_3,z_1;A) = -i\theta(z_3^+ - z_1^+)\int_{\substack{x_1^-(z_3^+)=z_3^- \\ x_1^-(z_1^+)=z_1^-}} \mathfrak{D}P_1^+\mathfrak{D}x_1^-\theta(P_1^+)\mathfrak{D}\psi_{+1}^\dagger\mathfrak{D}\psi_{+1} e^{iS_1}\Psi^*_{z_3^+,\alpha_3}(\psi_{+1},\psi_{+1}^\dagger)\Psi_{z_1^+,\alpha_1}(\psi_{+1},\psi_{+1}^\dagger), \tag{66}$$

$$S_1 = \int_{z_1^+}^{z_3^+} d\tau_1(-P_1^+\dot x_1^+ + \tfrac{1}{2}i\,\psi_{+1}^\dagger\overleftrightarrow{\partial}_{\tau_1}\psi_{+1} - H_1), \tag{67}$$

$$H_1 = \frac{M^2}{2P_1^+(\tau_1)} + e\psi_{+1}^\dagger(\tau_1)A^-(\tau_1,x_1^-(\tau_1))\psi_{+1}(\tau_1). \tag{68}$$

We turn our attention now to the "antiquark" Green's functional $\tilde G_F^{\alpha_2\alpha_4}(z_2,z_4;A)$. The previous $\theta(\dot x^+)$ chopping guaranteed that the quark moved always forward in τ; to guarantee that the antiquark moves always forward in τ, we must chop now with $\theta(-\dot x^+)$. We are studying then

$$\tilde G_{FC}^{\alpha_2\alpha_4}(z_2,z_4;A) = D_-^{z_2}\int_0^\infty dT\int_{\substack{x_2(T)=z_2 \\ x_2(0)=z_4}} \mathfrak{D}\psi_+^\dagger\mathfrak{D}\psi_+\mathfrak{D}x^-\mathfrak{D}x^+\mathfrak{D}P^-\mathfrak{D}P^+\theta(-\dot x^+)\Psi^*_{T,\alpha_2}(\psi_+,\psi_+^\dagger)\Psi_{0,\alpha_4}(\psi_+,\psi_+^\dagger)e^{iS}, \tag{69}$$

where S is given in Eq. (54). As above, we bring $\partial_-^{z_2}$ inside the functional integral, choose light-cone gauge, and do the P^- integral. The resulting δ functional $\delta[\epsilon_\tau(\dot x^+ + P^+)]$ this time brings the θ function to $\theta(+P^+)$. The desired rescaling is this time

$$\tau\equiv\int_0^\lambda \frac{d\bar\lambda}{\bar P^+(\lambda)},$$

$$T\equiv +\int_0^\Lambda \frac{d\bar\lambda}{\bar P^+(\lambda)}, \tag{70}$$

$$P^+(\tau)\equiv\bar P^+(\lambda), \quad x^\pm(\tau)=\bar x^\pm(\lambda),$$

and similarly for the fermionic variables, as in Eq. (61). The sign is again chosen to keep $\lambda\geq 0$. Because of this sign change relative to Eq. (60), the δ-functional identity is now

$$\delta[\epsilon_\tau(\dot x^+ + P^+)] = \delta(z_2^+ - z_4^+ + \Lambda) \times \prod_{0<\lambda<\Lambda}\delta[\bar x^+(\lambda) - z_4^+ + \lambda]. \tag{71}$$

The integration over Λ results in the factor $\theta(z_4^+ - z_2^+)$, and Λ is set equal to $z_4^+ - z_2^+$. The integration over $\bar x^+(\lambda)$ is simple, setting $\bar x^+(\lambda) = z_4^+ - \lambda$.

Finally, in analogy to Eq. (65), we make the change of variables

$$z_4^+ - \lambda \equiv \bar x^+(\lambda), \quad \overline{\text{variables}}\,(\lambda)\equiv\text{variables}_2(\tau_2) \tag{72}$$

obtaining our result, the chopped antiquark Green's functional:

$$\tilde{G}_{FC}^{\alpha_2\alpha_4}(z_2,z_4;A)=-i\theta(z_4^+-z_2^+)\int_{\substack{x_2^-(z_4^+)=z_4^-\\ x_2^-(z_2^+)=z_2^-}}\mathfrak{D}P_2^+\mathfrak{D}x_2^-\mathfrak{D}\psi_{+2}^\dagger\mathfrak{D}\psi_{+2}\theta(P_2^+)\Psi_{z_4^+,\alpha_4}(\psi_{+2},\psi_{+2}^\dagger)\Psi^*_{z_2^+,\alpha_2}(\psi_{+2},\psi_{+2}^\dagger)e^{iS_2}, \tag{73}$$

$$S_2=\int_{z_2^+}^{z_4^+}d\tau_2(-P_2^+\dot{x}_2^- -\tfrac{1}{2}i\,\psi_{+2}^\dagger\overleftrightarrow{\partial}_{\tau_2}\psi_{+2}-H_2), \tag{74}$$

$$H_2=\frac{M^2}{2P_2^+(\tau_2)}-e\psi_{+2}^\dagger(\tau_2)A^-(\tau_2,x_2^-(\tau_2))\psi_{+2}(\tau_2). \tag{75}$$

Since $z_4^+>z_2^+$ for the antiquark (moving forward in τ), we have taken the liberty of interchanging the positions of Ψ and Ψ^* (at the cost of one minus sign). Note that, for the antiquark, the final wave function is *not* complex-conjugated, while the initial wave function is. Also the derivative term differs in sign from the quark form. These effects are because (pure) antiquarks are like pure quarks moving backward in proper time: The roles of ψ_{+2}, $\psi_{+2}^\dagger$ are interchanged relative to the roles of ψ_{+1}, $\psi_{+1}^\dagger$ for the quark. (In operator form, $\psi_{+1,\mathrm{op}}|0\rangle=0$, but $\psi_{+2,\mathrm{op}}^\dagger|0\rangle=0$.)

We choose to make the formalism uniform for quark and antiquark by the fermonic change of variables

$$\psi_{+2}^\dagger\equiv\tilde{\psi}_{+2},\quad \psi_{+2}\equiv-\tilde{\psi}_{+2}^\dagger. \tag{76}$$

Thus,

$$\begin{aligned}
&\Psi_{z_4^+,\alpha_4}(\psi_{+2},\psi_{+2}^\dagger)=\Psi^*_{z_4^+,\alpha_4}(\tilde{\psi}_{+2},\tilde{\psi}_{+2}^\dagger),\\
&\Psi_{z_2^+,\alpha_2}(\psi_{+2},\psi_{+2}^\dagger)=-\Psi_{z_2^+,\alpha_2}(\tilde{\psi}_{+2},\tilde{\psi}_{+2}^\dagger),\\
&-\tfrac{1}{2}i\psi_{+2}^\dagger\overleftrightarrow{\partial}_{\tau_2}\psi_{+2}=+\tfrac{1}{2}i\tilde{\psi}_{+2}^\dagger\overleftrightarrow{\partial}_{\tau_2}\tilde{\psi}_{+2},\\
&\psi_{+2}^\dagger\tfrac{1}{2}\lambda_\alpha\psi_{+2}=+\tilde{\psi}_{+2}^\dagger\tfrac{1}{2}\lambda_\alpha^T\tilde{\psi}_{+2},\\
&\psi_{+2}^\dagger\psi_{+2}=\tilde{\psi}_{+2}^\dagger\tilde{\psi}_{+2}.
\end{aligned}\tag{77}$$

In terms of the variables with tildes then, the antiquark Green's functional has exactly the same form as the quark [except for the sign change of e and the transpose (T) on all λ matrices]. Dropping the tildes, we record our final result for the chopped antiquark Green's functional:

$$\tilde{G}_{FC}^{\alpha_2\alpha_4}(z_2,z_4;A)=i\theta(z_4^+-z_2^+)\int\mathfrak{D}P_2^+\mathfrak{D}x_2^-\mathfrak{D}\psi_{+2}^\dagger\mathfrak{D}\psi_{+2}\theta(P_2^+)\Psi^*_{z_4^+,\alpha_4}(\psi_{+2},\psi_{+2}^\dagger)\Psi_{z_2^+,\alpha_2}(\psi_{+2},\psi_{+2}^\dagger)e^{iS_2}, \tag{78}$$

$$S_2=\int_{z_2^+}^{z_4^+}d\tau_2(-P_2^+\dot{x}_2^-+\tfrac{1}{2}i\psi_{+2}^\dagger\overleftrightarrow{\partial}_{\tau_2}\psi_{+2}-H_2), \tag{79}$$

$$H_2=\frac{M^2}{2P_2^+(\tau_2)}-e\psi_{+2}^\dagger(\tau_2)A^{-T}(\tau_2,x_2^-(\tau_2))\psi_{+2}(\tau_2). \tag{80}$$

Now we are ready for the four-point function. Inserting Eqs. (66) and (78) into Eq. (50) and doing the functional integration over A^- (as in Ref. 2), the result is

$$\tilde{G}_4^{\alpha_1\alpha_2\alpha_3\alpha_4}(z_1,z_2,z_3,z_4)\Big|_{\substack{z_3^+=z_4^+\\ z_1^+=z_2^+\\ z_3^+>z_1^+}}=-\int_{\substack{x_1^-(z_3^+)=z_3^-\\ x_1^-(z_1^+)=z_1^-\\ x_2^-(z_3^+)=z_4^-\\ x_2^-(z_1^+)=z_2^-}}\mathfrak{D}x_1^-\mathfrak{D}P_1^+\mathfrak{D}x_2^-\mathfrak{D}P_2^+\,\theta(P_1^+)\theta(P_2^+)\mathfrak{D}\psi_{+1}^\dagger\mathfrak{D}\psi_{+1}\mathfrak{D}\psi_{+2}^\dagger\mathfrak{D}\psi_{+2}$$
$$\times\Psi^{(1)*}_{z_3^+,\alpha_3}\Psi^{(2)*}_{z_3^+,\alpha_4}\Psi^{(1)}_{z_1^+,\alpha_1}\Psi^{(2)}_{z_1^+,\alpha_2}e^{iS}, \tag{81}$$

$$S=\int_{z_1^+}^{z_3^+}d\tau(-P_1^+\dot{x}_1^- -P_2^+\dot{x}_2^-+\tfrac{1}{2}i\psi_{+1}^\dagger\overleftrightarrow{\partial}_\tau\psi_{+1}+\tfrac{1}{2}i\psi_{+2}^\dagger\overleftrightarrow{\partial}_\tau\psi_{+2}-H), \tag{82}$$

$$H=\frac{M^2}{2P_1^+}+\frac{M^2}{2P_2^+}+\tfrac{1}{2}e^2\psi_{+2}^\dagger\tfrac{1}{2}\lambda_\alpha^T\psi_{+2}|x_1^- -x_2^-|\,\psi_{+1}^\dagger\tfrac{1}{2}\lambda_\alpha\psi_{+1}. \tag{83}$$

The superscripts (1) and (2) on the external wave functions denote the factors involving ψ_1 and ψ_2, respectively.

At this point, we prefer to employ the equivalent operator Hamiltonian

$$H_{\mathrm{op}}=\frac{M^2}{2P_{1,\mathrm{op}}^+}+\frac{M^2}{2P_{2,\mathrm{op}}^+}+\tfrac{1}{2}e^2\psi_{2,\mathrm{op}}^\dagger\tfrac{1}{2}\lambda_\alpha^T\psi_{2,\mathrm{op}}|x_1^- -x_2^-|\,\psi_{1,\mathrm{op}}^\dagger\tfrac{1}{2}\lambda_\alpha\psi_{1,\mathrm{op}}. \tag{84}$$

If we choose the initial state as a color singlet, the functional integral will be expressible in terms of the eigenvalues spanned by the state(s) $(1/\sqrt{N})\,\psi^\dagger_{1\alpha,\mathrm{op}}\psi^\dagger_{2\alpha,\mathrm{op}}|0\rangle$. In the color-singlet sector H_{op} takes the form, then,

$$H_{\mathrm{op}} \sim \frac{M^2}{2P^+_{1,\mathrm{op}}} + \frac{M^2}{2P^+_{2,\mathrm{op}}} + \frac{e^2}{2N}\,\mathrm{Tr}(\tfrac{1}{2}\lambda_\alpha\tfrac{1}{2}\lambda_\alpha)\,|x_1^- - x_2^-|\,. \tag{85}$$

For U(N), the potential is thus $\frac{1}{4}(e^2N)|x_1^- - x_2^-|$, while for SU($N$) it is $\frac{1}{4}e^2(N-1/N)|x_1^- - x_2^-|$. This is the BBHP string Hamiltonian.

APPENDIX: DERIVATION OF FERMIONIC FUNCTIONAL INTEGRALS

Here we extend the techniques of Candlin[13] and Berezin[14] to derive the functional integral forms stated in the text.

We will need fermionic operators $\psi_{s,\mathrm{op}}, \overline{\psi}_{s,\mathrm{op}}$ satisfying $(\psi_{r,\mathrm{op}}, \overline{\psi}_{s,\mathrm{op}})_+ = \delta_{rs}$ and anticommuting c numbers $\psi_s, \overline{\psi}_s$. We assume that the appropriate Klein transformation has been done, so that the anticommuting c numbers also anticommute with the operators. The indices r, s subsume spin, color, flavor, etc. We will need a number of theorems.

Theorem 1:

$$[\psi_{m,\mathrm{op}}, e^{\overline{\psi}_{\mathrm{op}}\psi}] = \psi_m e^{\overline{\psi}_{\mathrm{op}}\psi} \tag{A1}$$

Here $\overline{\psi}_{\mathrm{op}}\psi = \sum_{s=1}^N \overline{\psi}_{s,\mathrm{op}}\psi_s$, and the proof is immediate using $e^ABe^{-A} = B + [A,B]$ (when $[A,B]$ is a c number).

Theorem 2:

$$e^{\overline{\psi}'\psi_{\mathrm{op}}}e^{\overline{\psi}_{\mathrm{op}}\psi} = e^{\overline{\psi}_{\mathrm{op}}\psi}e^{\overline{\psi}'\psi_{\mathrm{op}}}e^{\overline{\psi}'\psi}\,. \tag{A2}$$

This is also immediate, using $e^Ae^B = e^Be^Ae^{[A,B]}$ for $[A,B]$ a c number.

Theorem 3: Completeness. The ("coherent") states

$$|\psi\rangle \equiv e^{-\overline{\psi}\psi/2}e^{\overline{\psi}_{\mathrm{op}}\psi}|0\rangle, \quad \psi_{r,\mathrm{op}}|0\rangle = 0, \qquad \psi_{m,\mathrm{op}}|\psi\rangle = \psi_m|\psi\rangle\,, \tag{A3}$$

satisfy the completeness relation

$$1 = \int \prod_{i=1}^N d\overline{\psi}_i\,d\psi_i\,|\psi\rangle\langle\psi|\,. \tag{A4}$$

This can be shown term by term in a comparison with

$$1 = |0\rangle\langle 0| + \overline{\psi}_{r,\mathrm{op}}|0\rangle\langle 0|\,\psi_{r,\mathrm{op}} + \cdots\,. \tag{A5}$$

Now consider the object

$$\langle 0|\psi_{r,\mathrm{op}}e^{-iHT}\overline{\psi}_{s,\mathrm{op}}|0\rangle \equiv \langle r|e^{-iHT}|s\rangle\,, \tag{A6}$$

with $H = \overline{\psi}_{\mathrm{op}}\Gamma\psi_{\mathrm{op}}$ and Γ a matrix-valued function independent of $\psi_{\mathrm{op}}, \overline{\psi}_{\mathrm{op}}$. We introduce a grid of length $T = \epsilon M$, and spacing ϵ, by writing $e^{-iHT} = (e^{-iH\epsilon})^M$. Completeness is used repeatedly to obtain

$$\begin{aligned}\langle r|e^{-iHT}|s\rangle = \int d\overline{\psi}^M d\psi^M \cdots d\overline{\psi}^0 d\psi^0 \\ \times \langle 0|\psi_{r,\mathrm{op}}|\psi^M\rangle\langle\psi^M|e^{-iH\epsilon}|\psi^{M-1}\rangle \\ \times \cdots \langle\psi^1|e^{-iH\epsilon}|\psi^0\rangle\langle\psi^0|\overline{\psi}_{s,\mathrm{op}}|0\rangle\,,\end{aligned} \tag{A7}$$

where $|\psi^k\rangle$ is a complete set at the kth grid point. Using theorems 1 and 2, it is not hard to evaluate

$$\langle\psi^k|\psi^{k-1}\rangle = \exp[-\tfrac{1}{2}\overline{\psi}^k(\psi^k - \psi^{k-1}) + \tfrac{1}{2}(\overline{\psi}^k - \overline{\psi}^{k-1})\psi^{k-1}]\,, \tag{A8}$$

$$\langle\psi^k|H|\psi^{k-1}\rangle = \langle\psi^k|\psi^{k-1}\rangle\overline{\psi}^k\Gamma\psi^{k-1}\,. \tag{A9}$$

Thus, for small ϵ,

$$\langle\psi^k|e^{-iH\epsilon}|\psi^{k-1}\rangle \approx \exp[\tfrac{1}{2}i\overline{\psi}^k(\psi^k - \psi^{k-1}) - \tfrac{1}{2}i(\overline{\psi}^k - \overline{\psi}^{k-1})\psi^k - i\epsilon\overline{\psi}^k\Gamma\psi^{k-1}]\,. \tag{A10}$$

For the end points, we also need

$$\begin{aligned}\langle 0|\psi_{r,\mathrm{op}}|\psi^M\rangle = e^{-\overline{\psi}^M\psi^M/2}\psi^M_r\,,\\ \langle\psi^0|\overline{\psi}_{s,\mathrm{op}}|0\rangle = e^{-\overline{\psi}^0\psi^0/2}\overline{\psi}^0_s\,.\end{aligned} \tag{A11}$$

Putting everything together, we have

$$\langle r|e^{-iHT}|s\rangle = \prod_{n=0}^M \int d\overline{\psi}^n d\psi^n e^{-\overline{\psi}^M\psi^M/2}\psi^M_r \times \exp\left\{i\epsilon\sum_{k=1}^M\left[\frac{i}{2}\,\overline{\psi}^k\left(\frac{\psi^k - \psi^{k-1}}{\epsilon}\right) - \frac{i}{2}\left(\frac{\overline{\psi}^k - \overline{\psi}^{k-1}}{\epsilon}\right)\psi^{k-1} - \overline{\psi}^k\Gamma\psi^{k-1}\right]\right\} e^{-\overline{\psi}^0\psi^0/2}\overline{\psi}^0_s\,. \tag{A12}$$

As $\epsilon \to 0$ at fixed $T = \epsilon M$, we have finally

$$\langle 0|\psi_{r,\mathrm{op}}e^{-iHT}\overline{\psi}_{s,\mathrm{op}}|0\rangle = \int \mathfrak{D}\overline{\psi}\mathfrak{D}\psi\,\Phi^*_r(\overline{\psi}(T),\psi(T))\exp\left(i\int_0^T d\tau\,\mathfrak{L}\right)\Phi_s(\overline{\psi}(0),\psi(0))\,, \tag{A13}$$

$$\mathfrak{L} = \tfrac{1}{2}i\,\overline{\psi}\overleftrightarrow{\partial}_\tau\psi - \overline{\psi}\Gamma\psi\,. \tag{A14}$$

Here Φ^*, Φ are the external wave functions

$$\Phi_r^*(\bar\psi(T), \psi(T)) = e^{-\bar\psi(T)\psi(T)/2}\psi_r(T)\,,$$
$$\Phi_s(\bar\psi(0), \psi(0)) = e^{-\bar\psi(0)\psi(0)/2}\bar\psi_s(0)\,. \tag{A15}$$

With superposition of the usual coordinate-space structure, this is the result quoted in Sec. II of the text. Only minor notational changes are necessary to adapt this derivation to obtain the light-cone forms stated in Sec. III.

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After completion of the work, we learned that Bardakci and Samuel[15] have independently studied some aspects of the quark end-point terms. Also, following the completion of (Ref. 2 and) the present work, we learned of an investigation by Cornwall and Tiktopoulos,[16] which has some overlap with ours. We would also like to thank W. Siegel for a helpful discussion. One of us (A.J.) is grateful to Dr. H. Woolf for hospitality extended to him at the Institute of Advanced Study.

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[10]The chopping procedure is Lorentz invariant only in two dimensions.

[11]We are presently investigating the possibility of more extensive use of the N^{-1} expansion, in lieu of the chopping procedure.

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