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Fictitious-particle vertex in quantum gravity*

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We derive an exact expression for the fictitious-particle contribution to the pure graviton triangle diagram which is needed to verify, through the appropriate Slavnov-Taylor identities, the gauge invariance of the scattering matrix to third order in the gravitational coupling constant κ. The calculation is performed by working in the framework of covariant quantization and exploiting the technique of dimensional regularization which is known to preserve the local gauge symmetry of the underlying Einstein-Hilbert Lagrangian. The derivation is simplified by setting the energy-momentum of one of the external graviton lines equal to zero. It is shown that the final expression for the fictitious vertex is characterized by 14 invariant amplitudes and that the original divergences of the triangle graph reassert themselves as poles of Weierstrass's gamma function.

I. INTRODUCTION

In a previous paper,[1] henceforth referred to as I, we calculated in the context of covariant quantization[2-6] the one-loop contributions to the graviton self-energy by employing a modified technique of dimensional regularization[7,8] and by extending Goldberg's version[9] of the Einstein Lagrangian to 2ω dimensions. The regulating parameter ω is in general complex, with $\omega = 2$ corresponding to four-dimensional space-time. It was shown in I that the sum of the graviton and fictitious-particle contributions to the graviton propagator satisfies the Slavnov-Taylor identities[10,11] and that *pure* quantum gravity is renormalizable at least at the one-loop level.[12]

Next in line of complexity, after the self-energy loop, is the pure graviton triangle diagram (Fig. 1) with its associated contributions from fictitious particles (Fig. 2) also known as Feynman-DeWitt-Faddeev-Popov ghosts.[2,4,5,6] In pure quantum gravity these ghosts are unphysical massless vector particles which appear only in oriented, albeit closed, loops called fictitious "fermion" loops. The fictitious particles are needed to restore both the unitarity of the S matrix and the transversality of the scattering amplitudes.[5,6]

The purpose of this paper is to derive an expression for the fictitious-particle diagram, Fig. 3, by working again in the framework of covariant quantization and by employing the technique of dimensional regularization.[8,13] A knowledge of the detailed structure of the fictitious amplitude is essential in order to verify explicitly the gauge invariance of the scattering matrix to third order in the gravitational coupling constant κ. We recall that the technique of dimensional regularization preserves the local gauge symmetry of the underlying Lagrangian and thus allows for a consistent gauge-invariant treatment of divergent Feynman integrals. The continuous-dimension method may be applied not only to Abelian models, but also to non-Abelian massless spin-2 gauge theories such as quantum gravity.

The computation of the fictitious-particle amplitude is somewhat less complicated than that of

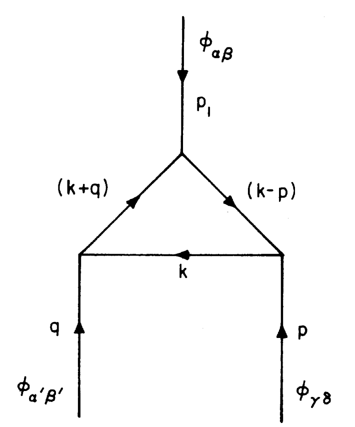

FIG. 1. The pure gravitation triangle diagram. All lines are graviton lines.

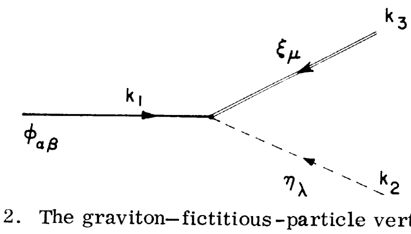

FIG. 2. The graviton–fictitious-particle vertex. The fictitious particles ξ and η have momentum labels k_3 and k_2 and polarization labels μ and λ, respectively. The graviton line is denoted by $\phi_{\alpha\beta}$.

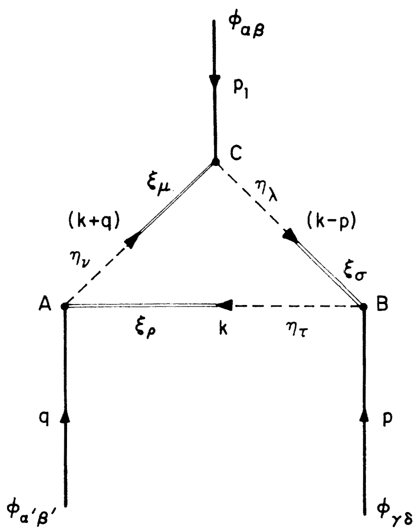

FIG. 3. Fictitious-particle triangle diagram. The fictitious particles ξ and η occur only in closed oriented "fermion" loops.

the corresponding amplitude for the pure graviton triangle diagram (Fig. 1) which contains, even in symmetrized form, over 150 000 terms.[14]

This article is organized as follows: In Sec. II we summarize the relevant Feynman rules. The general structure of the fictitious amplitude is stated in Sec. III, followed, in Sec. IV by the computation of the corresponding integrals over 2ω-dimensional Euclidean space. In Sec. V we give the final expression for the fictitious-particle contribution to the pure graviton triangle diagram. The article concludes in Sec. VI with a brief summary and discussion.

We work exclusively in Euclidean space and employ natural units $\hbar = c = 1$. We note that in four-space the gravitational coupling constant $\kappa^2 \propto G$, i.e., $\kappa^2 = 32\pi G \simeq 4\times 10^{-44}\ (m_e)^{-2} \simeq 4\times 10^{-38}$ GeV^{-2}, where G is Newton's constant and m_e is the mass of the electron. In 2ω-dimensional space, however,

$$\kappa^2 \propto G/(\mu^2)^{\omega-2};$$

μ is an arbitrary constant with the dimension of mass.[15]

II. FEYNMAN RULES

The pure Einstein-Hilbert Lagrangian density for the gravitational field reads

$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}, \tag{2.1}$$

where $g^{\mu\nu}$ is the metric tensor, $g \equiv \det g_{\mu\nu}$, and $R_{\mu\nu}$ is the Ricci tensor given by

$$R_{\mu\nu} = \Gamma_{\mu\rho,\nu}{}^{\rho} - \Gamma_{\mu\nu,\rho}{}^{\rho} - \Gamma_{\mu\nu}{}^{\sigma}\Gamma_{\sigma\rho}{}^{\rho} + \Gamma_{\sigma\nu}{}^{\rho}\Gamma_{\mu\rho}{}^{\sigma}, \tag{2.2}$$

$$\Gamma_{\mu\nu}{}^{\rho} = \tfrac{1}{2} g^{\rho\sigma}(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}). \tag{2.3}$$

It is convenient to rewrite the Lagrangian density (2.1) in terms of the tensor density $\tilde{g}^{\alpha\beta}$ of weight $+1$,[1]

$$\tilde{g}^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta}, \tag{2.4}$$

so that

$$\mathcal{L} = \frac{1}{2\kappa^2}\left(\tilde{g}^{\alpha\sigma}\tilde{g}_{\lambda\mu}\tilde{g}_{\beta\nu} - \frac{1}{2(\omega-1)}\tilde{g}^{\alpha\sigma}\tilde{g}_{\mu\beta}\tilde{g}_{\lambda\nu} - 2\delta_\beta{}^{\sigma}\delta_\lambda{}^{\alpha}\tilde{g}_{\mu\nu}\right) \times \tilde{g}_{,\alpha}{}^{\mu\beta}\tilde{g}_{,\sigma}{}^{\lambda\nu}. \tag{2.5}$$

If we define the graviton field $\phi^{\mu\nu}$ by

$$\tilde{g}^{\mu\nu} \equiv \delta^{\mu\nu} + \kappa\phi^{\mu\nu}, \tag{2.6}$$

where $\delta^{\mu\nu}$ is the 2ω-dimensional Kronecker δ, then

$$\tilde{g}_{\mu\nu} = \delta_{\mu\nu} - \kappa\phi_{\mu\nu} + \kappa^2\phi_{\mu\alpha}\phi_{\alpha\nu} - \kappa^3\phi_{\mu\alpha}\phi_{\alpha\beta}\phi_{\beta\nu} + O(\kappa^4). \tag{2.7}$$

The *inverse* of the fictitious-particle factor $\Delta(\tilde{g}^{\mu\nu})$ turns out to be [see Eq. (2.12) of I]

$$[\Delta(\tilde{g}^{\mu\nu})]^{-1} = \int d(\xi_\lambda)d(\eta_\nu)\exp\left\{ i\int dx\, \eta_\nu[\delta_{\nu\lambda}\Box - \kappa(\phi_{\mu\nu,\lambda\mu} - \phi_{\mu\rho}\delta_{\nu\lambda}\partial_\mu\partial_\rho - \phi_{\mu\rho,\mu}\delta_{\nu\lambda}\partial_\rho + \phi_{\mu\nu,\mu}\partial_\lambda)]\xi_\lambda\right\}, \tag{2.8}$$

where ξ_λ and η_ν are the fictitious particles discussed in the Introduction. Their Feynman propagator has, in momentum space k, the simple form

$$\langle T(\xi_\mu\eta_\nu)\rangle = \delta_{\mu\nu}/k^2, \tag{2.9}$$

with

$$\langle T(\xi_\mu\xi_\nu)\rangle = 0 = \langle T(\eta_\mu\eta_\nu)\rangle, \tag{2.10}$$

while the graviton-η-ξ vertex reads, again in momentum space,

$$V_{\alpha\beta,\lambda,\mu}(k_1,k_2,k_3) = \frac{\kappa}{2}[-(\delta_{\lambda\alpha}k_{1\beta} + \delta_{\lambda\beta}k_{1\alpha})k_{2\mu} + \delta_{\lambda\mu}(k_{2\alpha}k_{3\beta} + k_{2\beta}k_{3\alpha})]. \tag{2.11}$$

These Feynman rules are sufficient to derive, to order κ^3, the contributions to the fictitious-particle amplitude.

III. GENERAL STRUCTURE OF FICTITIOUS-PARTICLE AMPLITUDE

Consider the triangle diagram with vertices A, B, and C depicted in Fig. 3. According to Eqs. (2.9) and (2.11), the momentum-space propagators are

$$\langle T(\xi_\rho \eta_\tau)\rangle_{BA} = \delta_{\rho\tau}/k^2\,, \tag{3.1}$$

$$\langle T(\xi_\mu \eta_\nu)\rangle_{AC} = \delta_{\mu\nu}/(k+q)^2\,, \tag{3.2}$$

$$\langle T(\xi_\sigma \eta_\lambda)\rangle_{CB} = \delta_{\sigma\lambda}/(k-p)^2\,, \tag{3.3}$$

while the graviton–fictitious-particle vertices C, B, A are of the form

$$V_{\alpha\beta,\lambda,\mu}(p_1, -k+p, k+q) = \frac{\kappa}{2}\{(\delta_{\lambda\alpha} p_{1\beta} + \delta_{\lambda\beta} p_{1\alpha})(k-p)_\mu + \delta_{\lambda\mu}[(p-k)_\alpha (k+q)_\beta + (p-k)_\beta (k+q)_\alpha]\}\,, \tag{3.4}$$

$$V_{\gamma\delta,\tau,\sigma}(p, -k, k-p) = \frac{\kappa}{2}\{(\delta_{\tau\gamma} p_\delta + \delta_{\tau\delta} p_\gamma) k_\sigma + \delta_{\tau\sigma}[k_\gamma (p-k)_\delta + k_\delta (p-k)_\gamma]\}\,, \tag{3.5}$$

$$V_{\alpha'\beta',\nu,\rho}(q, -k-q, k) = \frac{\kappa}{2}\{(\delta_{\nu\alpha'} q_{\beta'} + \delta_{\nu\beta'} q_{\alpha'})(k+q)_\rho - \delta_{\nu\rho}[(k+q)_{\alpha'} k_{\beta'} + (k+q)_{\beta'} k_{\alpha'}]\}, \tag{3.6}$$

respectively. Consequently the amplitude for the fictitious loop drawn in Fig. 3 reads

$$F_{\alpha\beta,\gamma\delta,\alpha'\beta'}(p_1, p, q) = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} V_{\alpha'\beta',\nu,\rho}(q, -k-q, k)\, \frac{\delta_{\mu\nu}}{(k+q)^2}\, V_{\alpha\beta,\lambda,\mu}(p_1, -k+p, k+q)$$
$$\times \frac{\delta_{\sigma\lambda}}{(k-p)^2} V_{\gamma\delta,\tau,\sigma}(p, -k, k-p)\, \frac{\delta_{\rho\tau}}{k^2}\,, \tag{3.7}$$

or

$$F_{\alpha\beta,\gamma\delta,\alpha'\beta'}(p_1, p, q) = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega} k^2 (k+q)^2 (k-p)^2} V_{\alpha\beta,\lambda,\mu} V_{\gamma\delta,\tau,\lambda} V_{\alpha'\beta',\mu,\tau}\,. \tag{3.8}$$

Expression (3.8) is general. Unfortunately, it is also unwieldy since only some of the integrations in (3.8) can be performed in closed form. In order to display the explicit structure of the fictitious amplitude and to be able to analyze later the crucial Slavnov-Taylor identities, it is desirable to simplify the integrand of (3.8) without destroying any of its essential characteristics. This may be accomplished[16] by setting the energy-momentum of one of the external graviton lines, say of $\phi_{\alpha\beta}$, equal to zero: $p_1 = 0$ (see Fig. 4), in which case the fictitious amplitude (3.8) assumes the following structure [we replace the indices $(\gamma\delta)$ in (3.8) by $(\nu\sigma)$]:

$$F_{\alpha\beta,\nu\sigma,\alpha'\beta'}(0, p, -p) = 2\left(\frac{\kappa}{2}\right)^3 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega} k^2 (k-p)^4} \delta_{\lambda\mu} (k-p)_\alpha (k-p)_\beta \{k_\lambda(\delta_{\tau\nu} p_\sigma + \delta_{\tau\sigma} p_\nu) + \delta_{\tau\lambda}[k_\nu (p-k)_\sigma + k_\sigma (p-k)_\nu]\}$$
$$\times \{(\delta_{\mu\alpha'} p_{\beta'} + \delta_{\mu\beta'} p_{\alpha'})(k-p)_\tau + \delta_{\mu\tau}[(k-p)_{\alpha'} k_{\beta'} + (k-p)_{\beta'} k_{\alpha'}]\}\,, \tag{3.9}$$

or

$$F_{\alpha\beta,\nu\sigma,\alpha'\beta'}(0, p, -p) \equiv 2\left(\frac{\kappa}{2}\right)^3 \int \frac{d^{2\omega}k}{(2\pi)^{2\omega} k^2 (k-p)^4} V_{\alpha\beta\nu\sigma\alpha'\beta'}(\omega, p, k)\,, \tag{3.10}$$

where

$$V_{\alpha\beta\nu\sigma\alpha'\beta'}(\omega, p, k) \equiv \sum_{j=1}^{6} V^{(j)}_{\alpha\beta\nu\sigma\alpha'\beta'}(\omega, p, k) \tag{3.11}$$

consists of 56 terms. Whereas in four-space $(\omega = 2)$ the vertex integral appears, from power counting, to diverge quartically, in the spirit of dimensional regularization the integrals in (3.10) are now well defined.[7,8] Regrouping the various k terms in $V_{\alpha\beta\nu\sigma\alpha'\beta'}$ and remembering that $\delta_{\mu\mu} = 2\omega$, we obtain for the individual $V^{(j)}$'s

$$V^{(1)}_{\alpha\beta\nu\sigma\alpha'\beta'}(\omega, p, k) = -2(k_\nu p_\alpha p_\beta p_\sigma p_{\alpha'} p_{\beta'} + k_\sigma p_\alpha p_\beta p_\nu p_{\alpha'} p_{\beta'} + k_{\alpha'} p_\alpha p_\beta p_\nu p_\sigma p_{\beta'} + k_{\beta'} p_\alpha p_\beta p_\nu p_\sigma p_{\alpha'})\,, \tag{3.12}$$

$$V^{(2)}_{\alpha\beta\nu\sigma\alpha'\beta'}(\omega, p, k) = 2(k_\alpha k_{\alpha'} p_\nu p_\beta p_\sigma p_{\beta'} + k_\alpha k_{\beta'} p_\beta p_\nu p_\sigma p_{\alpha'} + k_\beta k_{\alpha'} p_\alpha p_\nu p_\sigma p_{\beta'} + k_\beta k_{\beta'} p_\alpha p_\nu p_\sigma p_{\alpha'} + 2k_\nu k_\sigma p_\alpha p_\beta p_{\alpha'} p_{\beta'}$$
$$+ k_\alpha k_\nu p_\beta p_\sigma p_{\alpha'} p_{\beta'} + k_\alpha k_\sigma p_\beta p_\nu p_{\alpha'} p_{\beta'} + k_\beta k_\nu p_\alpha p_\sigma p_{\alpha'} p_{\beta'} + k_\beta k_\sigma p_\alpha p_\nu p_{\alpha'} p_{\beta'})$$
$$+ (1-2\omega)(k_\nu k_{\alpha'} p_\alpha p_\beta p_\sigma p_{\beta'} + k_\nu k_{\beta'} p_\alpha p_\beta p_\sigma p_{\alpha'} + k_\sigma k_{\alpha'} p_\alpha p_\beta p_\nu p_{\beta'} + k_\sigma k_{\beta'} p_\alpha p_\beta p_\nu p_{\alpha'})\,, \tag{3.13}$$

$$
\begin{aligned}
V^{(3)}_{\alpha\beta\nu\sigma\alpha'\beta'}(\omega,p,k) = {} & -2(k_\alpha k_\beta k_{\alpha'} p_\nu p_\sigma p_{\beta'} + k_\alpha k_\beta k_{\beta'} p_\nu p_\sigma p_{\alpha'} + 2k_\alpha k_\nu k_\sigma p_\beta p_{\alpha'} p_{\beta'} + 2k_\beta k_\nu k_\sigma p_\alpha p_{\alpha'} p_{\beta'} + k_\alpha k_\beta k_\sigma p_\nu p_{\alpha'} p_{\beta'} \\
& + k_\alpha k_\beta k_\nu p_\sigma p_{\alpha'} p_{\beta'}) + 2(1+2\omega)(k_\nu k_{\alpha'} k_{\beta'} p_\alpha p_\beta p_\sigma + k_\sigma k_{\alpha'} k_{\beta'} p_\alpha p_\beta p_\nu) \\
& - (1-2\omega)(2k_\nu k_\sigma k_{\alpha'} p_\alpha p_\beta p_{\beta'} + 2k_\nu k_\sigma k_{\beta'} p_\alpha p_\beta p_{\alpha'} + k_\alpha k_\nu k_{\alpha'} p_\beta p_\sigma p_{\beta'} \\
& + k_\alpha k_\nu k_{\beta'} p_\sigma p_\beta p_{\alpha'} + k_\alpha k_\sigma k_{\alpha'} p_\beta p_\nu p_{\beta'} + k_\alpha k_\sigma k_{\beta'} p_\beta p_\nu p_{\alpha'} + k_\beta k_\nu k_{\alpha'} p_\alpha p_\sigma p_{\beta'} + k_\beta k_\nu k_{\beta'} p_\alpha p_\sigma p_{\alpha'} \\
& + k_\beta k_\sigma k_{\alpha'} p_\alpha p_\nu p_{\beta'} + k_\beta k_\sigma k_{\beta'} p_\alpha p_\nu p_{\alpha'}),
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
V^{(4)}_{\alpha\beta\nu\sigma\alpha'\beta'}(\omega,p,k) = {} & 4(k_\alpha k_\beta k_\nu k_\sigma p_{\alpha'} p_{\beta'} - 2\omega k_\nu k_\sigma k_{\alpha'} k_{\beta'} p_\alpha p_\beta) \\
& - 2(1+2\omega)(k_\alpha k_\nu k_{\alpha'} k_{\beta'} p_\beta p_\sigma + k_\beta k_\nu k_{\alpha'} k_{\beta'} p_\alpha p_\sigma + k_\alpha k_\sigma k_{\alpha'} k_{\beta'} p_\nu p_\beta + k_\beta k_\sigma k_{\alpha'} k_{\beta'} p_\nu p_\alpha) \\
& + (1-2\omega)(2k_\alpha k_{\alpha'} k_\nu k_\sigma p_\beta p_{\beta'} + 2k_\alpha k_\nu k_\sigma k_{\beta'} p_\beta p_{\alpha'} + 2k_\beta k_\nu k_\sigma k_{\alpha'} p_\alpha p_{\beta'} + 2k_\beta k_\nu k_\sigma k_{\beta'} p_\alpha p_{\alpha'} \\
& + k_\alpha k_\beta k_\nu k_{\alpha'} p_\sigma p_{\beta'} + k_\alpha k_\beta k_\nu k_{\beta'} p_\sigma p_{\alpha'} + k_\alpha k_\beta k_\sigma k_{\alpha'} p_\nu p_{\beta'} + k_\alpha k_\beta k_\sigma k_{\beta'} p_\nu p_{\alpha'}),
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
V^{(5)}_{\alpha\beta\nu\sigma\alpha'\beta'}(\omega,p,k) = {} & 8\omega(k_\alpha k_\nu k_\sigma k_{\alpha'} k_{\beta'} p_\beta + k_\beta k_\nu k_\sigma k_{\alpha'} k_{\beta'} p_\alpha) + 2(1+2\omega)(k_\alpha k_\beta k_\nu k_{\alpha'} k_{\beta'} p_\sigma + k_\alpha k_\beta k_\sigma k_{\alpha'} k_{\beta'} p_\nu) \\
& -2(1-2\omega)(k_\alpha k_\beta k_\nu k_\sigma k_{\alpha'} p_{\beta'} + k_\alpha k_\beta k_\nu k_\sigma k_{\beta'} p_{\alpha'}),
\end{aligned}
\tag{3.16}
$$

$$
V^{(6)}_{\alpha\beta\nu\sigma\alpha'\beta'}(\omega,p,k) = -8\omega k_\alpha k_\beta k_\nu k_\sigma k_{\alpha'} k_{\beta'}\,. \tag{3.17}
$$

Expressions (3.12) to (3.17) comprise in full the numerator in (3.10). The integration is summarized in the subsequent section.

IV. INTEGRATION OVER 2ω-DIMENSIONAL EUCLIDEAN SPACE

In this section we evaluate the various integrals occurring in the amplitude $F_{\alpha\beta,\nu\sigma,\alpha'\beta'}$ in Eq. (3.10). Since we operate in complex-dimensional Euclidean space, with ω acting as a regulating parameter, all integrals associated with the triangle graph, Fig. 4, are convergent. The *basic* integral

$$
I_1(\omega,p^2) \equiv \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}k^2(k-p)^2(k-p)^2} \tag{4.1}
$$

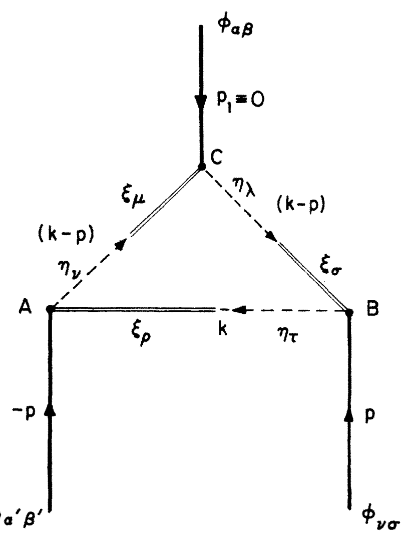

FIG. 4. Fictitious-particle diagram with the energy-momentum p_1 of the external graviton line $\phi_{\alpha\beta}$ set equal to zero.

is readily computed by employing three times the parametrization for *massless* momentum-space propagators,

$$
\frac{1}{k^2} = \int_0^\infty d\alpha \exp(-\alpha k^2), \quad k^2 > 0 \tag{4.2}
$$

together with the formula

$$
\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \exp(-ak^2 + 2b\cdot k) = \frac{1}{(2\pi)^{2\omega}} \left(\frac{\pi}{a}\right)^\omega \exp\left(\frac{b^2}{a}\right), \quad a > 0\,. \tag{4.3}
$$

Consequently $I_1(\omega,p^2)$ becomes

$$
I_1(\omega,p^2) = \frac{\pi^\omega}{(2\pi)^{2\omega}} \int_0^\infty \int_0^\infty \int_0^\infty d\alpha\, d\beta\, d\gamma (\alpha+\beta+\gamma)^{-\omega} \times \exp\left(\frac{-p^2(\alpha\beta+\alpha\gamma)}{\alpha+\beta+\gamma}\right). \tag{4.4}
$$

It is convenient[17] to introduce in (4.4) the new variables ξ, τ, λ defined by $\alpha = \xi\lambda$, $\beta = \tau\lambda$, and $\gamma = \lambda(1-\xi-\tau)$, with Jacobian $|J| = \lambda^2$, so that

$$
\int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma \to \int_0^1 d\xi \int_0^{1-\xi} d\tau \int_0^\infty \lambda^2 d\lambda\,.
$$

Subsequent integration over λ yields

$$
I_1(\omega,p^2) = \frac{(p^2)^{\omega-3}\Gamma(3-\omega)}{(4\pi)^\omega} \times \int_0^1 d\xi [\xi(1-\xi)]^{\omega-3} \int_0^{1-\xi} d\tau\,, \tag{4.5}
$$

or

$$
I_1(\omega,p^2) = (4\pi)^{-\omega}(p^2)^{\omega-3}\Gamma(3-\omega) \times \Gamma(\omega-2)\Gamma(\omega-1)/\Gamma(2\omega-3)\,, \tag{4.6}
$$

where

$$\Gamma(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(n+z)}+\int_1^{\infty}dt\,t^{z-1}e^{-t} \tag{4.7}$$

denotes the Weierstrass representation of the Γ function.[18]

The other integrals in (3.10), as seen from Eqs. (3.12) to (3.17), are of the form $(I_1k_{\alpha_1})$, $(I_1k_{\alpha_1}k_{\alpha_2})$, $\ldots$, and $(I_1k_{\alpha_1}\cdots k_{\alpha_6})$, where I_1 acts as an integral operator. These six integrals may be computed by repeatedly differentiating Eq. (4.3) partially with respect to b_μ to yield the following formulas (Ref. 16):

$$\int\frac{d^{2\omega}k\,k_\mu}{(2\pi)^{2\omega}k^2(k-p)^4}=p_\mu I_2(\omega,p^2), \tag{4.8}$$

$$\int\frac{d^{2\omega}k\,k_\mu k_\nu}{(2\pi)^{2\omega}k^2(k-p)^4}=p_\mu p_\nu I_3(\omega,p^2)+\delta_{\mu\nu}I_4(\omega,p^2), \tag{4.9}$$

$$\int\frac{d^{2\omega}k\,k_\mu k_\nu k_\sigma}{(2\pi)^{2\omega}k^2(k-p)^4}=p_\mu p_\nu p_\sigma I_5(\omega,p^2)+(p_\mu\delta_{\nu\sigma}+p_\nu\delta_{\mu\sigma}+p_\sigma\delta_{\mu\nu})I_6(\omega,p^2), \tag{4.10}$$

$$\int\frac{d^{2\omega}k\,k_\mu k_\nu k_\sigma k_\tau}{(2\pi)^{2\omega}k^2(k-p)^4}=p_\mu p_\nu p_\sigma p_\tau I_7(\omega,p^2)+\Big(\sum_{6\text{ perm}}\delta_{\mu\tau}p_\nu p_\sigma\Big)I_8(\omega,p^2)+\Big(\sum_{3\text{ perm}}\delta_{\mu\nu}\delta_{\tau\sigma}\Big)I_9(\omega,p^2), \tag{4.11}$$

$$\int\frac{d^{2\omega}k\,k_\mu k_\nu k_\sigma k_\tau k_\rho}{(2\pi)^{2\omega}k^2(k-p)^4}=p_\mu p_\nu p_\sigma p_\tau p_\rho I_{10}(\omega,p^2)+\Big(\sum_{10\text{ perm}}\delta_{\mu\nu}p_\sigma p_\tau p_\rho\Big)I_{11}(\omega,p^2)+\Big(\sum_{15\text{ perm}}\delta_{\mu\nu}\delta_{\sigma\tau}p_\rho\Big)I_{12}(\omega,p^2), \tag{4.12}$$

$$\begin{aligned}\int\frac{d^{2\omega}k\,k_\mu k_\nu k_\sigma k_\tau k_\rho k_\alpha}{(2\pi)^{2\omega}k^2(k-p)^4}&=p_\mu p_\nu p_\sigma p_\tau p_\rho p_\alpha I_{13}(\omega,p^2)+\Big(\sum_{15\text{ perm}}\delta_{\mu\nu}p_\sigma p_\tau p_\rho p_\alpha\Big)I_{14}(\omega,p^2)\\&\quad+\Big(\sum_{45\text{ perm}}\delta_{\mu\nu}\delta_{\sigma\tau}p_\alpha p_\rho\Big)I_{15}(\omega,p^2)+\Big(\sum_{15\text{ perm}}\delta_{\mu\nu}\delta_{\sigma\tau}\delta_{\rho\alpha}\Big)I_{16}(\omega,p^2),\end{aligned} \tag{4.13}$$

where $\sum_{N\text{ perm}}$ denotes the sum over N distinct permutations. The integrals $I_2(\omega,p^2),\ldots,I_{16}(\omega,p^2)$ are summarized in Appendix A and are seen to be expressible in terms of the basic integral $I_1(\omega,p^2)$ in Eq. (4.6).

V. TOTAL CONTRIBUTION FROM FICTITIOUS-PARTICLE LOOP

The machinery developed in the preceding section is sufficient to integrate the right-hand side of Eq. (3.10),

$$F_{\alpha\beta,\nu\sigma,\alpha'\beta'}(0,p,-p)\equiv 2\Big(\frac{\kappa}{2}\Big)^3[I_1V_{\alpha\beta\nu\sigma\alpha'\beta'}(\omega,p,k)],$$

over 2ω-dimensional Euclidean space with the numerator $V_{\alpha\beta\nu\sigma\alpha'\beta'}$ given explicitly by Eqs. (3.12) to (3.17), while the relevant momentum-space integrals are summarized between Eqs. (4.8) and (4.13). The final expression for the fictitious-particle contribution to the pure graviton triangle diagram then reads

$$\begin{aligned}F_{\alpha\beta,\nu\sigma,\alpha'\beta'}(0,p,-p)=2\Big(\frac{\kappa}{2}\Big)^3\Big\{&p_\alpha p_\beta p_\nu p_\sigma p_{\alpha'}p_{\beta'}F_1(\omega,p^2)+\delta_{\alpha\beta}p_\sigma p_\nu p_{\alpha'}p_{\beta'}F_2(\omega,p^2)+\delta_{\sigma\nu}p_\alpha p_{\alpha'}p_\beta p_{\beta'}F_3(\omega,p^2)\\
&+\delta_{\alpha'\beta'}p_\nu p_\sigma p_\alpha p_\beta F_4(\omega,p^2)\\
&+(\delta_{\alpha\sigma}p_\beta p_\nu p_{\alpha'}p_{\beta'}+\delta_{\alpha\nu}p_\beta p_{\beta'}p_\sigma p_{\alpha'}+\delta_{\nu\beta}p_\alpha p_{\alpha'}p_\sigma p_{\beta'}+\delta_{\sigma\beta}p_\alpha p_{\alpha'}p_\nu p_{\beta'})F_5(\omega,p^2)\\
&+(\delta_{\nu\beta'}p_\alpha p_{\alpha'}p_\sigma p_\beta+\delta_{\nu\alpha'}p_\alpha p_\beta p_\sigma p_{\beta'}+\delta_{\sigma\alpha'}p_\alpha p_\beta p_\nu p_{\beta'}+\delta_{\sigma\beta'}p_\alpha p_{\alpha'}p_\nu p_\beta)F_6(\omega,p^2)\\
&+(\delta_{\alpha\beta'}p_\nu p_\sigma p_{\alpha'}p_\beta+\delta_{\alpha\alpha'}p_\beta p_{\beta'}p_\nu p_\sigma+\delta_{\beta\beta'}p_\alpha p_{\alpha'}p_\nu p_\sigma+\delta_{\beta\alpha'}p_\alpha p_{\beta'}p_\nu p_\sigma)F_7(\omega,p^2)\\
&+p_\alpha p_\beta(\delta_{\alpha'\beta'}\delta_{\nu\sigma}+\delta_{\beta'\nu}\delta_{\alpha'\sigma}+\delta_{\beta'\sigma}\delta_{\alpha'\nu})F_8(\omega,p^2)\\
&+p_\nu p_\sigma(\delta_{\alpha\beta}\delta_{\alpha'\beta'}+\delta_{\alpha\beta'}\delta_{\beta\alpha'}+\delta_{\alpha\alpha'}\delta_{\beta\beta'})F_9(\omega,p^2)+p_{\alpha'}p_{\beta'}(\delta_{\alpha\sigma}\delta_{\beta\nu}+\delta_{\alpha\beta}\delta_{\sigma\nu}+\delta_{\alpha\nu}\delta_{\sigma\beta})F_{10}(\omega,p^2)\\
&+[p_\beta p_\sigma(\delta_{\alpha\beta'}\delta_{\nu\alpha'}+\delta_{\alpha\nu}\delta_{\alpha'\beta'}+\delta_{\alpha\alpha'}\delta_{\beta'\nu})+p_\alpha p_\sigma(\delta_{\beta\beta'}\delta_{\nu\alpha'}+\delta_{\beta\nu}\delta_{\alpha'\beta'}+\delta_{\beta\alpha'}\delta_{\beta'\nu})\\
&\quad+p_\nu p_\beta(\delta_{\alpha\beta'}\delta_{\alpha'\sigma}+\delta_{\alpha\sigma}\delta_{\alpha'\beta'}+\delta_{\alpha\alpha'}\delta_{\sigma\beta'})+p_\alpha p_\nu(\delta_{\beta\beta'}\delta_{\sigma\alpha'}+\delta_{\beta\sigma}\delta_{\alpha'\beta'}+\delta_{\beta\alpha'}\delta_{\beta'\sigma})]F_{11}(\omega,p^2)\\
&+[p_\beta p_{\beta'}(\delta_{\alpha\sigma}\delta_{\alpha'\nu}+\delta_{\alpha\alpha'}\delta_{\sigma\nu}+\delta_{\alpha\nu}\delta_{\sigma\alpha'})+p_\beta p_{\alpha'}(\delta_{\alpha\nu}\delta_{\sigma\beta'}+\delta_{\alpha\sigma}\delta_{\nu\beta'}+\delta_{\alpha\beta'}\delta_{\nu\sigma})\\
&\quad+p_\alpha p_{\beta'}(\delta_{\beta\alpha'}\delta_{\nu\sigma}+\delta_{\beta\nu}\delta_{\alpha'\sigma}+\delta_{\beta\sigma}\delta_{\alpha'\nu})+p_\alpha p_{\alpha'}(\delta_{\beta\beta'}\delta_{\nu\sigma}+\delta_{\beta\nu}\delta_{\beta'\sigma}+\delta_{\beta\sigma}\delta_{\beta'\nu})]F_{12}(\omega,p^2)\\
&+[p_\sigma p_{\beta'}(\delta_{\alpha\alpha'}\delta_{\beta\nu}+\delta_{\alpha\beta}\delta_{\alpha'\nu}+\delta_{\alpha'\beta}\delta_{\alpha\nu})+p_\sigma p_{\alpha'}(\delta_{\alpha\beta}\delta_{\nu\beta'}+\delta_{\alpha\nu}\delta_{\beta\beta'}+\delta_{\alpha\beta'}\delta_{\beta\nu})\\
&\quad+p_\nu p_{\beta'}(\delta_{\alpha\alpha'}\delta_{\beta\sigma}+\delta_{\alpha\beta}\delta_{\alpha'\sigma}+\delta_{\alpha\sigma}\delta_{\alpha'\beta})+p_\nu p_{\alpha'}(\delta_{\alpha\beta'}\delta_{\beta\sigma}+\delta_{\alpha\beta}\delta_{\beta'\sigma}+\delta_{\alpha\sigma}\delta_{\beta'\beta})]F_{13}(\omega,p^2)\\
&+\sum_{15\text{ perm}}\delta_{\alpha\beta}\delta_{\nu\sigma}\delta_{\alpha'\beta'}\;F_{14}(\omega,p^2)\Big\},\end{aligned} \tag{5.1}$$

where the invariant coefficients $F_j(\omega, p^2)$, $j=1, 2, \ldots, 14$, possess the following structure (note their dependence on the regulating parameter ω):

$$F_1(\omega,p^2) = -8[I_2 + (\omega-3)I_3 + (3-4\omega)I_5 + (6\omega-1)I_7 - 4\omega I_{10} + \omega I_{13}], \quad (5.2)$$

$$F_2(\omega,p^2) = -8[I_6 + (\omega-1)I_8 - 2\omega I_{11} + \omega I_{14}], \quad (5.3)$$

$$F_3(\omega,p^2) = 4[I_4 + (2\omega-3)I_6 + (3-6\omega)I_8 + (6\omega-1)I_{11} - 2\omega I_{14}], \quad (5.4)$$

$$F_4(\omega,p^2) = 4[(1+2\omega)I_6 - (2+6\omega)I_8 + (1+6\omega)I_{11} - 2\omega I_{14}], \quad (5.5)$$

$$F_5(\omega,p^2) = 2[I_4 + (2\omega-4)I_6 + (4-8\omega)I_8 + (10\omega-1)I_{11} - 4\omega I_{14}], \quad (5.6)$$

$$F_6(\omega,p^2) = (1-2\omega)I_4 + (12\omega-2)I_6 + (1-26\omega)I_8 + 24\omega I_{11} - 8\omega I_{14}, \quad (5.7)$$

$$F_7(\omega,p^2) = 2[I_4 + (2\omega-2)I_6 - 8\omega I_8 + (10\omega+1)I_{11} - 4\omega I_{14}], \quad (5.8)$$

$$F_8(\omega,p^2) = -8\omega(I_9 - 2I_{12} + I_{15}), \quad (5.9)$$

$$F_9(\omega,p^2) = 4[(1+2\omega)I_{12} - 2\omega I_{15}], \quad (5.10)$$

$$F_{10}(\omega,p^2) = 4[I_9 - (1-2\omega)I_{12} - 2\omega I_{15}], \quad (5.11)$$

$$F_{11}(\omega,p^2) = -2[(1+2\omega)I_9 - (6\omega+1)I_{12} + 4\omega I_{15}], \quad (5.12)$$

$$F_{12}(\omega,p^2) = 2[(1-2\omega)I_9 + (6\omega-1)I_{12} - 4\omega I_{15}], \quad (5.13)$$

$$F_{13}(\omega,p^2) = (1-2\omega)I_9 + 8\omega I_{12} - 8\omega I_{15}, \quad (5.14)$$

$$F_{14}(\omega,p^2) = -8\omega I_{16}. \quad (5.15)$$

The integrals $I_2(\omega,p^2), \ldots, I_{16}(\omega,p^2)$ are summarized, respectively, in Eqs. (A3) to (A17) of Appendix A. To study the analytic structure of the invariant coefficients F_j, $j=1,\ldots,14$, it is convenient to rewrite them as functions of the basic integral $I_1(\omega,p^2)$. It is evident from the "reduced" F's in Appendix B that the original ultraviolet divergences of the vertex graph manifest themselves, in the framework of the continuous dimension method, as poles of Weierstrass's partial fraction expansion of the Γ function (4.7). Thus

$$\Gamma(\omega-2) = \frac{1}{\omega-2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,(n+\omega-2)} + \int_1^{\infty} dt\, t^{z-1} e^{-t} \quad (5.16)$$

exhibits, for example, a pole at the "physical" value $\omega=2$. (We recall that $2\omega=4$ corresponds to four-dimensional space.)

VI. SUMMARY AND DISCUSSION

Working to third order in the gravitational coupling constant κ, we have derived in the context of dimensional regularization an explicit formula for the fictitious-particle contribution to the pure graviton triangle diagram. Our final expression $F_{\alpha\beta,\nu\sigma,\alpha'\beta'}$ in Eq. (5.1) is characterized by 14 invariant amplitudes $F_j(\omega,p^2)$. The latter are summarized in Eqs. (5.2) to (5.15) and again, in reduced form, in Appendix B, where they are seen to exhibit [through the basic integral $I_1(\omega,p^2)$] various poles, for example at $\omega=2$ and $\omega=1$. The pole at $\omega=1$ is connected with the fact that the Einstein-Hilbert Lagrangian $L=\int \mathcal{L} d^{2\omega}x$ collapses in two dimensions to a surface integral.[19] We also observe that the fictitious-particle loop associated with the graviton *self-energy*[20] yields, by comparison, only ten terms with five invariant coefficients.

The calculation presented here is the first step in a program designed to verify the gauge invariance of the scattering matrix to order κ^3. The second step consists of tackling the pure graviton vertex in Fig. 1, a somewhat tedious exercise since the corresponding amplitude contains, even in symmetrized form, over 150 000 terms.

The gauge invariance of the S matrix to order κ^3 may then be verified first, by adding the fictitious-particle amplitude $F_{\alpha\beta,\nu\sigma,\alpha'\beta'}$ to the contribution from the *pure* graviton triangle diagram and second, by ascertaining that the sum of the "real" graviton and the fictitious-particle contributions to the graviton vertex does indeed satisfy the appropriate Slavnov-Taylor identities. The latter can be derived from the general Slavnov-Taylor identity[10,11]

$$(2/\alpha)\langle T\phi_{\mu\nu,\mu}(x)\phi_{\lambda\beta,\lambda}(y)\rangle = -\delta_{\nu\beta}\delta(x-y), \quad \alpha\neq 0$$

which is valid to all orders in κ and which holds for any gauge specified by the parameter α. It is clear that the successful completion of the above two steps hinges decisively on a knowledge of the fictitious-particle contribution. The detailed structure of this contribution is given in Eq. (5.1).

It has been known for some time that calculations in gravity can be fun and the above computation is no exception. Yet our enthusiasm to attack Fig. 1 has diminished uniformly during the course of the present investigation and is now roughly proportional to $1/n^2$, where n, the total number of terms in the corresponding vertex amplitude, exceeds 150 000.

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APPENDIX A

In this appendix we express the integrals $I_2(\omega,p^2),\ldots,I_{16}(\omega,p^2)$, appearing between Eqs. (4.8) and (4.13), in terms of the basic integral $I_1(\omega,p^2)$, Eq. (4.6). The reduction is achieved with the help of the beta function

$$B(x,y)=\int_0^1 d\xi\, \xi^{x-1}(1-\xi)^{y-1},\quad \mathrm{Re}(x)>0,\quad \mathrm{Re}(y)>0 \tag{A1}$$

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{A2}$$

and eventually yields

$$I_2(\omega,p^2)=\frac{\omega-1}{2\omega-3}I_1(\omega,p^2), \tag{A3}$$

$$I_3(\omega,p^2)=\frac{\omega}{2(2\omega-3)}I_1(\omega,p^2), \tag{A4}$$

$$I_4(\omega,p^2)=\frac{-1}{4(2\omega-3)}p^2I_1(\omega,p^2), \tag{A5}$$

$$I_5(\omega,p^2)=\frac{\omega(\omega+1)}{2(2\omega-1)(2\omega-3)}I_1(\omega,p^2), \tag{A6}$$

$$I_6(\omega,p^2)=\frac{-\omega}{4(2\omega-1)(2\omega-3)}p^2I_1(\omega,p^2), \tag{A7}$$

$$I_7(\omega,p^2)=\frac{(\omega+1)(\omega+2)}{4(2\omega-1)(2\omega-3)}I_1(\omega,p^2), \tag{A8}$$

$$I_8(\omega,p^2)=\frac{-(\omega+1)}{8(2\omega-1)(2\omega-3)}p^2I_1(\omega,p^2), \tag{A9}$$

$$I_9(\omega,p^2)=\frac{1}{16(2\omega-1)(2\omega-3)}(p^2)^2I_1(\omega,p^2), \tag{A10}$$

$$I_{10}(\omega,p^2)=\frac{(\omega+1)(\omega+2)(\omega+3)}{4(4\omega^2-1)(2\omega-3)}I_1(\omega,p^2), \tag{A11}$$

$$I_{11}(\omega,p^2)=\frac{-(\omega+1)(\omega+2)}{8(4\omega^2-1)(2\omega-3)}p^2I_1(\omega,p^2), \tag{A12}$$

$$I_{12}(\omega,p^2)=\frac{(\omega+1)}{16(4\omega^2-1)(2\omega-3)}(p^2)^2I_1(\omega,p^2), \tag{A13}$$

$$I_{13}(\omega,p^2)=\frac{(\omega+2)(\omega+3)(\omega+4)}{8(4\omega^2-1)(2\omega-3)}I_1(\omega,p^2), \tag{A14}$$

$$I_{14}(\omega,p^2)=\frac{-(\omega+2)(\omega+3)}{16(4\omega^2-1)(2\omega-3)}p^2I_1(\omega,p^2), \tag{A15}$$

$$I_{15}(\omega,p^2)=\frac{(\omega+2)}{32(4\omega^2-1)(2\omega-3)}(p^2)^2I_1(\omega,p^2), \tag{A16}$$

$$I_{16}(\omega,p^2)=\frac{-1}{64(4\omega^2-1)(2\omega-3)}(p^2)^3I_1(\omega,p^2). \tag{A17}$$

APPENDIX B

By substituting the appropriate integrals from Appendix A into the right-hand sides of expressions (5.2) to (5.15), we may rewrite the invariant amplitudes $F_j(\omega,p^2)$, $j=1,\ldots,14$, explicitly as functions of ω, p^2, and the basic integral $I_1(\omega,p^2)$. Consequently,

$$F_1(\omega,p^2)=\frac{-(\omega-2)(\omega-1)(\omega^2+4\omega+2)}{(4\omega^2-1)(2\omega-3)}I_1(\omega,p^2), \tag{B1}$$

$$F_2(\omega,p^2)=\frac{(\omega-1)(\omega^2+4\omega+2)}{2(4\omega^2-1)(2\omega-3)}p^2I_1(\omega,p^2), \tag{B2}$$

$$F_3(\omega,p^2)=\frac{-(\omega-1)(\omega^2+\omega+1)}{2(4\omega^2-1)(2\omega-3)}p^2I_1(\omega,p^2), \tag{B3}$$

$$F_4(\omega,p^2)=\frac{-\omega(\omega^2-1)}{2(4\omega^2-1)(2\omega-3)}p^2I_1(\omega,p^2), \tag{B4}$$

$$F_5(\omega,p^2)=\frac{\omega(\omega-1)}{4(4\omega^2-1)(2\omega-3)}p^2I_1(\omega,p^2), \tag{B5}$$

$$F_6(\omega,p^2)=\frac{-(\omega-1)}{8(4\omega^2-1)(2\omega-3)}p^2I_1(\omega,p^2), \tag{B6}$$

$$F_7(\omega,p^2)=\frac{-\omega(\omega-1)}{4(4\omega^2-1)(2\omega-3)}p^2I_1(\omega,p^2), \tag{B7}$$

$$F_8(\omega,p^2)=\frac{-\omega^2}{4(4\omega^2-1)(2\omega-3)}(p^2)^2I_1(\omega,p^2), \tag{B8}$$

$$F_9(\omega,p^2)=\frac{(\omega^2+\omega+1)}{4(4\omega^2-1)(2\omega-3)}(p^2)^2I_1(\omega,p^2), \tag{B9}$$

$$F_{10}(\omega,p^2)=\frac{\omega(\omega+1)}{4(4\omega^2-1)(2\omega-3)}(p^2)^2I_1(\omega,p^2), \tag{B10}$$

$$F_{11}(\omega,p^2)=\frac{-\omega}{8(4\omega^2-1)(2\omega-3)}(p^2)^2I_1(\omega,p^2), \tag{B11}$$

$$F_{12}(\omega,p^2)=\frac{\omega}{8(4\omega^2-1)(2\omega-3)}(p^2)^2I_1(\omega,p^2), \tag{B12}$$

$$F_{13}(\omega,p^2)=\frac{1}{16(4\omega^2-1)(2\omega-3)}(p^2)^2I_1(\omega,p^2), \tag{B13}$$

and finally,

$$F_{14}(\omega,p^2)=\frac{\omega}{8(4\omega^2-1)(2\omega-3)}(p^2)^3I_1(\omega,p^2). \tag{B14}$$

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