Geometrical mass in the Reissner-Nordström solution

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The Reissner-Nordström (RN) solution is obtained from the Schwarzschild solution by a method due to Teixeira, Wolk, and Som (TWS). It then follows that the mass parameter, m, of the RN solution is inseparably connected with the charge parameter, q, and $m = 0$ only when $q = 0$. It is thus suggested that the TWS method picks out, from among all possible RN solutions, just those of physical interest.

I. INTRODUCTION

The only spherically symmetric asymptotically flat solution of the Einstein-Maxwell equations is that of Reissner and Nordström (RN). It represents the space-time outside a spherically symmetric charged body. There exist coordinates in which the metric has the form

$$ds^2=\left(1-\frac{2m_R}{r}+\frac{q^2}{r^2}\right)(dx^0)^2 - \left(1-\frac{2m_R}{r}+\frac{q^2}{r^2}\right)^{-1}dr^2-r^2d\Omega^2\,,$$
$$d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2\,,$$

where m_R represents the geometrical mass and q the electric charge of the body. However, these two parameters are found to be unrelated. Indeed one can put either m_R or q equal to zero. The invariant Kretschmann scalar α has the form

$$\alpha=R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}=\frac{8}{r^6}\left(6m_R{}^2-\frac{12m_Rq^2}{r}+\frac{7q^4}{r^2}\right).$$

From this scalar it is evident that the geometry is not flat when any one of the two parameters is different from zero. It is natural then to conclude that the Einstein-Maxwell equations admit solutions corresponding to a spherically symmetric massless charged body. The massless charge, however, is not a peculiarity of the RN solution. In a paper, Som and Raychaudhuri[1] demonstrated that the massless charged dust under rigid rotation can exist in equilibrium in its own magnetic field.

In nature the existence of massless charge is yet unknown. If one takes this fact as an undeniable physical situation, one should expect that the mass parameter in the RN solution should take account of the contribution from the electrostatic energy. However, when $q=0$ one has $m_R=m_S$, where m_S is the Schwarzschild mass of the neutral system. If one adds to it a further restriction (on physical grounds) that the charge does not exist without its geometrical mass, then m_R assumes an invariant significance. If one works entirely classically, one finds that, for a charged sphere, the effective mass is given by $Gm_0+\Lambda q^2/a$, where m_0 is the bare mass of the charged body and a is the radius of the sphere. The effective mass is never zero, unless the bare mass is negative.

One way to find the explicit expression of the mass parameter is to obtain the interior solution. Bonnor[2] studied the interior solutions corresponding to a spherically symmetric charge distribution. For $q\leq m_R$ he found that the electrical energy contributes to gravitational mass. The mass parameter m_R cannot be put equal to zero unless the matter density is negative. This solution is unsatisfactory for a point charge, since $m_R\to\infty$ as $r_0\to 0$, where r_0 is the radius of the charged sphere. For $q\geq m_R$, the only interior solution known by the present authors has been given recently by Teixeira, Wolk, and Som.[3] However, their solution corresponds to an unphysical source of a long-range scalar field. For an attractive scalar field these solutions admit the case where m_R can be put equal to zero. In this case one finds that the classical condition of balance holds for $|q| = |b|$, where $|b|$ is the scalar charge strength.

In the present work we are tempted to investigate the same problem from a different point of view. In a recent work, Teixeira, Wolk, and Som[4] showed how a static solution of the Einstein-Maxwell equations may be derived from any static vacuum solution. Henceforth we call it the TWS method. In the literature there are other methods (Bonnor[5], Janis, Robinson, and Winicour[6]); however, the interesting feature of the TWS method lies in the fact it is a generalization of the methods previously given by Bonnor and by Janis, Robinson, and Winicour. All the known solutions of the Einstein-Maxwell equations for source-free fields can be obtained quite easily by this method, and further one can recover the original exterior solution in a straightforward way. If the RN solution is obtained from the Schwarzschild solution by the TWS method, it then follows that the mass parameter cannot be zero unless the charge parameter is zero. Though it is a particular way of obtaining the RN solution from the Schwarzschild solution, the method picks out, from among all possible RN solutions, just those of physical interest.

In Sec. II we shall present a brief review of the TWS method for developing coupled field solutions for the electromagnetic field from known vacuum solutions. In Sec. III we have used this technique to obtain the Reissner-Nordström solution from the Schwarzschild solution.

II. TWS METHOD

In the present section we review the TWS method for developing coupled-field solutions for the Einstein-Maxwell equations.

If the metric of the line element

$$ds^2 = e^{2v}(dx^0)^2 - e^{-2v}h_{ij}dx^i dx^j\,, \tag{2.1}$$

where v and h^{ij} are functions of x^i (latin indices vary from 1 to 3), represents the vacuum solutions of the Einstein equations, then a static solution of the Einstein-Maxwell equations is given by

$$ds^2 = e^{2\Psi}(dx^0)^2 - e^{-2\Psi}h_{ij}dx^i dx^j\,, \tag{2.2}$$

where

$$\Psi = -\ln(A\cosh v + B\sinh v)\,. \tag{2.3}$$

The electrostatic potential $\phi(x^i)$ and the electrostatic field $F_{0i}(x^i)$ are

$$\phi = -ae^{\Psi}\sinh v\,, \tag{2.4}$$

$$F_{0i} = ae^{2\Psi}v_{,i}\,. \tag{2.5}$$

A, B, and a are real constants of integrations, related by

$$B^2 - A^2 = a^2\,. \tag{2.6}$$

One can easily generalize (2.3), by including a magnetostatic field

$$F^{ij} = \frac{b}{a}\sqrt{-g}\,\epsilon^{0ijk}\phi_{,k}\,, \tag{2.7}$$

where b is a constant related with the angle of the duality rotation θ by $\tan\theta = -b/a$. The relation (2.6) then takes the form

$$B^2 - A^2 = a^2 + b^2\,. \tag{2.8}$$

We shall consider here only the electrostatic field so that the constants A, B, and a always satisfy the relation (2.6). The different methods in the literature can be obtained by suitably choosing the constants.

Case 1. When $A = a\sinh c$, one has $B = a\cosh c$; then Eq. (2.3) reduces to

$$\Psi = -\ln|a\sinh(v+c)|\,. \tag{2.9}$$

Bonnor[3] obtained the result in this form, which expresses the solution of the Einstein-Maxwell equations in terms of the known vacuum solution. In this form, the field refers to a set of particles for each of which the specific charge is the same, and such that the gravitational and electric forces on each particle balance. However, there is no straightforward way to switch back to the original vacuum solution.

Case 2. When $A = 0$, we have $B^2 = a^2$. If $a = 1$, then Eq. (2.3) takes the form

$$\Psi = -\ln\sinh v\,. \tag{2.10}$$

Equation (2.10) is equivalent to the result obtained by Janis, Robinson, and Winicour.[6] One can remark that such a field is due to the existence of a source-free electrostatic field, and vanishes as soon as the electric field vanishes.

Case 3. When $A = 1$, we have $B = \pm(1+a^2)^{1/2}$; then Eq. (2.3) is given by

$$\Psi = -\ln[\cosh v - (1+a^2)^{1/2}\sinh v]\,. \tag{2.11}$$

We have chosen the negative value of B, because when $a = 0$, then Eq. (2.11) corresponds to the original vacuum solution. In this form Teixeira, Wolk, and Som[3] presented the result which gives the solutions of the Einstein-Maxwell equations in terms of the known vacuum solutions.

III. REISSNER-NORDSTRÖM SOLUTION

To obtain the RN metric by the above TWS prescription we start from the Schwarzschild line element

$$ds^2 = \left(1 - \frac{2m_S}{r}\right)(dx^0)^2 - \left(1 - \frac{2m_S}{r}\right)^{-1}dr^2 - r^2 d\Omega^2\,. \tag{3.1}$$

Next we perform the coordinate transformation

$$\exp(-2m_S/\rho) = 1 - 2m_S/r\,. \tag{3.2}$$

The line element (3.1) takes the form

$$ds^2 = e^{-2m_S/\rho}(dx^0)^2 - e^{2m_S/\rho}[(m_S/\rho)^4\sinh^{-4}(m_S/\rho)d\rho^2 + m_S{}^2\sinh^{-2}(m_S/\rho)d\Omega^2]\,. \tag{3.3}$$

From this static, spherically symmetric vacuum solution the prescription given by the TWS method leads immediately to

$$ds^2 = [\cosh(m_S/\rho) + (1+a^2)^{1/2}\sinh(m_S/\rho)]^{-2}(dx^0)^2 - [\cosh(m_S/\rho) + (1+a^2)^{1/2}\sinh(m_S/\rho)]^2 \times[(m_S/\rho)^4 \sinh^{-4}(m_S/\rho)d\rho^2 + m_S{}^2\sinh^{-2}(m_S/\rho)d\Omega^2]. \quad (3.4)$$

The line element (3.4) can be written in the familiar form

$$ds^2 = \left(1 - \frac{2m_R}{r} + \frac{q^2}{r^2}\right)(dx^0)^2 - \left(1 - \frac{2m_R}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (3.5)$$

by making the coordinate transformation

$$r = m_S[\coth(m_S/\rho) + (1+a^2)^{1/2}], \quad (3.6)$$

where

$$m_R = (m_S{}^2 + q^2)^{1/2}, \quad q = m_S a. \quad (3.7)$$

The equations (2.4) and (2.5) then reduce to

$$\phi = -\frac{q}{r}, \quad (3.8)$$

$$F_{10} = \frac{q}{r^2}. \quad (3.9)$$

The transformation (3.6) is a simple change of radial coordinate whose meaning is clear in the asymptotic region, where

$$r = \rho + (m_S{}^2 + q^2)^{1/2}.$$

IV. CONCLUSION

Starting from the Schwarzschild solution we obtained the solution (3.4) of the Einstein-Maxwell equations for the field of a spherically symmetric charged mass point. In regions far from the source, the geometry of the space-time is flat. By a real coordinate transformation (3.6) the solution (3.4) can be reduced to (3.5), which is similar in form to that of the RN solution. However, in this case the geometrical mass is given by $m_R = (m_S{}^2 + q^2)^{1/2}$. If one puts $q = 0$, one immediately gets back the Schwarzschild solution. The vanishing of m_R now implies that both m_S and q must vanish identically, provided one accepts both of them real which is the case for the TWS method. In this case the geometrical mass of the charged mass point is no more found to be independent of the electric charge q. When $m_S \to 0$, the effective mass $m_R \to |q|$. The coordinate transformation (3.6) reduces to $r = \rho + |q|$, and $a \to \infty$ as $m_S \to 0$ in such a way that $q = m_S a$ is finite. The line element (3.5) tends to the well-known form

$$ds^2 = (1 - |q|/r)^2(dx^0)^2 - (1 - |q|/r)^{-2}dr^2 - r^2 d\Omega^2, \quad (4.1)$$

and the Kretschmann scalar

$$\alpha \to \frac{8}{r^6}\left(6q^2 - 12\frac{|q|^3}{r} + 7\frac{q^4}{r^2}\right). \quad (4.2)$$

which never goes to zero for any real value of $q > 0$ in finite regions. One recovers the flat space-time geometry only when $m_R = 0$. However, if one does not want to give any invariant significance to the parameters m_S and q, one might put $m_R = 0$ and $q \neq 0$. In this case we have from (3.6)

$$m_S = iq, \quad a = \pm i, \quad r = q\cot q/\rho, \quad (4.3)$$

which asymptotically reduces to $r = \rho$. This gives rise to a peculiar situation. In the near regions we find that r might take up positive as well as negative values as ρ monotonically increases, which implies an unusual behavior of the radial coordinate.

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